Notes on the Thomas and Worrall paper
Econ 8801

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1 Introduction

The basic reference for these notes is:

Thomas, J. and T. Worrall (1990): "Income Fluctuation and Asymmetric Information: An Example of a Repeated Principal-Agent Problem". Journal of Economic Theory, 51, 367-90.

This is based on and is a considerable generalization of the paper:
Green, E. J., (1987), "Lending, and the Smoothing of Uninsurable Income," in Edward C. Prescott and Neil Wallace, Eds., Contractual Arrangements for Intertemporal Trade, 3-25. University of Minnesota Press, Minnenapolis

The basic idea in both of these papers is to study the form of the efficient method of providing insurance to income shocks under private information in a repeated setting. In each paper, this is modeled as a single good setting in which an agent has an endowment of the good in each period that is subject to period by period shocks. The difference between them is that Green assumed that utility is CARA and consumption was allowed to be negative, while in TW, the assumptions on utility are fairly general. In particular, most CRRA utility functions are allowed, as are many others.

There are two important results in the Thomas and Worrall paper:

1. The model has a recursive structure in which the state varibable is 'promised future utility.' This allows the solution to have a useful characterization – some properties of the optimal contract can be derived.

2. When the time horizon is infinite, promised future utility has a negative drift and converges to its lower bound almost surely.

Section 7 of the paper deals in detail with an example like that in Green (i.e., CARA utility) and characterizes the optimal arrangement as one in which there is 'debt' which limits the amount of loans and that the size of this debt converges to infinitiy as time goes on (a.s.).

2 The Basic Setup

Two players – Borrower and Lender.

$T+1$ periods $t = 0, 1, ...T$.

Random income by the borrower: $s_t \in \Theta = \{\theta_1 < \theta_2 < ... < \theta_n\}$. The s_t are assumed i.i.d.

$\pi_i = P(s_t = \theta_i)$.

Preferences, Borrower: $\sum_{t=0}^{T} \alpha^t v(c_t)$

Assumptions:

1. $v : (0, \infty) \to R$ is C^2,
2. $\sup_c v(c) < \infty$,
3. $\inf_c v(c) = -\infty$
4. $v' > 0, v'' < 0$
5. $\frac{v'}{v''}$ non-increasing
6. $\lim_{c \to a} v'(c) = \infty$

NOTE: T&W use $v : (a, \infty) \to R$.

Preferences, Lender: $\sum_{t=0}^{T} \alpha^t c_t$

History: $\theta^t = (\theta_0, \theta_1, ..., \theta_t) \in \Theta^{t+1}$.

Reported History: $r^t = (r_0, r_1, ..., r_t) \in \Theta^{t+1}$.

Game:

Stage I: Lender offers a contract b^{T+1}:

Contract: $b^{T+1} = (b_0^{T+1}, b_1^{T+1}, ..., b_T^{T+1})$ where $b_t^{T+1} : \Theta^{t+1} \to (-\infty, \infty)$, $b_t^{T+1}(\theta^t) \geq a - \theta_t$ (so that consumption is at least a).

$b_t^{T+1}(r^t)$ is the amount of the consumption good transferred from the lender to the borrower if the reported history up to and including date t is r^t.

Stage II: Borrower chooses 'yes' or 'no'

Stage IIIA: If Stage II was 'no', game ends

Stage IIIB, If Stage II was 'yes':
a) at date t, after history θ^{t-1}, Nature chooses θ_t and tells the borrower,
b) borrower announces a 'report' $r_t \in \Theta$,
c) transfer from lender to borrower of $b_t^{T+1}(r^t)$
d) Flow utility to Lender (time zero value): $-\alpha^t b_t^{T+1}(r^t)$
e) Flow utility to Borrower (time zero value): $\alpha^t v(\theta_t + b_t^{T+1}(r^t))$

NOTE: The effect of stage II is to guarantee a minimal equilibrium payoff for the Borrower. WHEN the Lender optimizes, he soaks as much surplus from the Borrower as he can – thus, he leaves the Borrower with the same overall utility level as he would get under autarchy. Thus, this determines an initial condition, in equilibrium, for the 'continuation utility' state variable that will be used below – V_0 is the level of utility that the Borrower gets under autarchy.

Equivalently:

Stage I: Lender offers Contract (as above)

Stage II: Borrower says yes or no (as above)

Stage III: Borrower chooses a 'reporting strategy' – a date/state contingent strategy: $r_t(\theta^t) : \Theta^{t+1} \rightarrow \Theta^{t+1}$

The meaning here is 'if the history is θ^t, the borrower tells the lender that the history is $r_t(\theta^t)$.

Because of this, the history in the game theoretic sense is (θ^t, r^t).

Payoffs are the sum of date/state contingent payoffs given by:

c) Flow utility to Lender (time zero value): $-\alpha^t b_t^{T+1}(r^t)$
d) Flow utility to Borrower (time zero value): $\alpha^t v(\theta_t + b_t^{T+1}(r^t))$

Given any contract, there is a best reporting strategy $r^*(b^{T+1})$. I.e., r^* satisfies:

for all t, θ^t, and for all reporting strategies, $r = r_{0,} r_1, ...$

$$(*) \qquad \frac{1}{\alpha^t} \sum_{s=t}^{T} \frac{\pi(\theta^s)}{\pi(\theta^t)} \alpha^{s-1} v(\theta_s + b_s^{T+1}(r_s^*(\theta^t))) \geq \frac{1}{\alpha^t} \sum_{s=t}^{T} \frac{\pi(\theta^s)}{\pi(\theta^t)} \alpha^{s-1} v(\theta_s + b_s^{T+1}(r_s(\theta^t)))$$

An equilibrium is then a contract that maximizes the lender's utility given this anticipated response. I.e., it maximizes:

$\sum_{t=0}^{T} \sum_{\theta^t} \pi(\theta^t)\alpha^t(-b_t^{T+1}(r_t^*(\theta^t))$

Notice that we have imposed sequential rationality on the borrower since (*) is required to hold at each date/node of the tree. This is either sequential, ... There is only one party taking an action, and that party is perfectly informed.

