Econ 8801: The Public Finance of Redistribution

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1 A Simple, Static Model of Labor Supply

A continuum of mass one of individual agents, with identical preferences over consumption, c, and hours worked, l, given by $U(c, l)$ (increasing and concave, decreasing and convex – i.e., $V(c, \ell) \equiv U(c, 1-\ell)$ is increasing and concave in (c, ℓ)).

Each agent has a distinctive labor productivity, θ, so that if l hours are worked, an agent of type θ produces θl units of effective labor.

The distribution of $\theta' s$ across agents is given by the probability $dG(\theta)$.

2 Competitive Equilibrium Version: Ex Post

There is a consumption good, and the different types of labor. We will use the consumption good, c, as numeraire.

The agent of productivity type θ solves:

$$\begin{array}{lccl} & & \max\limits_{c,l} & U(c,l) \\ s.t. & c & \leq & w(\theta)l. \end{array}$$

We will assume that there is a CRS production function in the input 'vector' $l_f(\theta)$, where $l_f(\theta)$ is the number of hours worked by each worker of type θ.

The firm's problem is:

$$\begin{array}{lccl} & & \max\limits_{y,l_f(\theta)} & y - \int w(\theta) l_f(\theta) dG(\theta) \\ s.t. & y & \leq & \int \theta l_f(\theta) dG(\theta) \end{array}$$

Hence, profit maximization requires that

$$w(\theta) = \theta \text{ for all } \theta.$$

What is the firm's problem?

Thus, an equilibrium can be described by a wage function, $w(\theta)$, and an all allocation for each type $(c(\theta), l(\theta))$ (why must each agent of type θ get the same thing?) and an allocation for the representative firm, $(y, l_f(\theta))$, such that:

1. For each θ, $(c(\theta), l(\theta))$ solves:

   $\max_{c,l} \qquad U(c,l)$

   s.t. $\qquad c \leq w(\theta)l.$

2. $w(\theta) = \theta$

   i.e., the firm problem:

   $\max_{y, l_f(\theta)} \qquad y - \int w(\theta) l_f(\theta) dG(\theta)$

   s.t. $\qquad y \leq \int \theta l_f(\theta) dG(\theta)$

3. Markets Clear:

   $\int c(\theta) dG(\theta) = y;$

   $l_f(\theta) = l(\theta)$ for all θ;

Proposition 1 *** is an Ex Post CE if and only if:*

1. *$(c(\theta), l(\theta))$ solves:*

$$\begin{aligned} \max_{c,l} \quad & U(c,l) \\ \textit{s.to.} \quad c \leq \; & \theta l \end{aligned}$$

2. *$l_f(\theta) = l(\theta)$ for all θ*
3. *etc.*

2.1 Characterizing Equilibrium: General

Notice that utility is increasing in θ. That is, let $(c(\theta), l(\theta))$ denote the solution to the consumer's problem. And let $W(\theta) = U(c(\theta), l(\theta))$. Then, $W(\theta)$ is increasing in θ.

2.2 Special Case I: Log Utility

Suppose that $U = \alpha \log c + (1-\alpha)\log(1-l)$.

Then CP is:

$$\begin{aligned} & \max_{c,l} \quad \alpha \log c + (1-\alpha)\log(1-l) \\ s.to.\ c \ & \leq \ \theta l \end{aligned}$$

$$\max_{c,l} \quad \alpha \log \theta l + (1-\alpha)\log(1-l)$$

FOC is:

$$\frac{\alpha\theta}{\theta l} = \frac{1-\alpha}{(1-l)}$$

$$\frac{\alpha}{1-\alpha} = \frac{l}{(1-l)}$$

Thus, $l(\theta)$ doesn't depend on θ. Indeed, $l = \alpha$ for all $\theta' s$.

It follows that $c(\theta) = \alpha\theta$ which is increasing in θ.

Basically, the wealth of a θ agent is θ and he splits this between c and leisure according to weights in utility as is standard with log utility.

Also, $W(\theta) = \alpha \log \theta + (1-\alpha)\log(1-\alpha)$, which is increasing in θ.

Proposition 2 *Those individuals who are 'born' with low $\theta' s$, would have been happier if they had been born with higher $\theta' s$.*

Proposition 3 *The Income Distribution is F.*

2.2.1 Adding Linear Income Taxes with Log Utility

The new CP is:

$$\begin{aligned} &\max_{c,l} \quad \alpha \log c + (1-\alpha)\log(1-l) \\ s.to.\ c \ &\leq \ (1-\tau)\theta l + T \end{aligned}$$

Note: Need to add extra equilibrium condition here that the governments budget balances:

$$\int \tau\theta l(\theta) dF(\theta) = \int T(\theta) dF(\theta) = T.$$

$$\max_{c,l} \quad \alpha \log\left((1-\tau)\theta l + T\right) + (1-\alpha)\log(1-l)$$

FOC is:

$$\frac{\alpha(1-\tau)\theta}{((1-\tau)\theta l + T)} = \frac{1-\alpha}{(1-l)}$$

$$\alpha(1-\tau)\theta(1-l) = (1-\alpha)\left((1-\tau)\theta l + T\right)$$

$$\alpha(1-\tau)(1-l) = (1-\alpha)\left((1-\tau)l + \frac{T}{\theta}\right)$$

$$LHS(l;\theta) = \alpha(1-\tau)(1-l) = (1-\alpha)\left((1-\tau)l + \frac{T}{\theta}\right) = RHS(l;\theta)$$

$LHS(l;\theta)$ is decreasing in l as long as $0 < \tau < 1$.

$RHS(l;\theta)$ is increasing in l as long as $0 < \tau < 1$.

So there is a unique solution (if there is one).

LHS doesn't change across $\theta's$ and RHS falls as θ goes up. Thus, $l(\theta)$ is increasing in θ.

IF $\tau > 0$ and $T = 0$, $l(\theta)$ is unchanged for all θ. Adding in $T > 0$ shifts RHS up for all $\theta's$ causing ALL $l(\theta)'s$ to fall. How much depends on how big θ is. For example, IF $\theta = \infty$, $l(\theta; \tau > 0, T = 0) = l(\theta; \tau > 0, T > 0)$.

Who is better off and who is worse off?
For sure the net transfer satisfies:

$$T - \theta_{\max}\tau l(\theta_{\max}) \leq 0 \leq T - \theta_{\min}\tau l(\theta_{\min})$$

since $l(\theta)$ is increasing in θ (see above).
If everyone is the same (i.e., $G = \delta_{\{\theta\}}$ for some θ), then everyone is worse off (First Welfare Theorem) if $\tau > 0$.

If $\theta = 0$ is possible, they are are obviously helped.

If θ is low enough, $l(\theta; \tau, T) = 0$ This will hold for all $\theta's$ such that the marginal value of leisure at $\ell = 1$ (so that $l = 0$) exceeds the marginal utility of an additional unit of consumption, per unit of work.

The marginal value of leisure is $\frac{1-\alpha}{1-l}$, which at $l = 0$ is $1 - \alpha$. While the marginal value of extra consumption from working just a bit is $\frac{\alpha(1-\tau)\theta}{((1-\tau)\theta l + T)}$, which, when $l = 0$ is $\frac{\alpha(1-\tau)\theta}{T}$. I.e., $l = 0$ for all $\theta's$ such that:

$$1 - \alpha \geq \frac{\alpha(1-\tau)\theta}{T}$$

$$\theta \leq \frac{(1-\alpha)T}{\alpha(1-\tau)}$$

This is not as simple as it sounds since T and τ are linked through the Gov't Budget constraint.

Problems Show how the equilibrium depends on the distribution of $\theta' s$. E.g., suppose $P(\theta = .5) = \varepsilon$, and $P(\theta = 1.5) = 1 - \varepsilon$. How does the equilibrium depend on ε? When is it true that $l(.5) = 0$? Who is helped and hurt by a given τ, T scheme?

What does the Income Distribution look like in this case? $y(\theta) = \theta l(\theta)$ and $l(\theta)$ depends on τ, T, F.

What about non-linear tax systems?

What if it is a disutility of work shock?

Can we show that the highest type is always made worse off?

What if $\theta \sim U[0, 2]$? What is the equilibrium? Who is made better off and who is made worse off?