It can be shown that given any contract, b^{T+1} and the allocation resulting from the equilibrium choice of r, $r^*(b^{T+1})$, there is another contract, b^{*T+1} which for which the equilibrium reporting strategy, $r^*(b^{*T+1})$ is given by 'truthful reporting' – $r_t^*(b^{*T+1};\theta^t) = \theta^t$ – and for which, the allocation is the same as that from b^{T+1}, $r^*(b^{T+1})$.

This is a version of the 'Revelation Principle.' Intuitively, let b^{*T+1} be defined by $b_t^{*T+1}(\theta^t) = b_t^{T+1}(r_t^*(b^{T+1};\theta^t))$.

Actually, the Revelation Principle implies a bit more than this. It also says that looking only at revelation mechanisms is without loss of generality:

1. Given any other system of messages, any equilibrium outcome can be obtained by a revelation mechanism – i.e., one in which the message space at date t is Θ^{t+1};
2. This equilibrium outcome can be obtained by a mechanism in which it is optimal for all informed parties to tell the truth. I.e., $r^t(\theta^t) = \theta^t$.

Thus, it follows that, for any equilibrium outcome from any mechanism (i.e., contract contingent on possibly more than the realization of the $\theta's$), it can be realized from a contract (as above) that satisfies:

(1) for all k, for all θ^k and for all reporting strategies r^t,

$$IC \quad \sum_{t=k}^{T} \sum_{\theta^t > \theta^k} \frac{\pi(\theta^t)}{\pi(\theta^k)} \alpha^t v(\theta_t + b_t^{T+1}(\theta^t)) \geq \sum_{t=k}^{T} \sum_{\theta^t > \theta^k} \frac{\pi(\theta^t)}{\pi(\theta^k)} \alpha^t v(\theta_t + b_t^{T+1}(r_t(\theta^t)))$$

B^{T+1} is the set of contracts that satisfy Incentive Compatability, IC, i.e., (1).

Notice that this formulation of the problem is FULL COMMITTMENT on the part of both parties – i.e., there is no 'renegotiation' after a string of good/bad shocks and neither party can 'leave' the agreement if they reach a node where it looks to them like the continuation of the game is worse than autarchy.

Since the Borrower gets to choose actions down the tree the 'commitment' on his part is only to the 'Yes' made at Stage II

Since the Lender never gets to 'move' again, it is 'commitment' to the entire, future, state/date contingent path of the contract.

3 With full information

Best allocation from Borrower's point of view is subject to zero profits for the Lender is given by:

$c_t(\theta^t) = E(\theta)$ for all t, θ^t.

This gives the lender $\theta^t - E(\theta)$ in each state and hence zero expected profits.

Add a constant that is independent of date and state to get other PO allocations.

What about savings? Assume that saving is not possible.

4 Simple Examples

Assume that there are only two types:

$\Theta = \{\theta_L, \theta_H\}$ with $\theta_L < \theta_H$.

Let $\pi_L = P(\theta_t = \theta_L)$, $\pi_H = P(\theta_t = \theta_H)$.

4.1 One Period in the Two θ example

Only IC contract is $b(r) = 0$ for all r.

**** show this ****

4.2 Two Periods

IC contracts in the two period case:

Can't have borrowing depend on second period (last period) report. That is, $b_1^1(r_0, r_1) = b_1^1(r_0, r_1')$ for all r_1, r_1'.

4.2.1 Effect of Discounting in Two Period Example

Suppose that there are only two periods so that $T = 1$ and suppose that the second period is a very long time after the first (i.e., α is very low). Consider the following two contracts:

Contract 1: $b_t^1(r) = 0$ for $t = 0, 1$ – borrower is in autarchy in all periods

Contract 2: $b_0^1(\theta_L) = -b_0^1(\theta_H) = \frac{\theta_L+\theta_H}{2} + \theta_L$, $b_1^1(\theta_0, \theta_1) = -\delta$ – i.e., full insurance in period 0, and a payment in period 1 no matter what θ is.

The utility of the borrower under contract 1 is:

$$U^1 = \pi_L v(\theta_L) + \pi_H v(\theta_H) + \alpha [\pi_L v(\theta_L) + \pi_H v(\theta_H)].$$

Under contract 2 it is:

$$U^2 = v\left(\frac{\theta_L+\theta_H}{2}\right) + \alpha [\pi_L v(\theta_L - \delta) + \pi_H v(\theta_H - \delta)].$$

(This assumes truth telling or full info.)

Notice that since v is strictly concave, $v\left(\frac{\theta_L+\theta_H}{2}\right) > \pi_L v(\theta_L) + \pi_H v(\theta_H)$.

Also, since $\delta > 0$, it follows that $\alpha [\pi_L v(\theta_L) + \pi_H v(\theta_H)] < \alpha [\pi_L v(\theta_L - \delta) + \pi_H v(\theta_H - \delta)]$.

BUT, when α is small enough (period 1 is far enough in the future), this second term is about 0. Thus, when α is low enough, $U^2 > U^1$. This holds no matter how big δ is. In particular, as an example it holds even if δ is so large that $\theta_H - \delta < \theta_L$. I.e., in this case, the borrower gets less than θ_L for sure!

5 Characterization of Optimal Contract

5.1 Recursive Nature of Optimal Contracts

The main idea used in characterization is that Optimal Contracts have a recursive structure – a version of BE must hold for something.