Special Case II: CES Utility

Consider the problem:

$$\max_{c,\ell} \qquad u(c, \ell) = \tfrac{a_1}{1-\sigma} c^{1-\sigma} + \tfrac{a_2}{1-\sigma} \ell^{1-\sigma}$$

$$\text{s.t.} \qquad c + \theta\ell \leq \theta.$$

$$\max_{\ell} \qquad u(c, \ell) = \tfrac{a_1}{1-\sigma} \left[\theta(1-\ell)\right]^{1-\sigma} + \tfrac{a_2}{1-\sigma} \ell^{1-\sigma}$$

$$\text{FOC} \qquad a_1 \theta^{1-\sigma} (1-\ell)^{-\sigma} = a_2 \ell^{-\sigma};$$

$$\text{FOC} \qquad a_1 \theta^{1-\sigma} \left[\tfrac{\ell}{1-\ell}\right]^{\sigma} = a_2;$$

$$\text{FOC} \qquad \left[\tfrac{\ell}{1-\ell}\right]^{\sigma} = \tfrac{a_2}{a_1 \theta^{1-\sigma}};$$

$$\text{FOC} \qquad \tfrac{\ell}{1-\ell} = \left[\tfrac{a_2}{a_1}\right]^{1/\sigma} \theta^{(\sigma-1)/\sigma}.$$

Thus, $\frac{\ell}{1-\ell}$ and hence ℓ is decreasing in θ if and only if $\sigma < 1$. I.e., $l = 1 - \ell$ is increasing in θ if and only if $\sigma < 1$.

This is easy to see by looking at the two extremes, $\sigma = 0$, perfect subsitutes between c and ℓ and $\sigma = \infty$, perfect complements between c and ℓ.

3 What if there were Insurance against θ?

The simplest Mirlees example is.

Government can see everyone's θ.

Two different formulations:

1. Government observes each persons θ and tells them what to do – $(l(\theta), c(\theta))$.

2. Government announces a tax schedule as a function of income and type – $T(y, \theta)$ – and individual agents solve:

$$\max_{l,c} \qquad U(c, 1-l) \qquad s.t.\ c \leq \theta l(\theta) - T(\theta l(\theta), \theta).$$

1. Any allocation of the form in 1 that satisfies: $U(c(\theta), 1-l(\theta))$ that satisfies $U(c(\theta), l(\theta)) \geq U(0, 1)$ can be realized as an equilibrium with a tax schedule:

Just have the tax schedule give a θ type 0 consumption and $l - 0$ if they don't work the right amount? I.e., suppose you want to implement $(l^*(\theta), c^*(\theta))$ then let $T(\theta l^*(\theta), \theta) = \theta l^*(\theta) - c^*(\theta)$ if $l = l^*$ and $T(\theta l(\theta), \theta) = \theta l(\theta)$ if $l \neq l^*$.

2. Any equilibrium of the form in 2 can be realized as a order by the government – if the equilibrium is $(l(\theta), c(\theta))$, let this be what is dictated.

3.1 Benevolent Government With Full Information:

$$\max_{\{c(\theta), l(\theta)\}} \qquad \int U(c(\theta), 1-l(\theta)) dG(\theta)$$

$$\int c(\theta) dG(\theta) \leq \int \theta l(\theta) dG(\theta)$$

Special Utility: Assume that $U(c, 1-l) = u(c) - v(l)$ where for usual preference properties, we need u to be increasing and concave, and v to be increasing and convex.

$$\max_{\{c(\theta), l(\theta)\}} \qquad \int \left[u(c(\theta)) - v(l(\theta))\right] dG(\theta)$$

$$\int c(\theta) dG(\theta) \leq \int \theta l(\theta) dG(\theta).$$

Assume that there is a density for θ, $g(\theta) = dG(\theta)$. Then, this becomes:

$$\max_{\{c(\theta), l(\theta)\}} \qquad \int \left[u(c(\theta)) - v(l(\theta))\right] g(\theta) d\theta$$

$$\int c(\theta) g(\theta) d\theta \leq \int \theta l(\theta) g(\theta) d\theta.$$

FOC's

$c(\theta) \qquad u'(c(\theta))g(\theta) = \lambda g(\theta)$

$l(\theta) \qquad v'(l(\theta))g(\theta) = \lambda\theta g(\theta)$

$c(\theta) \qquad u'(c(\theta)) = \lambda$

$l(\theta) \qquad v'(l(\theta)) = \lambda\theta$

$c(\theta) \qquad c(\theta) = \bar{c}$

$\qquad u'(\bar{c}) = \lambda;$

$l(\theta) \qquad v'(l(\theta)) = u'(\bar{c})\theta$

I.e., c doesn't depend on θ, $l(\theta)$ is increasing in θ.

Thus: $U(c(\theta), 1 - l(\theta))$ is decreasing in θ.

I.e., it's not 'incentive compatible' if θ is only privately observed.

3.1.1 Implementing the Optimum with Taxes

The consumer's problem is:

$$\max_{l,c} \qquad U(c, 1-l) \qquad s.t.\ c \leq \theta l(\theta) - T(\theta l(\theta), \theta).$$

$$\max_{l} \qquad U(\theta l - T(\theta l, \theta), 1 - l).$$

The FOC is:

$$\theta U_1 - \theta T' U_1 = U_2.$$

For our utility function this becomes:

$$\theta u'(c(\theta)) \left[1 - T'(\theta l(\theta), \theta)\right] = v'(l(\theta)) \text{ for all } \theta.$$

From the FOC's of the Planner's Problem above, we have that:

$c(\theta) \qquad u'(c(\theta)) = \lambda$

$l(\theta) \qquad v'(l(\theta)) = \lambda\theta$

so that

$$v'(l(\theta)) = \theta u'(c(\theta))$$

for all θ.

Thus, to implement this with a (smooth) tax system, it must be true

$$1 - T'(\theta l(\theta), \theta) = 1 \text{ for all } \theta$$

i.e., $T'(\theta l(\theta), \theta) = 0$

Thus, all taxes are lump sum, but these lump sum amounts are type specific, $T(y, \theta) = T(\theta)$.

Since everyone has the same c and higher types have higher l, it follows that $T(\theta)$ is decreasing in θ.

What is income distribution? With log utility, high types have higher pre-tax income (they work more), all types have the same after tax income (since it is given by c).

What if U is not separable?

Who is happy, who is not?

3.1.2 Insurance

Why is this called insurance?

Suppose that we gave each person, ex ante a choice between the government 'contract' and the Ex Post CE. Ex ante, every agent would choose to go with the gov't contract. It insures agents against the future realization of their type.

Indeed, you could rephrase the gov't's contracting problem as one of a profit maximizing insurance company. You'd get exactly the same FOC's if you also imposed a minimum expected utility on the contracts that they could offer.

3.2 Private Information version:

Assume that only the agent can see θ and hours, $l(\theta)$, but that the government can see output – $\theta l(\theta) = y(\theta)$ – and consumption $c(\theta)$.

Again, there are two versions of the plan:

1. The government announces a contract $(l(\theta), c(\theta))$, but is restricted in that it can only choose contracts where it is in the agents own self-interest (given that θ is private info) to deliver on those contracts:

2. The government announces a tax function, $T(y)$ and then agents optimize to get $(l(\theta), c(\theta))$:

$$\max_{c,l} \qquad U(c,l) \qquad s.t. \qquad c \leq \theta l - T(\theta l).$$

I.e., T can no longer depend on θ.

CONTRACT VERSION:

$$\max_{\{c(\theta),l(\theta)\}} \qquad \int [u(c(\theta)) - v(l(\theta))]\, g(\theta) d\theta$$

$$\int c(\theta) g(\theta) d\theta \leq \int y(\theta) g(\theta) d\theta$$

$$u(c(\theta)) - v(l(\theta)) \geq u(c(\hat{\theta})) - v(\tfrac{\hat{\theta}}{\theta} l(\hat{\theta})) \text{ for all } \theta, \hat{\theta}.$$

I.e., output, $\theta l(\theta)$ is directly observable for each person, but neither θ nor $l(\theta)$ is. Thus, a person of productivity θ can 'pretend' to be a person of type $\hat{\theta}$ by producing the same output as is required of a person of type $\hat{\theta}$. This requires $\frac{\hat{\theta}}{\theta} l(\hat{\theta})$ from a person of type θ.

3.3 Simple version #1:

Assume that there are exactly two types, θ_H and θ_L with $\theta_H > \theta_L$ and $P(\theta = \theta_H) = \pi_H$.

Then, the problem becomes:

$$\max_{\{c_H,l_H,c_L,l_L\}} \qquad \pi_H [u(c_H) - v(l_H)] + \pi_L [u(c_L) - v(l_L)]$$

FEAS $\quad \pi_H c_H + \pi_L c_L \leq \pi_H \theta_H l_H + \pi_L \theta_L l_L$

IC1 $\quad u(c_H) - v(l_H) \geq u(c_L) - v(\frac{\theta_L}{\theta_H} l_L)$

IC2 $\quad u(c_L) - v(l_L) \geq u(c_H) - v(\frac{\theta_H}{\theta_L} l_H)$.

3.3.1 Step 1: Rewrite the Problem

Rewrite the problem in terms of output requirements by types. Let y_H be the output required of a high type and y_L that required of a low type.

$$\max_{\{c_H,y_H,c_L,y_L\}} \qquad \pi_H \left[u(c_H) - v(\tfrac{y_H}{\theta_H})\right] + \pi_L \left[u(c_L) - v(\tfrac{y_L}{\theta_L})\right]$$

FEAS $\quad \pi_H c_H + \pi_L c_L \leq \pi_H y_H + \pi_L y_L$

IC1 $\quad u(c_H) - v(\frac{y_H}{\theta_H}) \geq u(c_L) - v(\frac{y_L}{\theta_H})$

IC2 $\quad u(c_L) - v(\frac{y_L}{\theta_L}) \geq u(c_H) - v(\frac{y_H}{\theta_L})$.

3.3.2 Step 2: Note some properties of Feasible Contracts

Here we note some simple properties of contracts – i.e., combinations of (c_L, y_L) and (c_H, y_H) that must be true if FEAS, IC1 and IC2 are all satisfied.

1. Suppose $c_H > c_L$ but $y_H \leq y_L$. If this were true, then:

   $u(c_H) - v(\frac{y_H}{\theta_L}) > u(c_L) - v(\frac{y_H}{\theta_L})$ since $c_H > c_L$ and u is monotone;

   $u(c_L) - v(\frac{y_H}{\theta_L}) \geq u(c_L) - v(\frac{y_L}{\theta_L})$ since $y_L \geq y_H$ and v is monotone;

   Thus,

   $u(c_H) - v(\frac{y_H}{\theta_L}) > u(c_L) - v(\frac{y_L}{\theta_L})$,

   But this violates IC2.

2. A similar argument holds if $c_H \geq c_L$ and $y_H < y_L$.

3. Suppose $c_H < c_L$ but $y_L \leq y_H$. If this were true, as above, we would have:

   $u(c_L) - v(\frac{y_L}{\theta_H}) > u(c_H) - v(\frac{y_H}{\theta_H})$

   I.e., IC1 would be violated.

4. A similar argument holds if $c_H \leq c_L$ but $y_L < y_H$.

   Let's summarize:

Lemma 4 *If the contract (c_L, y_L) and (c_H, y_H) satisfies FEAS, IC1 and IC2, then one of the following three configurations must hold:*

1. $c_H > c_L$ *and* $y_H > y_L$;

2. $c_L > c_H$ *and* $y_L > y_H$;

3. $c_L = c_H$ *and* $y_L = y_H$.

3.3.3 Step 3: Showing that Configuration #2 is not feasible

First some simple intuition – this is that the indifference curves of the high type are, at every point steeper (in (c, y) space) than those of the low type:

Fix a pair, (c, y). At this point, what is the slope of an indifference curve of a type θ agent? (Note that indifference curves are upward sloping in this space – draw a picture.) This is given by:

$$\frac{dy}{dc} = -\frac{U_c(\theta)}{U_y(\theta)} = -\frac{\partial U(c,y;\theta)/\partial c}{\partial U(c,y;\theta)/\partial y} = -\frac{u'(c)}{-\frac{1}{\theta}v'(y/\theta)} = \theta\frac{u'(c)}{v'(y/\theta)}.$$

Since v is convex and y/θ is decreasing in θ, $v'(y/\theta)$ is decreasing in θ holding y fixed. Thus, increasing θ increases $\frac{1}{v'(y/\theta)}$ and increases θ. Thus, increasing θ increases $\theta\frac{1}{v'(y/\theta)}u'(c) = \frac{dy}{dc}$.

In sum, higher θ types have steeper indifference curves than lower θ types through any (c, y) pair.

The reason that this matters is:

Suppose that it is true that $(c_L, y_L) > (c_H, y_H)$.

Consider the indifference curve of the L type that passes through the point (c_H, y_H). By IC2, this corresponds to a (weakly) lower level of utility to the low type. But, since the indifference curves of the high type are everywhere steeper than those of the low, it follows that the IC of the high type through the point (c_H, y_H) passes ABOVE (c_L, y_L). This implies that the high type prefers the choice (c_L, y_L) – IC1 is violated. Thus, it must be true that $(c_H, y_H) > (c_L, y_L)$ – the optimal contract is monotone increasing in type.

Next, we'll do this formally:

Assume $(c_L, y_L) > (c_H, y_H)$. By IC2,

$$U_L(c_L, y_L) - U_L(c_H, y_H) = u(c_L) - u(c_H) - \left[v(\tfrac{y_L}{\theta_L}) - v(\tfrac{y_H}{\theta_L})\right] \geq 0;$$

or,

$$u(c_L) - u(c_H) - \tfrac{1}{\theta_L}\int_{y_H}^{y_L} v'(\tfrac{y}{\theta_L})dy \geq 0;$$

$$u(c_L) - u(c_H) \geq \frac{1}{\theta_L} \int_{y_H}^{y_L} v'(\frac{y}{\theta_L})dy.$$

Notice that the first term, $u(c_L) - u(c_H)$ is positive since c_L is assumed to be larger than c_H.

Also, since $v' > 0$ and $y_L > y_H$ it follows that the second term is also positive.

But since $\theta_H > \theta_L$ and v is convex, it follows that $v'(y/\theta_H) < v'(y/\theta_L)$ for all y and so,

$$\int_{y_H}^{y_L} v'(\frac{y}{\theta_H})dy < \int_{y_H}^{y_L} v'(\frac{y}{\theta_L})dy.$$

Since $\theta_H > \theta_L$, we also have:

$$\frac{1}{\theta_H} \int_{y_H}^{y_L} v'(\frac{y}{\theta_H})dy < \frac{1}{\theta_L} \int_{y_H}^{y_L} v'(\frac{y}{\theta_L})dy.$$

Thus,

$$u(c_L) - u(c_H) \geq \frac{1}{\theta_L} \int_{y_H}^{y_L} v'(\frac{y}{\theta_L})dy > \frac{1}{\theta_H} \int_{y_H}^{y_L} v'(\frac{y}{\theta_H})dy,$$

so,

$$u(c_L) - u(c_H) - \frac{1}{\theta_H} \int_{y_H}^{y_L} v'(\frac{y}{\theta_H})dy > 0,$$

$$U_H(c_L, y_L) - U_H(c_H, y_H) = u(c_L) - u(c_H) - \left[v(\frac{y_L}{\theta_H}) - v(\frac{y_H}{\theta_H})\right] > 0.$$

That is H prefers (c_L, y_L) too, violating IC1.

Thus, either $c_L = c_H$, or $c_H > c_L$ and $y_H > y_L$.

3.3.4 Step 4: Show that we can drop IC2

From here on, we'll ignore the possible situation in which $c_L = c_H$ and $y_L = y_H$.

The Planner's Problem is:

(PP) $\quad \max_{c_L, y_L, c_H, y_H} \quad \pi_H \left[u(c_H) - v(\frac{y_H}{\theta_H})\right] + \pi_L \left[u(c_L) - v(\frac{y_L}{\theta_L})\right]$

Subject to:

FEAS $\quad \pi_H c_H + \pi_L c_L \leq \pi_H y_H + \pi_L y_L;$

IC1 $\quad u(c_H) - v(\frac{y_H}{\theta_H}) \geq u(c_L) - v(\frac{y_L}{\theta_H});$

IC2 $\quad u(c_L) - v(\frac{y_L}{\theta_L}) \geq u(c_H) - v(\frac{y_H}{\theta_L})$;

MONOT $\quad c_L < c_H$ and $y_H < y_L$.