Intuitively, in an efficient contract, it must be true that the continuation of it from any date/node is also efficient in some way. If it were not, one could replace the continuation with a better contract leaving the borrower indifferent, but strictly improving things for the Lender. There are two issues:

1. The first difficulty is that for this to work, it must be done in such a way so as to not destroy incentives at some node earlier on in the tree – That is, can one find a better contract such that using it as the continuation doesn't screw up incentives someplace before that node?
2. A second issue is 'which' efficient contract should be the continuation. I.e., we don't want to use the same continuation after a string of θ_H's as we do after a string of θ_L's – If it was the same, intuitively, no one would ever admit to being θ_H today! What this comes down to is: What is the 'state' variable for the problem? Once we know that, the answer to this part will be just use the efficient contract for that version of the state.

The problem from the Lender's point of view is to

$$\max \quad -\sum_{t=0}^{T}\sum_{\theta^s}\alpha^s\pi(\theta^s)b_t^{T+1}(\theta^s)$$

$$\text{s.t. } b^{T+1}\in B^{T+1}.$$

This is the 'Sequence Problem' version.

Where we can use truth telling in the OBJ (i.e., $b_t^{T+1}(\theta^s)$ and not $b_t^{T+1}(r_t(\theta^s;b))$ because in B^{T+1} the optimal response to any contract is to tell the truth.

From this, it is clear that the objective function of the Lender satisfies what is needed for the problem to be recursive.

So, the hard part is to figure out a way of rewriting the constraint set so that it also has a recursive structure. Of course, the first thing that must be done then is figure out whan an appropriate state variable might be. There is no 'natural' one present in the problem as written above.

Recall that B^{T+1} is the set of contracts that satisfy:

(1) for all t, for all θ^t and for all reporting strategies r,

$$IC \quad \sum_{s=t}^{T} \sum_{\theta^s > \theta^t} \frac{\pi(\theta^s)}{\pi(\theta^t)} \alpha^s v(\theta_s + b_s^{T+1}(\theta^s)) \geq \sum_{s=t}^{T} \sum_{\theta^s > \theta^t} \frac{\pi(\theta^s)}{\pi(\theta^t)} \alpha^s v(\theta_s + b_s^{T+1}(r_s(\theta^s)))$$

5.1.1 Definition of State Variable

Recall that in the game theoretic sense, the history up to and including date $t-1$ is given by (r^{t-1}, θ^{t-1}).

For any reported history up to date $t-1$, r^{t-1}, and any actual history θ^{t-1} up to date $t-1$, suppose that the next period report is $r_t = \theta_i$ – so $r^t = (r^{t-1}, \theta_i)$.

Let $V_i(r^{t-1}, \theta^{t-1})$ be the highest expected continuation utility that the Borrower can obtain from date t forward for any continuation reporting strategy for the remainder of the period, $\hat{r} = (\hat{r}_{t+1}, \hat{r}_{t+2}, ..., \hat{r}_T)$, when he reports that the outcome is θ_i in period t:

$$V_i(r^{t-1}; \theta^{t-1}) = \sup_{\hat{r}_t} \qquad \frac{1}{\alpha^t} \sum_{s=t}^{T} \sum_{\theta^s > \theta^t} \frac{\pi(\theta^s)}{\pi(\theta^t)} \alpha^{s-1} v(\theta_s + b_t^{T+1}(r^{t-1}, \hat{r}_s))$$

Since the s_t are i.i.d., it follows that V_i depends only on r^{t-1} and not on θ^{t-1}.

Formally, this is because of the fact that the only way that θ^{t-1} enters this is through the $\frac{\pi(\theta^s)}{\pi(\theta^t)}$ in each term. But, by independence, this is

$$\frac{\Pi_{j=1}^{s} \pi(\theta_j)}{\Pi_{j=1}^{t} \pi(\theta_j)} = \Pi_{j=t+1}^{s} \pi(\theta_j)$$

which does not depend on θ^{t-1}.

Thus, we'll write this as $V_i(r^{t-1})$.

5.1.2 Rewrite the Constraint Set (B^{T+1}) in Terms of State Variable

As with V_i, let b_i denote the transfer received by the Borrower if he reports $r_t = \theta_i$ in period t:

$$b_i = b_t^{T+1}(r_0, r_1, ..., r_{t-1}, \theta_i).$$

Note that the transfers don't depend on the $\theta's$ by definition.

Given these definitions of b_i and V_i we can rewrite the IC constraints as:

$$IC \qquad v(b_i + \theta_i) + \alpha V_i \geq v(b_j + \theta_i) + \alpha V_j. \; \forall r^{t-1}, \theta_i, \theta_j$$

By (1), if $b^{T+1} \in B^{T+1}$ telling the truth is optimal from t on, so that

$$V_i(r^{t-1}; \theta^{t-1}) = \frac{1}{\alpha^t} \sum_{s=t}^{T} \sum_{\theta^s > \theta^t} \frac{\pi(\theta^s)}{\pi(\theta^t)} \alpha^{s-1} v(\theta_s + b_t^{T+1}(r^{t-1}, \theta_i, ..., \theta_s))$$

5.1.3 Intuition?

There are two parts to this:

1. The first is that V_i can be found from the truth telling strategy going forward. This is true even if $r^{t-1} \neq \theta^{t-1}$ – for contracts in B^{T+1}, it's best to tell the truth in the future, even at nodes where you haven't told the truth in the past. This is true even without independence.
2. The second part is that V_i only depends on r^{t-1}, not on θ^{t-1}. I.e., even if you've lied in the past so that $r^{t-1} \neq \theta^{t-1}$, the best you can expect in the future only depends on what you've said. This second part would not be true if the θ_t were not independent.

Because of this, we can rewrite (1) as:

(1) for all t, for all θ^t, for all reporting strategies, r, and for all (θ_i, θ_j),

$$v(\theta_i + b_t^{T+1}(\theta^{t-1}, \theta_i)) + \alpha V_i(r^{t-1}) \geq v(\theta_i + b_t^{T+1}(\theta^{t-1}, \theta_j)) + \alpha V_j(r^{t-1})$$

I guess there could be different probabilities? How about: by IC, the best is the truth from now on, this fixes the reports from now on, independence then fixes the probabilities from now on????