Consider the following Relaxed Version of this maximization problem:

(RP) $\quad \max_{c_L, y_L, c_H, y_H} \quad \pi_H \left[u(c_H) - v(\frac{y_H}{\theta_H}) \right] + \pi_L \left[u(c_L) - v(\frac{y_L}{\theta_L}) \right]$

Subject to:

FEAS $\quad \pi_H c_H + \pi_L c_L \leq \pi_H y_H + \pi_L y_L$;

IC1 $\quad u(c_H) - v(\frac{y_H}{\theta_H}) \geq u(c_L) - v(\frac{y_L}{\theta_H})$;

MONOT $\quad c_L < c_H$ and $y_H < y_L$.

Since there are strictly less constraints for this problem, it follows that if a solution exists, and if that solution satisfies IC2, it is a solution for PP too.

We will show that this is true.

Suppose that the solution to (RP) is (c_L, y_L) and (c_H, y_H).

We'll show that IC1 is satisfied at equality. We will do this by supposing it is false and constructing a better contract. The better contract that we will construct will have better insurance over c without disrupting IC1.

Suppose that IC1 is NOT satisfied at equality – suppose that

$$u(c_H) - v(\frac{y_H}{\theta_H}) > u(c_L) - v(\frac{y_L}{\theta_H}).$$

Notice that if this holds, by continuity, it will still hold if we add a bit to c_L and subract a bit from c_H –

$$u(c_H - \varepsilon) - v(\frac{y_H}{\theta_H}) > u(c_L + \delta) - v(\frac{y_L}{\theta_H})$$

as long as ε and δ are small enough.

Consider the alternative contract given by $(c_H - \varepsilon, y_H)$ and $(c_L + \delta, y_L)$

Choose $\delta = \frac{\pi_H}{\pi_L}\varepsilon$.

Then, if ε is small enough, IC1 will still hold.

FEAS becomes:

$$\pi_H(c_H - \varepsilon) + \pi_L(c_L + \delta) = \pi_H c_H + \pi_L c_L + \pi_H \varepsilon - \pi_L \frac{\pi_H}{\pi_L}\varepsilon = \pi_H c_H + \pi_L c_L.$$

Thus, FEAS will hold because we didn't change y_L or y_H and because of the way we constructed δ.

So, we only need to show that welfare goes up from this change, even when ε is small and positive. To see this note that the change in welfare is given by:

$$\Delta W = \pi_H \left[u(c_H - \varepsilon) - u(c_H)\right] + \pi_L \left[u(c_L + \tfrac{\pi_H}{\pi_L}\varepsilon) - u(c_L)\right].$$

The terms involving the $y's$ do not appear in this since they are unchanged.

$$\frac{d\Delta W}{d\varepsilon}|_{\varepsilon=0} = -\pi_H u'(c_H) + \pi_L \frac{\pi_H}{\pi_L} u'(c_L) = -\pi_H u'(c_H) + \pi_H u'(c_L) = \pi_H \left[u'(c_L) - u'(c_H)\right] > 0$$

since $c_L < c_H$ and u is strictly concave.

Thus, (RP) is equivalent to:

(RP') $\quad \max_{c_L, y_L, c_H, y_H} \quad \pi_H \left[u(c_H) - v(\frac{y_H}{\theta_H})\right] + \pi_L \left[u(c_L) - v(\frac{y_L}{\theta_L})\right]$

Subject to:

FEAS $\quad \pi_H c_H + \pi_L c_L \leq \pi_H y_H + \pi_L y_L;$

IC1 $\quad u(c_H) - v(\frac{y_H}{\theta_H}) = u(c_L) - v(\frac{y_L}{\theta_H});$

MONOT $\quad c_L < c_H$ and $y_H < y_L$.

To finish the proof that IC2 is redundant, it suffices to show that at the solution to (RP'), IC2 is satisfied.

I.e., we want to show that if:

$$u(c_H) - v(\tfrac{y_H}{\theta_H}) = u(c_L) - v(\tfrac{y_L}{\theta_H});$$

and

$$c_L < c_H \text{ and } y_L < y_H$$

then,

$$u(c_L) - v(\tfrac{y_L}{\theta_L}) > u(c_H) - v(\tfrac{y_H}{\theta_L}).$$

The proof of this follows from the fact that the indifference curves of the low type are flatter than those of the high type. See Step 3 for the Details.

3.3.5 Summary

Proposition 5 *The solution so (PP) is the same as the solution to (RP).*

3.3.6 Next: Characterize Solution.

Given the steps above, the problem becomes:

$$\max_{\{c_H, l_H, c_L, l_L\}} \qquad \pi_H \left[u(c_H) - v(l_H)\right] + \pi_L \left[u(c_L) - v(l_L)\right]$$

FEAS $\qquad \pi_H c_H + \pi_L c_L \leq \pi_H \theta_H l_H + \pi_L \theta_L l_L \qquad \lambda$

IC1 $\qquad u(c_H) - v(l_H) \geq u(c_L) - v(\frac{\theta_L}{\theta_H} l_L) \qquad \mu$

The Lagrangian is:

$$\begin{array}{l} \pi_H \left[u(c_H) - v(l_H)\right] + \pi_L \left[u(c_L) - v(l_L)\right] \\ +\lambda \left[\pi_H \theta_H l_H + \pi_L \theta_L l_L - \pi_H c_H - \pi_L c_L\right] \\ +\mu \left[u(c_H) - v(l_H) - u(c_L) + v(\dfrac{\theta_L}{\theta_H} l_L)\right]. \end{array}$$

FOC's:

$c_H: \qquad \pi_H u'(c_H) + \mu u'(c_H) = \pi_H \lambda;$

$l_H: \qquad \pi_H v'(l_H) + \mu v'(l_H) = \lambda \pi_H \theta_H;$

$c_L: \qquad \pi_L u'(c_L) - \mu u'(c_L) = \pi_L \lambda;$

$l_L: \qquad \pi_L v'(l_L) - \mu \frac{\theta_L}{\theta_H} v'(\frac{\theta_L}{\theta_H} l_L) = \pi_L \theta_L \lambda.$

From the FOC's for c_H and l_H, we get:

$\frac{[\pi_H + \mu] v'(l_H)}{[\pi_H + \mu] u'(c_H)} = \frac{\lambda \pi_H \theta_H}{\pi_H \lambda}$, or,

$\frac{v'(l_H)}{u'(c_H)} = \theta_H.$

$\frac{v'(l_H)}{\theta_H} = \frac{u'(c_H)}{1}.$

This is the same FOC that we get in the Full Info case.

And from those for c_L and l_L, we get:

$$\frac{\pi_L v'(l_L)-\mu\frac{\theta_L}{\theta_H}v'(\frac{\theta_L}{\theta_H}l_L)}{[\pi_L-\mu]u'(c_L)} = \frac{\pi_L\theta_L\lambda}{\pi_L\lambda}, \text{ or,}$$

$$\frac{\pi_L v'(l_L)-\mu\frac{\theta_L}{\theta_H}v'(\frac{\theta_L}{\theta_H}l_L)}{[\pi_L-\mu]u'(c_L)} = \theta_L;$$

$$\frac{\left[\pi_L-\mu\frac{\theta_L}{\theta_H}\frac{v'(\frac{\theta_L}{\theta_H}l_L)}{v'(l_L)}\right]}{[\pi_L-\mu]}v'(l_L) = \theta_L u'(c_L).$$

Next, we show that $\mu < \pi_L$ and that $\frac{\theta_L}{\theta_H}\frac{v'(\frac{\theta_L}{\theta_H}l_L)}{v'(l_L)} < 1$.