**** What's missing here is something like Prop: $b^{t+1} \in B^{T+1}$ if and only if b_i, V_i etc....?******

We will use something like V_i as the state variable in the Lender's problem to find the best (most profitable) contract.

Intuitively, an optimal contract has a natural recursivity property: From any date, node, t, θ^t, the continuation of the optimal contract must be optimal for the remainder of the contracting period. This doesn't uniquely solve the problem however since there are many different ones depending on the implicit outside option value.... i.e., how much utility it is supposed to deliver over the remainder of the contract.

More importantly perhaps – the only relevant thing at a node (θ^t, r^t) is what the B expects to get over the rest of the contract – i.e., if there are two different nodes, (θ^t, r^t) and $(\theta^t, r^t)'$, with the same expected continuation value from then on, the continuation of the contract must be the same. (This assumes uniqueness – strict convexity of $U's$.)

5.2 Definition of Lender's Value Functions

So, consider the problem from period $T - k + 1$ on. This has k more periods to go. And fix a level of utility, V.

Define $B^k(V)$ to be the set of IC contracts of length k that give a net increase in utility over autarchy to the borrower of V :

$$B^k(V) = \{b^k \in B^k| \quad \sum_{s=0}^{k-1} \sum_{\theta^s} \pi(\theta^s)\alpha^s \left[v(\theta_s + b_s^k(\theta^s)) - v(\theta_s)\right] = V\}$$

And define the Lender's value function over the remainder of the contracting period as:

$$U_k(V) = \sup_{b^k \in B^k(V)} \qquad - \sum_{s=0}^{k-1} \sum_{\theta^s} \alpha^s b_s^k(\theta^s)$$

This is defined for any $V \in (-\infty, d_k)$ where $d_k = \frac{1-\alpha^k}{1-\alpha} \sup\left[v(c) - \sum_\theta \pi(\theta)v(\theta)\right]$. I.e., this is the net gain to B if he was given the highest possible utility ($\sup_c v(c)$) forever.

d_k is the largest increase in utility it is possible to give the Borrower if there are k periods to go.

Extend U_k to $V's$ above d_k by setting $U_k(V) = -\infty$ for $V > d_k$.

And call U_* the value function when $T = \infty$.

5.3 Operator Equation for Lender's Value

Define the operator L by:

$$L(U)(V) = \sup_{\{b_i,V_i\}\in\Lambda} \qquad \textstyle\sum_{\theta_i} \pi(\theta_i)\left(-b_i + \alpha U(V_i)\right)$$

where

$$\begin{aligned}
\Lambda &= \{b_i \in (a-\theta_i,\infty), V_i \in (-\infty, d_\infty) \\
\sum_{\theta_i} \pi(\theta_i)\left(v(b_i+\theta_i) - v(\theta_i) + \alpha V_i\right) &= V \\
v(b_i+\theta_i) + \alpha V_i &\geq v(b_j+\theta_i) + \alpha V_j \ \textit{all}\ i,j\}
\end{aligned}$$

Clearly, $U_0 = 0$, this is the essence of the one period example above – in a one period world, the only IC borrowing plan is zero.

Then, it follows as usual that $U_1 = L(U_0)$, etc.

Lemmas 1-3 prove the basic results from dynamic programming in this setting.

I.e.,

Lemma 1 *i)* $U_k(V) \equiv L(U_{k-1})(V)$*, ii)* U_* *is a fixed point of* L*.*

'Proof' that $U_k(V) = L(U_{k-1})(V)$.
The meaning of this statement is:

The best contract for the L that delivers V to the B in k periods can be found by first finding the best contract for the L that delivers V in $k-1$ periods (i.e., the one that delivers $U_{k-1}(V)$ to the L) and then apply L

Two technical difficulties that have to be addressed here:

1. U_k and U_{k-1} are not naturally defined on the same set of $V's$ – the best you can give the Borrower in terms of utility gain (d_k) depends on the length of the horizon.

2. The set of relevant potential Value functions on which we would define the operator is not bounded – e.g., $U_k(d_k) = -\infty$ – the only way to give the Borrower d_k is to have $c = \infty$ always, and this is infinitely expensive for the Lender.

To help handle these problems proceed as follows:

U_k can be bounded above and below by two contracts. The first gives the Borrower full insurance at the level of consumption, c_k, and total utility V while the second transfers a constant amount from the Lender to the Borrower in all dates and states, y_k, and gives the Borrower total utility V.

Formally, define $y_k(V)$ as the value of y_k that satisfies:

$$\frac{1-\alpha^k}{1-\alpha}\sum_{\theta}\pi(\theta)\left(v(y_k+\theta)-v(\theta)\right)=V$$

And $c_k(V)$ as the value of c_k that satisfies:

$$\frac{1-\alpha^k}{1-\alpha}\sum_{\theta}\pi(\theta)\left(v(c_k)-v(\theta)\right)=V$$

Thus,

$$-\frac{1-\alpha^k}{1-\alpha}y_k(V)\leq U_k(V)\leq\frac{1-\alpha^k}{1-\alpha}\sum_{\theta}\pi(\theta)\left(\theta-c_k(V)\right)$$

Similarly,

$$-\frac{1}{1-\alpha}y_\infty(V)\leq U_\infty(V)\leq\frac{1}{1-\alpha}\sum_{\theta}\pi(\theta)\left(\theta-c_\infty(V)\right)\quad ***$$

where, $c_k \to c_\infty$, and $y_k \to y_\infty$.