To see the first part, use the FOC's for c_H and c_L :

c_H : $\quad \pi_H u'(c_H) + \mu u'(c_H) = \pi_H\lambda$

c_L : $\quad \pi_L u'(c_L) - \mu u'(c_L) = \pi_L\lambda$

So,

$$\frac{\pi_H u'(c_H)+\mu u'(c_H)}{\pi_L u'(c_L)-\mu u'(c_L)} = \frac{\pi_H}{\pi_L}$$

$$\frac{[\pi_H+\mu]u'(c_H)}{[\pi_L-\mu]u'(c_L)} = \frac{\pi_H}{\pi_L}$$

$$[\pi_H+\mu]\,u'(c_H) = \frac{\pi_H}{\pi_L}\,[\pi_L-\mu]\,u'(c_L)$$

Since $[\pi_H+\mu] > 0$, $u'(c_H) > 0$, $\frac{\pi_H}{\pi_L} > 0$, and $u'(c_L) > 0$ it follows that $\mu < \pi_L$.

Recall from above that we had:

$$\frac{\left[\pi_L-\mu\frac{\theta_L}{\theta_H}\frac{v'(\frac{\theta_L}{\theta_H}l_L)}{v'(l_L)}\right]}{[\pi_L-\mu]}v'(l_L) = \theta_L u'(c_L).$$

Now, if $\theta_L < \theta_H$, since v' is strictly increasing, we have, $\frac{\theta_L}{\theta_H} < 1$ and so, $v'(\frac{\theta_L}{\theta_H}l_L) < v'(l_L)$, which implies that $\frac{v'(\frac{\theta_L}{\theta_H}l_L)}{v'(l_L)} < 1$ and hence, $\frac{\theta_L}{\theta_H}\frac{v'(\frac{\theta_L}{\theta_H}l_L)}{v'(l_L)} < 1$.

Thus, $\mu\frac{\theta_L}{\theta_H}\frac{v'(\frac{\theta_L}{\theta_H}l_L)}{v'(l_L)} < \mu$.

From this it follows that $\frac{\left[\pi_L-\mu\frac{\theta_L}{\theta_H}\frac{v'(\frac{\theta_L}{\theta_H}l_L)}{v'(l_L)}\right]}{[\pi_L-\mu]} > 1$, or $\frac{[\pi_L-\mu]}{\left[\pi_L-\mu\frac{\theta_L}{\theta_H}\frac{v'(\frac{\theta_L}{\theta_H}l_L)}{v'(l_L)}\right]} < 1$ since $\mu < \pi_L$.

Thus, we have:

$$v'(l_L) = \frac{[\pi_L - \mu]}{\left[\pi_L - \mu \frac{\theta_L}{\theta_H} \frac{v'(\frac{\theta_L}{\theta_H} l_L)}{v'(l_L)}\right]} \theta_L u'(c_L) < \theta_L u'(c_L).$$

Or,

$$\frac{v'(l_L)}{\theta_L} < \frac{u'(c_L)}{1}.$$

I.e., the value of leisure to the low type household is less than it's productivity at the margin. If:

$$\frac{v'(l_L)}{(1-\tau)\theta_L} < \frac{u'(c_L)}{1},$$

Then, $0 < \theta < 1$.

From this comes the standard intuition about optimal contracts:

The high type is undistorted at the margin, but the low type is.

As we can see, if $\frac{\theta_L}{\theta_H} = 1$, i.e., there is no private information, then

$$v'(l_L) = \frac{[\pi_L - \mu]}{\left[\pi_L - \mu \frac{\theta_L}{\theta_H} \frac{v'(\frac{\theta_L}{\theta_H} l_L)}{v'(l_L)}\right]} \theta_L u'(c_L) = \frac{[\pi_L - \mu]}{[\pi_L - \mu]} \theta_L u'(c_L) = \theta_L u'(c_L)$$

and so the low type is also undistorted.

3.3.7 Implementing the Optimal Contract with Taxes

Next, we discuss how to implement the contractual outcome characterized above through decentralized decisions by workers subject to income taxes.

That is, we want to find an income tax schedule, $T(y)$, such that for each θ_i, the contractual allocation, (c_i, y_i) is the solution to:

$$(\text{HP}\theta_i) \qquad \max_{c,y} \qquad u(c) - v(\tfrac{y}{\theta_i})$$

$$\text{s.t.} \qquad c \leq y - T(y)$$

In general there is no unique way of doing this, i.e., there are many different functions $T(y)$ such that this is true.

For example, suppose $T(y) = y$ for all y other than y_L and y_H and that $T(y_L) = y_L - c_L$, $T(y_H) = y_H - c_H$ If this is $T(y)$, it follows immediately that no one would ever choose any y level other than y_L or y_H since they would get $c = 0$. Thus, the only relevant options given this tax scheme are (c_L, y_L) and (c_H, y_H). Then, from IC1 and IC2, it follows that the low type chooses (c_L, y_L) and the hight type chooses (c_H, y_H).

A second alternative is to choose $T(y)$ so that the mapping $(y, y - T(y))$ follows the indifference curve of the low type for $y's$ below y_L and follows that of the high type for $y's$ above y_L. I.e.:

1. For $y \leq y_L$, $u(y - T(y)) - v(\frac{y}{\theta_L}) = u(y_L - T(y_L)) - v(\frac{y_L}{\theta_L})$;
2. For $y \geq y_H$, $u(y - T(y)) - v(\frac{y}{\theta_H}) = u(y_H - T(y_H)) - v(\frac{y_H}{\theta_H})$;

This makes the low type indifferent between picking any $y \leq y_L$, and the high type indifferent between picking any $y \geq y_L$. Moreover, given the characterization of the contract above, it follows that the low type is strictly worse off by picking any $y > y_L$ and the high type is strictly worse off by picking any $y < y_L$.

3.3.8 Labor Supply Implications

We know that $y_H > y_L$, but can we tell who works more? I.e., is it also true that $l_H > l_L$?

This is not an easy problem and not much is known about it.

In these notes I'll do two things related to this. First I'll use what we did above concerning the marginal tax rates for the two types along with the ASSUMPTION that there is a transfer from the high type to the low type.

Below, I'll turn to trying to show that $T_H < 0$ and $T_L > 0$ for the special case of log utility.

First something to note. Suppose the income tax function $T(y)$ implements the optimal contract and is differentiable (from the left) for both types, then, from above we have. In particular, suppose $T(y)$ is the tax function given by 1 and 2 above. Then:

$$\frac{d(y-T(y))}{dy}|_{y=y_H} = 1 - T'(y)|_{y=y_H} = 1 - 0 = 1.$$

Where $T'(y)|_{y=y_H} = 0$ follows from the argument above that the high type is undistorted at the margin.

Similarly,

$\frac{d(y-T(y))}{dy}|_{y=y_L} = 1 - T'(y)|_{y=y_L} < 1,$

Since $T'(y)|_{y=y_L} > 0$ follows from above.

Define $T_L = c_L - (1 - T'(y_L))y_L$. I.e., this is just enough of a transfer to allow the low type to afford c_L given that he is producing y_L and given that he faces the constant marginal tax rate $T'(y_L)$ on all income.

Define $T_H = c_H - y_H$. Again, this is exactly the transfer that the high type would need so that he can exactly afford to buy c_H given he is producing y_H and given that he faces a linear tax on all income of $T'(y_H) = 0$.

Now, construct the two part tax function:

1'. $T^*(y) = -T_L + T'(y_L)y_L$ for all $y \leq y_L$. this is an affine tax function, with a non-zero intercept.

2'. $T^*(y) = T_H$ for all $y \geq y_L$. I.e., this is a lump sum tax function.

DRAW PICTURE. This is a two part income tax scheme such that each party would pick their component of the optimal contract. It has the advantage that each component is linear, and they differ in their lump sum transfers.

ASSUMPTION: Assume that $T_H < 0$ (i.e., $c_H < y_H$) and that $T_L > 0$ (i.e., $c_L > (1 - T'(y_L))y_L$).

Finally, define the labor supply function, $l((1-\tau)w, T)$ by:

$$\max_{c,l} \quad u(c) - v(l)$$

$$\text{s.t.} \quad c \leq (1-\tau)wl + T.$$

a) Note that leisure is a normal good here and hence, $l((1-\tau)w, T)$ is strictly decreasing in T, holding w and τ constant.

b) $l_H = \frac{y_H}{\theta_H} = l(\theta_H, 0, T_H) > l(\theta_H, 0, 0)$. I.e., since the high type has a zero marginal tax rate and faces a negative transfer, he works more than he would without taxes at all.

ASSUMPTION: Assume that the labor supply function is upward sloping in net wages– $l((1-\tau)w, T)$ is increasing in $(1-\tau)w$ holding T fixed.

c) Under this assumption, it follows that $l_L = l((1 - T'(y_L))\theta_L, T_L) < l(\theta_L, T_L) < l(\theta_H, T_L) < l(\theta_H, 0)$, where the last step comes from the assumption that $T_L > 0$ given that leisure is a normal good.

d) Under these two assumptions, it follows that $l_L < l(\theta_H, 0) < l(\theta_H, T_H) = l_H$.

3.3.9 Log Utility

Intuitively, I think it should work with log utility. Why? First consider the CE outcome. This has $l_H = l_L$ since income and substitution effects cancel out in this case. And $c_H = \theta_H l_H > \theta_L l_L$. This should lead the Planner to want to distribute from the H to the L. As noted above, he does this by leaving a (positive?) lump sum tax on the high type. This shifts the equilibrium budget constraint in (c, ℓ) space inward in a parallel way. Thus, the H consumes less leisure and less c than without the redistribution. This revenue is rebated to the L and a labor income tax is also levied against him. In the end, this twists the BC of the low type to make it flatter (since the labor income tax is positive, see above), but also shifts it out. Budget balance requires that the vertical increase in BC_L is equal to the vertical decrease in BC_H. Also, it must be true that the consumption leisure allocation of the L must be on the IC of the H. Since this BC is flatter, it must be to the right. But positive labor taxes reduce labor supply of individuals when they are getting lump sum transfers. I.e., they consume more leisure.

Thus, the high type consumes less leisure that at the CE, while the low type consumes more. Since the consume the same amount at the CE, it follows that at the Mirrlees allocation, $l_H > l_L$.

In what follows we give a proof of this result by using the implementation of the Mirrlees allocation using taxes. We assume that $T_H < 0$ and $T_L > 0$, i.e., that the planner moves resources from the high type to the low type. This is still left to be shown.

Assume that $U(c, l) = \alpha \log(c) + (1-\alpha) \log(1-l)$. In the Competitive Equilibrium allocation without insurance we have the following allocation (we have normalized the price of the consumption good to $p_c = 1$):

$$c_H^{ce} = \alpha\theta_H,\ \ell_H^{ce} = 1-\alpha,\ l_H^{ce} = \alpha;$$

$$c_L^{ce} = \alpha\theta_L,\ \ell_L^{ce} = 1-\alpha,\ l_L^{ce} = \alpha;$$

The important thing to note is that $l_L^{ce} = l_H^{ce} = \alpha$ and $\ell_H^{ce} = \ell_L^{ce} = 1-\alpha$.

Next, consider the optimal decision of an agent of type s when faced with an income tax rate of τ_s and with a lump sum transfer of T_s:

$$\max_{\{c,\ell\}} \qquad \alpha \log(c) + (1-\alpha)\log(1-l)$$

$$\text{s.t.} \qquad c \le (1-\tau_s)\theta_s l + T_s.$$

$$\max_{\{c,\ell\}} \qquad \alpha \log((1-\tau_s)\theta_s l + T_s) + (1-\alpha)\log(1-l)$$

FOC is:

$$\frac{\alpha(1-\tau_s)\theta_s}{c_s} = \frac{1-\alpha}{1-l_s};$$

$$c_s = \frac{\alpha}{1-\alpha}(1-\tau_s)\theta_s(1-l_s);$$

and

$$c_s = (1-\tau_s)\theta_s l_s + T_s.$$

Thus,

$$(1-\tau_s)\theta_s l_s + T_s = \frac{\alpha}{1-\alpha}(1-\tau_s)\theta_s(1-l_s);$$

$$l_s = \frac{\alpha}{1-\alpha}(1-l_s) - \frac{T_s}{(1-\tau_s)\theta_s};$$

$$l_s + \frac{\alpha}{1-\alpha}l_s = \frac{\alpha}{1-\alpha} - \frac{T_s}{(1-\tau_s)\theta_s};$$

$$l_s\left[\frac{1-\alpha+\alpha}{1-\alpha}\right] = \frac{\alpha}{1-\alpha} - \frac{T_s}{(1-\tau_s)\theta_s};$$

$$l_s = \alpha - \frac{(1-\alpha)T_s}{(1-\tau_s)\theta_s};$$

Note that if τ_s and $T_s < 0$ it follows that $l_s > \alpha$. While, if $T_s > 0$, $l_s < \alpha$ and it is decreasing in τ_s as long as $\tau_s < 1$.

Thus, IF $T_H < 0$ and $T_L > 0$ then $l_H^{mir} > \alpha > l_L^{mir}$ as desired.

3.3.10 Showing that $T_H < 0$ and $T_L > 0$ in the Log Case

To see that $T_H < 0$ and $T_L > 0$, note that the CE allocation above is incentive compatible. Next we will argue that an allocation in which $T_H > 0$ and $T_L < 0$ is worse for the planner. Consider first an allocation with no income taxes but with lump sum redistribution from the L to the H. The welfare benefit of this transfer to the Planner coming from the increased utility to the H is less than the loss to the Planner coming from the decreased utility to the L type by concavity of the indirect utility function over wealth by the agents. Thus, this makes the planner worse off than just having the CE outcome. Having a positive income tax on labor income of the low type only lowers the utility of the $L's$ further, so that makes the Planner even worse off. I.e., having a transfer from the L to the H coupled with $\tau_L > 0$ (which we know must hold) is worse than just having a transfer directly from the L to the H which is worse than the CE allocation. But, the CE allocation is Incentive Feasible, hence, transferring from the L to the H is never optimal.

Doing the details:

Define $V(\theta, \tau, T)$ by:

$$V(\theta, \tau, T) = \max_{\{c,\ell\}} \qquad \alpha \ln(c) + (1-\alpha)\ln(\ell)$$

$$\text{s.t.} \qquad c + (1-\tau)\theta\ell \leq (1-\tau)\theta + T.$$

As normal,

$$c = \alpha\left[(1-\tau)\theta + T\right]; \text{ and,}$$

$$(1-\tau)\theta\ell = (1-\alpha)\left[(1-\tau)\theta + T\right]$$

$$\ell = (1-\alpha)\left[1 + \tfrac{T}{(1-\tau)\theta}\right].$$

Thus,

$$V(\theta, \tau, T) = \alpha \ln(c) + (1-\alpha)\ln(\ell)$$

$$= \alpha \ln(\alpha\left[(1-\tau)\theta + T\right]) + (1-\alpha)\ln((1-\alpha)\left[1 + \tfrac{T}{(1-\tau)\theta}\right])$$

$$= D + \alpha \ln(\left[(1-\tau)\theta + T\right]) + (1-\alpha)\ln(\left[1 + \tfrac{T}{(1-\tau)\theta}\right]).$$

Where D is a constant.

A feasible scheme has to have the property that:

$$\pi_H T_H + \pi_L T_L \leq \pi_L \tau_L y_L.$$

Or,

$$T_L = -\tfrac{\pi_H}{\pi_L} T_H + \tau_L y_L.$$

First, hold fixed $\tau_L > 0$, and consider varying T_H and T_L so as to keep budget balance.

The utility to the planner from using such a scheme is:

$$V^p(T_H) = \pi_L V(\theta_L, \tau_L, -\tfrac{\pi_H}{\pi_L} T_H + \tau_L y_L) + \pi_H V(\theta_H, 0, T_H).$$

What we want to show is that the best thing for the Planner is to have $T_H < 0$. We will do this by showing that $\partial V^p / \partial T_H|_{T_H=0} < 0$.