To show:

$$-\tfrac{1-\alpha^k}{1-\alpha}y_k(V)\leq\tfrac{1-\alpha^k}{1-\alpha}\sum_{\theta}\pi(\theta)\left(\theta-c_k(V)\right)=\tfrac{1-\alpha^k}{1-\alpha}\left(E(\theta)-c_k(V)\right)$$

use definitions of y_k and c_k and the fact that it must be true that $E(y_k+\theta) = y_k + E(\theta) > E(c_k)$ because c_k involves no risk and $y_k+\theta$ has risk and the both give expected utility of $V + E(v(\theta))$.

How can we show that $\sup_V \left[\frac{1-\alpha^k}{1-\alpha}\sum_\theta \pi(\theta)\,(\theta - c_k(V)) - \left(-\frac{1-\alpha^k}{1-\alpha} y_k(V)\right)\right]$ is finite???

Because of the properties assumed about v, and the equations defining y_k, c_k we can find some properties of U_k:

1. Since $\lim_{c\to a} v'(c) = \infty$, $\lim_{V\to -\infty} U_k'(V) = 0$
2. $\lim_{V\to d_k} U_k'(V) = -\infty$
3. If $a = -\infty$, $\lim_{V\to -\infty} U_k(V) = \infty$
4. If $a > -\infty$, $\lim_{V\to -\infty} U_k(V) < \infty$
5. $\lim_{V\to d_k} U_k(V) = -\infty$

**** Add more discussion of these things here. ****

**** Also, what is the most that the Lender could every get from the Borrower? I think it is $\frac{1-\alpha^k}{1-\alpha}E(\theta)$ in a k period contracting problem. *****

PICTURE HERE

Lemma 2 *Let F be the set of continuous functions described by ***. It's a metric space under the* sup *metric. L is a contraction. U_* is the unique fixed point of L and $U_k(V) \to U_*(V)$.*

Lemmas 1, 2 and 3 are the results about standard DP holding

5.4 Concave and Which IC constraints are binding

Next we want to show that the $U's$ are strictly concave.

First step: Rewrite IC's:

$$C_{ij} = v(b_i + \theta_i) + \alpha V_i - (v(b_j + \theta_i) + \alpha V_j) \geq 0$$

Lemma 3 *Local Implies Global: Assume that the local IC's are satisfied –* $C_{i-1,i} \geq 0$ *and* $C_{i,i-1} \geq 0$*. Then,* $C_{i,j} \geq 0$*.*

To see this:

We want to show:

$$v(\theta_{i+1} + b_{i+1}) + \alpha V_{i+1} \geq v(\theta_{i+1} + b_{i-1}) + \alpha V_{i-1}$$

We know that:

$$v(\theta_{i+1} + b_{i+1}) + \alpha V_{i+1} \geq v(\theta_{i+1} + b_i) + \alpha V_i$$

And because of this, it's enough to show that:

$$v(\theta_{i+1} + b_i) + \alpha V_i \geq v(\theta_{i+1} + b_{i-1}) + \alpha V_{i-1}$$

This can be rewritten as:

$$v(\theta_{i+1} + b_i) - v(\theta_{i+1} + b_{i-1}) + \alpha\,(V_i - V_{i-1}) \geq 0.$$

NOTE: Since, in the end, $v(\theta_{i+1} + b_{i+1}) + \alpha V_{i+1} \geq v(\theta_{i+1} + b_i) + \alpha V_i$ is supposed to hold with equality, we sort of have to show this anyway.

We also have:

$$v(\theta_i + b_i) - v(\theta_i + b_{i-1}) + \alpha\,(V_i - V_{i-1}) \geq 0.$$

So, we will be done if we can show that

$$v(\theta_{i+1} + b_i) - v(\theta_{i+1} + b_{i-1}) \geq v(\theta_i + b_i) - v(\theta_i + b_{i-1})$$

To see that this holds, we will use the extra assumption that $b_{i-1} \geq b_i$ and the concavity of v.

Notice that this is equivalent to:

$$v(\theta_{i+1}+b_i)-v(\theta_{i+1}+b_{i-1}) = \int_{b_{i-1}}^{b_i} v'(\theta_{i+1}+x)dx \geq \int_{b_{i-1}}^{b_i} v'(\theta_i+x)dx = v(\theta_i+b_i)-v(\theta_i+b_{i-1})$$

Thus, it is sufficient to show that

$$\int_{b_{i-1}}^{b_i} v'(\theta_{i+1}+x)dx \geq \int_{b_{i-1}}^{b_i} v'(\theta_i+x)dx$$

To see that this holds, notice that $v'(\theta_i + x) \geq v'(\theta_{i+1} + x)$ for all $x \in [b_i, b_{i-1}]$ and, because of this we have that:

$$\int_{b_i}^{b_{i-1}} v'(\theta_i+x)dx \geq \int_{b_i}^{b_{i-1}} v'(\theta_i+x)dx$$

and so,

$$\int_{b_{i-1}}^{b_i} v'(\theta_{i+1}+x)dx = -\int_{b_i}^{b_{i-1}} v'(\theta_i+x)dx \geq -\int_{b_i}^{b_{i-1}} v'(\theta_i+x)dx = \int_{b_{i-1}}^{b_i} v'(\theta_i+x)dx$$

as desired.

NOTE: Notice that this adds the extra assumption, $b_{i-1} \geq b_i$ to prove that local implies global. Perhaps there is another way to show the above without assuming $b_{i-1} \geq b_i$ first. But, reading the proof of the next lemma below seems to say this is fine – i.e., it only uses Local IC's, not Global IC's.

Thus, I think the outline of the proof should be:

1. Global IC's implies Local IC's
2. Local IC's implies monotonicity of the contract (Lemma below)
3. Local IC's plus monotonicity implies Global IC's.

So, Global IC's $\Longleftrightarrow$ Local IC's and monotonicity.

Lemma 4 *$b_{i-1} \geq b_i$ and $V_i \geq V_{i-1}$.*

Adding $C_{i,i-1} \geq 0$ and $C_{i-1,i} \geq 0$, we obtain:

$$v(b_i+\theta_i)+\alpha V_i-(v(b_{i-1}+\theta_i)+\alpha V_{i-1})+v(b_{i-1}+\theta_{i-1})+\alpha V_{i-1}-(v(b_i+\theta_{i-1})+\alpha V_i) \geq 0$$

$$v(b_i+\theta_i)-(v(b_{i-1}+\theta_i))+v(b_{i-1}+\theta_{i-1})-(v(b_i+\theta_{i-1})) \geq 0$$

$$v(b_i+\theta_i)-(v(b_i+\theta_{i-1}))-(v(b_{i-1}+\theta_i))+v(b_{i-1}+\theta_{i-1}) \geq 0$$

$$v(b_i+\theta_i)-v(b_i+\theta_{i-1}) \geq v(b_{i-1}+\theta_i)-v(b_{i-1}+\theta_{i-1}) \quad ****$$

I.e., but, since v is concave and $\theta_i > \theta_{i-1}$, it follows that

$$v(b+\theta_i)-v(b+\theta_{i-1})$$

is decreasing in b. Hence, it follows from **** that $b_i \leq b_{i-1}$.

From this it follows that $V_i \geq V_{i-1}$ from IC.

Lemma 5 *Assume $U_{k-1}(V)$ is strictly concave, then, i) $C_{i,i-1}=0$, ii) $-b_i+\alpha U(V_i) \geq -b_{i-1}+\alpha U(V_{i-1})$, and $v(b_i+\theta_i)+\alpha V_i > v(b_{i-1}+\theta_{i-1})+\alpha V_{i-1}$, iii) $C_{i-1,i} > 0$. I.e., local down one are binding, local up are slack.*

Step 1, $C_{i,i-1}=0$. Suppose $C_{i,i-1}>0$.

We'll construct a change in the contract that keeps the Borrower indifferent and improves things for the Lender. Make the following change to the contract:

1. Leave all the b_i unchanged.
2. Keep V_1 fixed and reduce V_2 to V_2' so that $C_{2,1}=0$. Notice that this makes $V_2'-V_1'=V_2'-V_1'$ smaller than V_2-V_1. Then, reduce V_3 until $C_{3,2}=0$, etc., until $C_{i,i-1}=0$ for all i.
3. Next, add a constant, d, to all V_i', $V_i''=V_i'+d$ to keep $\sum_i \pi(\theta_i)V_i''=\sum_i \pi(\theta_i)V_i$, i.e., this is unchanged.
4. Because we haven't changed the b_i and we haven't changed $\sum_i \pi(\theta_i)V_i$, promise keeping still holds
5. IC holds by construction.
6. Notice that after these changes, $V_i''-V_{i-1}''=V_i'-V_{i-1}'$, has gone down.
7. 6 implies that Lender's utility has gone up since the $b's$ are unchanged and $U_{k-1}(V)$ is decreasing. I don't see why. So far, all we've shown is that U_{k-1} is decreasing... and V_1 has gone up to $V_1'=V_1+d$. I.e.,

$$U_k(V) = \sum_{\theta_i} \pi(\theta_i) \left(-b_i + \alpha U_{k-1}(V_i)\right)$$

This sounds like it might work if we already knew that U_{k-1} is concave – i.e., same mean V but less variance?

8. $\theta_i \geq \theta_{i-1}$ and $C_{i,i-1} = 0$ implies $C_{i,i+1} > 0$.

To see this, note that:

$$C_{i-1,i} = v(b_{i-1} + \theta_{i-1}) + \alpha V_{i-1} - (v(b_i + \theta_{i-1}) + \alpha V_i)$$

$$-C_{i-1,i} = (v(b_i + \theta_{i-1}) + \alpha V_i) - (v(b_{i-1} + \theta_{i-1}) + \alpha V_{i-1})$$

$$= (v(b_i + \theta_{i-1}) + \alpha V_i) - (v(b_{i-1} + \theta_{i-1}) + \alpha V_{i-1})$$

$$= (v(b_i + \theta_i) - v(b_i + \theta_i) + v(b_i + \theta_{i-1}) + \alpha V_i) - (v(b_{i-1} + \theta_{i-1}) + \alpha V_{i-1})$$

$$= (v(b_i + \theta_i) + \alpha V_i) - (v(b_{i-1} + \theta_{i-1}) + \alpha V_{i-1}) - v(b_i + \theta_i) + v(b_i + \theta_{i-1})$$

$$= C_{i,i-1} - v(b_i + \theta_i) + v(b_i + \theta_{i-1}) = v(b_i + \theta_{i-1}) - v(b_i + \theta_i) < 0$$

where the last inequality follows from the facts that v is increasing and $\theta_{i-1} < \theta_i$.

(We used $C_{i,i-1} = v(b_i + \theta_i) + \alpha V_i - (v(b_{i-1} + \theta_i) + \alpha V_{i-1})$ and $C_{i,i-1} = 0$ from above.)

Thus, all of the local IC's bind, hence, all of them bind.

Step 2. Proof of ii). The second part of ii) follows from $b_{i-1} \geq b_i$ and Step 1. To see that the first part holds, suppose $-b_i + \alpha U(V_i) < -b_{i-1} + \alpha U(V_{i-1})$. If this holds, we'll find a better contract. In this case, replace b_i by b_{i-1} and V_i by V_{i-1}. Since $C_{i,i-1} = 0$, this leaves the Borrower indifferent. But, it increases the utility of the Lender (why?****)

Step 3. Proof of iii).

**** Ali thinks this step is wrong in TW ****

1. Suppose we ignore the constraint $C_{i-1,i} \geq 0$. If $b_{i-1} \geq b_i$ it follows from Step 1 that the upward IC is binding (why?****), and we are done.