Now,

$\frac{\partial V^p}{\partial T_H}|_{T_H=0} = -\pi_H V_3(\theta_L, \tau_L, \tau_L y_L) + \pi_H V_3(\theta_H, 0, 0).$

From the expression for V above, we get that:

$V_3 = \frac{\alpha}{[(1-\tau)\theta+T]} + \frac{1}{(1-\tau)\theta}\frac{1-\alpha}{\left[1+\frac{T}{(1-\tau)\theta}\right]} = \frac{\alpha}{[(1-\tau)\theta+T]} + \frac{1-\alpha}{[(1-\tau)\theta+T]} = \frac{1}{[(1-\tau)\theta+T]},$

And hence,

$\frac{\partial V^p}{\partial T_H}|_{T_H=0} = -\pi_H V_3(\theta_L, \tau_L, \tau_L y_L) + \pi_H V_3(\theta_H, 0, 0)$

$= -\pi_H \frac{1}{[(1-\tau_L)\theta_L+\tau_L y_L]} + \pi_H \frac{1}{[\theta_H]} = \pi_H \left[\frac{1}{\theta_H} - \frac{1}{[(1-\tau_L)\theta_L+\tau_L y_L]}\right].$

Thus, $\frac{\partial V^p}{\partial T_H}|_{T_H=0} < 0$ if and only if:

$\left[\frac{1}{\theta_H} - \frac{1}{[(1-\tau_L)\theta_L+\tau_L y_L]}\right] < 0;$

$\frac{1}{\theta_H} < \frac{1}{[(1-\tau_L)\theta_L+\tau_L y_L]};$

$[(1-\tau_L)\theta_L + \tau_L y_L] < \theta_H.$

To see that this must hold, note that $l_L \leq 1$ so that $y_L \leq \theta_L$, and hence we have:

$[(1-\tau_L)\theta_L + \tau_L y_L] \leq (1-\tau_L)\theta_L + \tau_L\theta_L = \theta_L < \theta_H,$

as desired.

Hence, at the Mirlees optimum, $T_H < 0$ and $T_L = -\frac{\pi_H}{\pi_L}T_H + \tau_L y_L > 0.$

As in the discussion above, this should hold more generally due to the concavity of V. This must be the argument in the paper: Stiglitz, J. E. (1982), Self-Selection and Pareto Efficient Taxation, JPublE 17, 213 - 240.

3.3.11 Another Special Case

This section looks to be a dead end and nothing was done with it. It is just here for now for possible future use.

Suppose $v(l) = a_2 l^\alpha$ with $\alpha > 1$. Then, $v'(l) = \alpha a_2 l^{\alpha-1} = x$ and $l = v'^{-1}(x) = \left[\frac{x}{\alpha a_2}\right]^{1/(\alpha-1)}.$

In this case,

$$\text{(FOCL)} \qquad l_L = \left[\left[\frac{\theta_H}{\theta_L} \right]^{\alpha-1} \right]^{1/(\alpha-1)} \left[\frac{1}{\alpha a_2} \frac{[\pi_L - \mu]}{\left[\pi_L - \mu \frac{\theta_L}{\theta_H}\right]} \theta_L u'(c_L) \right]^{1/(\alpha-1)}$$

$$= \left[\frac{1}{\alpha a_2} \left[\frac{\theta_H}{\theta_L} \right]^{\alpha-1} \frac{[\pi_L - \mu]}{\left[\pi_L - \mu \frac{\theta_L}{\theta_H}\right]} \theta_L u'(c_L) \right]^{1/(\alpha-1)}$$

$$u'(c_H) > \frac{\theta_L}{\theta_H} \left[\frac{\theta_H}{\theta_L} \right]^{\alpha-1} \frac{[\pi_L - \mu]}{\left[\pi_L - \mu \frac{\theta_L}{\theta_H}\right]} u'(c_L);$$

$$u'(c_H) > \left[\frac{\theta_L}{\theta_H} \right]^{1+(1-\alpha)} \frac{[\pi_L - \mu]}{\left[\pi_L - \mu \frac{\theta_L}{\theta_H}\right]} u'(c_L);$$

and

$$\text{(FOCH)} \qquad l_H = v'^{-1} \left[\theta_H u'(c_H) \right] = \left[\frac{1}{\alpha a_2} \theta_H u'(c_H) \right]^{1/(\alpha-1)}.$$

Thus, $l_H > l_L$ if and only if:

$$\left[\frac{1}{\alpha a_2} \theta_H u'(c_H) \right]^{1/(\alpha-1)} > \left[\frac{1}{\alpha a_2} \left[\frac{\theta_H}{\theta_L} \right]^{\alpha-1} \frac{[\pi_L - \mu]}{\left[\pi_L - \mu \frac{\theta_L}{\theta_H}\right]} \theta_L u'(c_L) \right]^{1/(\alpha-1)};$$

Not clear if it is worth it to do this case any further?

3.4 What if $c_H = c_L$ and $y_H = y_L$?

Maybe you can fill this in.

3.5 Generalizations

Most of these arguments seem pretty general. That the solution is monotone increasing in type and that no one ever pretends up. Still would need to show that it is one IC down that is the binding constrained (i.e., θ_H does not pretend to be θ_{H-2}). See the Appendix that Anderson added on doing this.

The exception to this is the details of showing that $l_H > l_L$. The current argument for this depends heavily on the assumption of log utility. It seems that much of this argument could be extended, at least in the two type case, to situations in which the labor supply curve is upward sloping:

Assuming that the Planner wants $T_H < 0$ and $T_L > 0$,

1) $l_H^{mirr} < l_H^{ce}$ since leisure is a normal good and the high type is poorer;

2) Since leisure is normal, $T_L > 0$, the labor supply curve slopes up and $(1 - \tau_L)\theta_L < \theta_L$, we have that $l_L^{mirr} < l_L^{ce}$;

3) Since the labor supply curve is upward sloping, and $(1 - \tau_L)\theta_L < \theta_H = (1 - \tau_H)\theta_H$, we have that $l_L^{ce} < l_H^{ce}$.

4) Altogether then, $l_L^{mirr} < l_L^{ce} < l_H^{ce} < l_H^{mirr}$ as desired.

In terms of what kinds of utility functions are included in this, suppose:

$$u(c, \ell) = \tfrac{a_1}{1-\sigma} c^{1-\sigma} + \tfrac{a_2}{1-\sigma} \ell^{1-\sigma} \text{ with } \sigma \geq 0.$$

Then the labor supply curve slopes up if and only if $0 \leq \sigma < 1$. Thus, this corresponds to the low curvature case. Not the most natural assumption. See the Appendix for this.

4 Implementation in Contracts

Could either of these allocations (Full Info or Priv Info) be implemented by Private Firms offering wage contracts (with IC constraints where relevant)?

Need the ability to sign contracts BEFORE the realization of the shock. Is this possible in the real world?

Adverse Selection and Forced Participation as an advantage for Government provision.

5 Appendix

In this section it is presented the Lemma about the sufficiency of checking only the IC constraints between the neighbor types.

Assumption 1: $U(c, l) = u(c) - v(l)$ with $u(\cdot)$ being st. concave and $v(\cdot)$ st. convex.

Lemma 2. Under A1, for any $\theta_j > \theta_i$:

(I) If $u(c) - v(y/\theta_i) \geq u(\hat{c}) - v(\hat{y}/\theta_i)$, $c \geq \hat{c}$ and $y \geq \hat{y}$
then $u(c) - v(y/\theta_j) \geq u(\hat{c}) - v(\hat{y}/\theta_j)$

(II) If $u(c) - v(y/\theta_j) \geq u(\hat{c}) - v(\hat{y}/\theta_j)$, $c \leq \hat{c}$ and $y \leq \hat{y}$ then $u(c) - v(y/\theta_i) \geq u(\hat{c}) - v(\hat{y}/\theta_i)$

Proof:

I'll prove only part (I), part (II) is the same but with the sign reversed. So, suppose not, then there exists $(c, y) \geq (\hat{c}, \hat{y})$ and $\theta_j > \theta_i$, such that

$$u(c) - v(y/\theta_i) \geq u(\hat{c}) - v(\hat{y}/\theta_i) \tag{1}$$

but

$$u(c) - v\left(\frac{y}{\theta_j}\right) < u(\hat{c}) - v\left(\frac{\hat{y}}{\theta_j}\right)$$

or

$$u(c) - u(\hat{c}) < v\left(\frac{y}{\theta_j}\right) - v\left(\frac{\hat{y}}{\theta_j}\right) \tag{2}$$

Now define the function

$$F(x, \theta_i, \theta_j) = v\left(\frac{x}{\theta_j}\right) - v\left(\frac{x}{\theta_i}\right)$$

and notice that:

$$\frac{\partial F(x, \theta_i, \theta_j)}{\partial x} = v'\left(\frac{x}{\theta_j}\right)\frac{1}{\theta_j} - v'\left(\frac{x}{\theta_i}\right)\frac{1}{\theta_i} < 0$$

Where the last inequality follows from the convexity of $v(\cdot)$. Then, since $F(x,\mathrm{w}_i,\mathrm{w}_j)$ is decreasing in x it follows that

$$F(x, \theta_i, \theta_j) \leq F(\hat{x}, \theta_i, \theta_j)$$

or

$$v\left(\frac{y}{\theta_j}\right) - v\left(\frac{y}{\theta_i}\right) \leq v\left(\frac{\hat{y}}{\theta_j}\right) - v\left(\frac{\hat{y}}{\theta_i}\right)$$

$$v\left(\frac{\hat{y}}{\theta_i}\right) - v\left(\frac{y}{\theta_i}\right) \leq v\left(\frac{\hat{y}}{\theta_j}\right) - v\left(\frac{y}{\theta_j}\right)$$

$$v\left(\frac{y}{\theta_i}\right) - v\left(\frac{\hat{y}}{\theta_i}\right) \geq v\left(\frac{y}{\theta_j}\right) - v\left(\frac{\hat{y}}{\theta_j}\right)$$

Using the last inequality and (1) we get:

$$u(c) - u(\hat{c}) \geq v\left(\frac{y}{\theta_i}\right) - v\left(\frac{\hat{y}}{\theta_i}\right) \geq v\left(\frac{y}{\theta_j}\right) - v\left(\frac{\hat{y}}{\theta_j}\right)$$

which contradicts (2).

It turns out that a strict version for Lemma 2 also holds.

Lemma 2. (Strict version) Under A1, for any $\theta_j > \theta_i$:

(I) If $u(c) - v(y/\theta_i) > u(\hat{c}) - v(\hat{y}/\theta_i)$, $c \geq \hat{c}$ and $y \geq \hat{y}$
then $u(c) - v(y/\theta_j) > u(\hat{c}) - v(\hat{y}/\theta_j)$

(II) If $u(c) - v(y/\theta_j) > u(\hat{c}) - v(\hat{y}/\theta_j)$, $c \leq \hat{c}$ and $y \leq \hat{y}$
then $u(c) - v(y/\theta_i) > u(\hat{c}) - v(\hat{y}/\theta_i)$

Proof: I'll prove only part (I), part (II) is the same but with the sign reversed. So, suppose not, then there exists $(c, y) \geq (\hat{c}, \hat{y})$ and $\theta_j > \theta_i$, such that

$$u(c) - v(y/\theta_i) > u(\hat{c}) - v(\hat{y}/\theta_i) \tag{3}$$

but

$$u(c) - v\left(\frac{y}{\theta_j}\right) \leq u(\hat{c}) - v\left(\frac{\hat{y}}{\theta_j}\right)$$

or

$$u(c) - u(\hat{c}) \leq v\left(\frac{y}{\theta_j}\right) - v\left(\frac{\hat{y}}{\theta_j}\right) \tag{4}$$

Now define the function

$$F(x, \theta_i, \theta_j) = v\left(\frac{x}{\theta_j}\right) - v\left(\frac{x}{\theta_i}\right)$$

and notice that:

$$\frac{\partial F(x, \theta_i, \theta_j)}{\partial x} = v'\left(\frac{x}{\theta_j}\right)\frac{1}{\theta_j} - v'\left(\frac{x}{\theta_i}\right)\frac{1}{\theta_i} < 0$$

Where the last inequality follows from the single crossing property. Then, since $F(x, \theta_i, \theta_j)$ is decreasing in x it follows that

$$F(x, \theta_i, \theta_j) \leq F(\hat{x}, \theta_i, \theta_j)$$

then

$$v\left(\frac{y}{\theta_j}\right)-v\left(\frac{y}{\theta_i}\right)\leq v\left(\frac{\hat{y}}{\theta_j}\right)-v\left(\frac{\hat{y}}{\theta_i}\right)$$

$$v\left(\frac{\hat{y}}{\theta_i}\right)-v\left(\frac{y}{\theta_i}\right)\leq v\left(\frac{\hat{y}}{\theta_j}\right)-v\left(\frac{y}{\theta_j}\right)$$

$$v\left(\frac{y}{\theta_i}\right)-v\left(\frac{\hat{y}}{\theta_i}\right)\geq v\left(\frac{y}{\theta_j}\right)-v\left(\frac{\hat{y}}{\theta_j}\right)$$

Using the last inequality and (3) we get:

$$u(c)-u(\hat{c})>v\left(\frac{y}{\theta_i}\right)-v\left(\frac{\hat{y}}{\theta_i}\right)>v\left(\frac{y}{\theta_j}\right)-v\left(\frac{\hat{y}}{\theta_j}\right)$$

which contradicts (4).

Lemma 3. Under A1 we have that $(c(\theta),y(\theta))>(c(\hat{\theta}),y(\hat{\theta}))$ whenever $\theta>\hat{\theta}$.

Lemma 4. Under A1 we have:

$$u(c(\theta_i))-v\left(\frac{y(\theta_i)}{\theta_i}\right)\geq u(c(\theta_j))-v\left(\frac{y(\theta_j)}{\theta_i}\right)\text{ for } j=i-1,i+1$$

implies

$$u(c(\theta_i))-v\left(\frac{y(\theta_i)}{\theta_i}\right)\geq u(c(\theta_j))-v\left(\frac{y(\theta_j)}{\theta_i}\right)\ \forall j$$

Proof: Suppose not, and WLOG assume that is not true for $j=i+2$.Then we have:

$$u(c(\theta_i))-v\left(\frac{y(\theta_i)}{\theta_i}\right)\geq u(c(\theta_{i+1}))-v\left(\frac{y(\theta_{i+1})}{\theta_i}\right)$$

and

$$u(c(\theta_{i+2}))-v\left(\frac{y(\theta_{i+2})}{\theta_i}\right)>u(c(\theta_i))-v\left(\frac{y(\theta_i)}{\theta_i}\right)$$

Together the last two inequalities imply

$$u(c(\theta_{i+2}))-v\left(\frac{y(\theta_{i+2})}{\theta_i}\right)>u(c(\theta_{i+1}))-v\left(\frac{y(\theta_{i+1})}{\theta_i}\right)\tag{5}$$

By Lemma 3 we know that $(c(\theta_{i+2}), y(\theta_{i+2})) > ((c(\theta_{i+1}), y(\theta_{i+1})))$. Since $\theta_{i+1} > \theta_i$ we can use Lemma 2, part (I), applied to equation (5). It yields:

$$u(c(\theta_{i+2})) - v\left(\frac{y(\theta_{i+2})}{\theta_{i+1}}\right) > u(c(\theta_{i+1})) - v\left(\frac{y(\theta_{i+1})}{\theta_{i+1}}\right)$$

which violates the IC for type θ_{i+1}.

Now, suppose that is not true for $j = i - 2$, then as before we have:

$$u(c(\theta_i)) - v\left(\frac{y(\theta_i)}{\theta_i}\right) \geq u(c(\theta_{i-1})) - v\left(\frac{y(\theta_{i-1})}{\theta_i}\right)$$

and

$$u(c(\theta_{i-2})) - v\left(\frac{y(\theta_{i-2})}{\theta_i}\right) > u(c(\theta_i)) - v\left(\frac{y(\theta_i)}{\theta_i}\right)$$

Together the last two inequalities imply

$$u(c(\theta_{i-2})) - v\left(\frac{y(\theta_{i-2})}{\theta_i}\right) > u(c(\theta_{i-1})) - v\left(\frac{y(\theta_{i-1})}{\theta_i}\right) \tag{6}$$

Since by Lemma 3 $(c(\theta_{i-2}), y(\theta_{i-2})) < (c(\theta_{i-1}), y(\theta_{i-1}))$ and because $\theta_{i-1} < \theta_i$, using Lemma 2, part (II), we have that equation (6) implies:

$$u(c(\theta_{i-2})) - v\left(\frac{y(\theta_{i-2})}{\theta_{i-1}}\right) > u(c(\theta_{i-1})) - v\left(\frac{y(\theta_{i-1})}{\theta_{i-1}}\right)$$

which violates the IC for type θ_{i-1}.

5.1 The Slope of the Labor Supply Curve