2, So, suppose that $b_i > b_{i-1}$. In this case, $V_i < V_{i-1}$ and $C_{i-1,i} < 0$. In this case, replace b_{i-1} by b_i and V_{i-1} by V_i. This will not decrease L's utility and will not violate IC.

But, since v is concave and $\theta_i > \theta_{i-1}$, $v(b_i + \theta_{i-1}) - v(b_{i-1} + \theta_{i-1}) > v(b_i + \theta_i) - v(b_{i-1} + \theta_i) = \alpha(V_{i-1} - V_i)$.

So, $v(b_i + \theta_{i-1}) + \alpha V_i > v(b_{i-1} + \theta_{i-1}) + \alpha V_{i-1}$. Thus, we've improved the welfare of the B in state $i - 1$ by making this change.

This is a contradiction. (why??*****)

Proposition 6 *i) There is a unique* (b_i, V_i) *solving RHS of BE; ii)* $U_k(V)$ *is strictly decreasing, strictly concave and cont. differentiable; iii)* $U_*(V)$ *is strictly decreasing, weakly concave and cont. differentiable.*

Step 1. $U_k(V)$ is strictly decreasing. Obvious, lower utility promises everywhere.

Step 2.

a) Assume that $U_{k-1}(V)$ is strictly concave.

b)Consider two values of continuation utility, V and V' and let $(b_i, V_i)_{i\in\Theta}$, $(b_i', V_i')_{i\in\Theta}$ be contracts that attain the sup.

c) Choose $\delta \in (0,1)$ and let $V_i^* = \delta V_i + (1-\delta)V_i'$.

d) Define b_i^* by:

$$v(b_i^* + \theta_i) = \delta v(b_i + \theta_i) + (1-\delta)v(b_i' + \theta_i)$$

e) This choice of $(b_i^*, V_i^*)_{i\in\Theta}$ gives the B the average utility of what he gets under the two, i.e., he gets $\delta V + (1-\delta)V'$ under this contract. (I.e., he gets $\sum_i \pi_i \left(v(b_i^* + \theta_i) + \alpha V_i^*\right) = \sum_i \pi_i \left(\delta v(b_i + \theta_i) + (1-\delta)v(b_i' + \theta_i) + \alpha\left(\delta V_i + (1-\delta)V_i'\right)\right) = \delta V + (1-\delta)V'$ from the promise keeping constraint for each of the two contracts.)

f) This also gives the L at least the average value of what he gets under the two: by strict concavity of v it follows that $b_i^* < \delta b_i + (1-\delta)b_i'$. Thus, $\sum_i -\pi_i b_i^* > -\sum_i \pi_i \left(\delta b_i + (1-\delta)b_i'\right)$ and by the strict concavity of U_{k-1} we have $\sum_i \pi_i U_{k-1}(V_i^*) = \sum_i \pi_i U_{k-1}(\delta V_i + (1-\delta)V_i') > \sum_i \pi_i \left(\delta U_{k-1}(V_i) + (1-\delta)U_{k-1}(V_i')\right)$. Thus,

$$\sum_i -\pi_i b_i^* + \alpha \sum_i \pi_i U_{k-1}(V_i^*) > -\sum_i \pi_i \left(\delta b_i + (1-\delta)b_i'\right) + \alpha \sum_i \pi_i \left(\delta U_{k-1}(V_i) + (1-\delta)U_{k-1}(V_i')\right)$$

as desired.

So, we need to show that it is IC.

g) It follows that:

$$C_{i,i-1}^* = \delta C_{i,i-1} + (1-\delta)C_{i,i-1}^* + \delta v(b_{i-1} + \theta_i) + (1-\delta)v(b_{i-1}' + \theta_i) - v(b_i^* + \theta_i)$$

h) From above, $C_{i,i-1} = C_{i,i-1}^* = 0$, so that:

$$C_{i,i-1}^* = \delta v(b_{i-1} + \theta_i) + (1-\delta)v(b_{i-1}' + \theta_i) - v(b_i^* + \theta_i)$$

i) Since the risk premium is a decreasing function (NIARA), the last term:

$$\delta v(b_{i-1} + \theta_i) + (1-\delta)v(b_{i-1}' + \theta_i) - v(b_i^* + \theta_i) \geq 0$$

Thus, the downward IC's are binding for the contract $(b_i^*, V_i^*)_{i\in\Theta}$.

j) Upward IC's may be violated, but we can build a new one as we did in Lemma 4 that gives both at least as high utility and satisfies all IC's:

1. Keep V_1 fixed and lower V_2 until $C_{2,1} = 0$ or $V_1 = V_2$, then V_3 the same etc.
2. Add a constant to V_i so that $E(V_i)$ is unchanged.
3. This will not make the L worse off and keeps the downward IC's
4. If $V_1 = V_2$ it must be that (why???) $b_2^* > b_1^*$ and so, keep b_1^* fixed and lower b_2^* until $C_{2,1} = 0$, etc.
5. Since $b_i + \theta_i > b_{i-1} + \theta_{i-1}$ adding a constant to each b_i to restore $E(b_i)$ unchanged will not make the B worse off. (So???)
6. In this new contract, $C_{i,i-1} = 0$ and $b_{i-1} \geq b_i$ so that the upward IC's also hold.

Step 3. Strict concavity of U_k now follows since it is not possible to have all three of $b_i = b_i'$ for all i, $V_i = V_i'$ for all i and $V \neq V'$. Thus, it must be that the contract $(b_i^*, V_i^*)_{i\in\Theta}$ gives the L strictly higher payoff –

$$U_k(\delta V + (1-\delta)V') \geq U_k(\textit{this contract}) > \delta U_k(V) + (1-\delta)U_k(V').$$

Step 4. Note that $U_0 \equiv 0$ and hence $U_1(V)$ is trivially strictly concave.

Step 5. It also follows that U_* is concave since it is the pointwise limit of the U_k.

Step 6. Optimal contract is unique. It is not possible to have both $b_i = b_i'$ and $V_i \neq V_i'$ for all i and the same V. So, non-uniqueness would necessarily imply that $b_i \neq b_i'$ and then the constructed contract above would be strictly better, a contradiction of optimality of either or both of the contracts. This also works for the infinite horizon????

***** suppose U is weakly concave and (b_i, V_i) and (b_i', V_i') are both solutions to the RHS of BE for this choice of U. IF $b_i \neq b_i'$ for all i, use the argument above to show $\delta * + (1-\delta) **$ is strictly better. If $b_i = b_i'$ for all i then it follows from $C_{i,i-1} = 0$ that $V_i = V_i'$ for all i I think. Thus it holds when I put in U_* on the RHS of BE since U_* is alread known to be weakly concave.

Step 7. Continuous Differentiability. Fix a V' and a neighborhood of $V's$ around it. Construct an IC contract for each V by taking the best contract at V' and keeping the $V_i's$ fixed, but varying the b_i so as to

i) maintain IC

ii) give V overall.

There is a unique way of doing this (use argument above???)

Call the realized value to L $\tilde{U}$. This new function is concave, lower than U_k and equal to U_k at $V = V'$. Finally, $\tilde{U}$ is C^1. Thus, following Benveniste and Scheinkman, U_k is C^1.

**** looks like a lot of this comes ONLY from the fact that you're doing the RHS of BE with U weakly concave?**** State a lemma like that????****

5.5 Summary

Proposition 7 *i)* $b_i \leq b_{i-1}$*; ii)* $V_i \geq V_{i-1}$*; iii)* $C_{i,i-1} = 0$*, iv)* $C_{i,i+1} > 0$*.*

5.6 Infinite Horizon Properties

Equation (8), the Martingale Property.

Remark 8 $\sum_\theta \pi(\theta) U_*'(V(\theta)) = U_*'(V)$*. That is,* $U_*'(V)$ *is a Martingale.*

Basically, this is the envelope theorem.

Consider alternative ways of increasing V to the B by one unit. The cost, at the optimal contract, of doing this is $U_*'(V)$.

One way of doing this is to increase $V(\theta)$ by $1/\alpha$ for every θ leaving the $b(\theta)$ unchanged.

This keeps all of the IC's satisfied, and hence, is a feasible way to increase V to $V + dV$. By the envelope theorem, it is also locally optimal.

(What is the Enevelope Theorem exactly?)
The cost of making this change is:

$\sum_\theta \pi(\theta) U_*'(V(\theta))$.

Thus, we have

$\sum_\theta \pi(\theta) U_*'(V(\theta)) = U_*'(V)$.

Some Properties of the Optimum:

Let's define f_i as the solution to the RHS of BE, i.e., $f_i(V)$ is the V_i that, along with b_i is the optimal contract when promised value is V and the current realization is $\theta = \theta_i$.

From above, f_i is a function. By the Theorem of the Max, it is a continuous function of V.

Lemma 9 *i)* $\lim_{V\to-\infty} U_*'(V) = 0$
ii) $\lim_{V\to d_\infty} U_*'(V) = -\infty$
iii) f_i *is continuous and a function*
iv) $V_1 = f_1(V) < V < V_N = f_N(V)$ *there is spreading in continuation utility*
v) $U_*'(f_N(V)) < U_*'(V) < U_*'(f_1(V)) < 0$

5.6.1 Aside on Martingales

Let $X_1, X_2, ...$ be a stochastic process.

It is a 'Martingale' if:

$$E(X_{n+1}|X_n, X_{n-1}, ..., X_1) = X_n \quad a.s.$$

Theorem 10 *(Martingale Convergence Theorem) If* $X_1, X_2, ...$ *is a non-negative Martingale with finite mean, there exists a random variable,* X^* *such that* $X_n \to X^*$ *a.s.*

5.6.2 Back to Main Development

Proposition 11 *If* $T = \infty$, $V^t \to -\infty$ *a.s.*

Recall that U_* is decreasing. Thus, from the remark, the sequence of random variables, $U_*'(V(\theta^t))$ is a negative martingale.

By the Martingale Convergence Theorem, there exists some r.v., R such that $U_*'(V(\theta^t)) \to R$ a.s.

Of course, $R \leq 0$ a.s.

The rest of the proof is to show that $R = 0$ a.s. From this it follows that $V(\theta^t) \to -\infty$ a.s.

Intuition is that if $R \neq 0$ on a set of positive measure, then it would have to be true that $V(\theta^t) \to U_*'^{-1}(R)$. But then, $V(\theta^t) \approx V(\theta^{t+1})$. But, then there are no incentives for B to tell the truth since it must be true that $V(\theta^t, \theta_i) \approx V(\theta^t, \theta_j)$

Now, consider a path such that $\lim\ U_*'(V^t) = C < 0$.

Note that a.s. these paths have $\theta_t = \theta_N$ infinitely often.

Choose a τ so that for $t > \tau$, $U_*'(V^t) \in [C - \varepsilon, C + \varepsilon]$.

Thus, there is a subsequence, t_k, such that $V^{t_k} \to W$ for some W.

Since f_N is continuous, the sequence $f_N(V^{t_k}, \theta_N)$ converges to $f_N(\lim_k V^{t_k}, \theta_N) = f_N(W, \theta_N)$.

By definition, $V^{t_k+1} = f_N(V^{t_k}, \theta_N)$, $V^{t_k+1} \to f_N(W, \theta_N)$ also.

Thus, we have:

i) $\lim_k U'_*(V^{t_k}) = C$ by assumption and
ii) $\lim_k U'_*(V^{t_k+1}) = C$

Thus, by continuity of U'_*, we have

$$U'_*(W) = U'_*(f_N(W, \theta_N)) = C$$

But, by Lemma 5i) it follows that $f_N(W, \theta_N) > W$ a contradiction.

This contradiction completes the proof.