

Class Notes, Econ 8801

Lump Sum Taxes are Awesome - Simple Economies with Taxes and Spending

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1 Basic Definitions

Assume that there are k private consumption goods and m public consumption goods which can be consumed in any non-negative amounts. These will be denoted by x_i , for the private goods and g_i for the public goods. In addition, consumers are endowed with leisure which can be used to produce either the x'_j 's, or g'_j 's. Leisure will be denoted by ℓ and work will be denoted by n .

- 1) I consumers indexed by i ;
- 2) $\bar{n}_i > 0$ is i 's endowment of leisure;
- 3) $x_i = (x_{i1}, \dots, x_{ik}) \in R^k$ is the private consumption of i ;
- 4) $g_i = (g_{i1}, \dots, g_{im}) \in R^m$ is the public consumption of i ;
- 5) $U_i = U_i(\ell_i, x_i; g) : R_+^{1+k+m} \rightarrow R$ is i 's utility function;
- 6) There is one firm with a CRS production function that can turn labor into any of the other goods. W.L.O.G, we can (normalization) assume that this is of the form:

$$\sum_{j=1}^k x_j^f + \sum_{j=1}^m g_j^f \leq n^f \quad (*prod*)$$

Denote this as $(n^f; x^f; g^f) = (n^f; x_1^f, x_2^f, \dots, x_k^f; g_1^f, \dots, g_m^f)$ where x_j^f is the amount of output of the j - th private good produced by the firm, $j = 1, \dots, k$, g_j^f is the amount of the j - th public good produced by the firm and n^f is the amount of the labor that the firm uses as an input in the production process.

Allocation: $((\ell_1, x_1; g_1), \dots, (\ell_I, x_I; g_I); (n^f; x^f, g^f)) \in R_+^{I(1+k+m)+1+k+m}$

Feasibility:

$$\sum_{ij} x_{ij} = x_j^f, \quad j = 1, \dots, k$$

$$g_1 = g_2 = \dots = g^f$$

$$\sum_i (\bar{n}_i - \ell_i) = n^f$$

$$\sum_j x_j^f + \sum_j g_j^f = n^f$$

The condition that $g_1 = g_2 = \dots$ captures the idea that the g 's are public goods plus the fact that they only have to be made once, not once per consumer.

Definition 1 *Conditional Pareto Optimality (given g , ℓ_i 's, x_i 's are optimal): A feasible allocation, $((\ell_1, x_1; g_1), \dots, (\ell_I, x_I; g_I); (n^f; x^f; g^f))$, is CPO if there is not another feasible allocation, $((\ell'_1, x'_1; g_1), \dots, (\ell'_I, x'_I; g_I); (n^f; x'^f; g^f))$, (note that g has not been changed) with $U_i(\ell'_i, x'_i; g) \geq U_i(\ell_i, x_i; g)$ for all i and strict for at least one i .*

Definition 2 *Pareto Optimality (both the choice of g and the distribution of the x_i 's are optimal): A feasible allocation, $((\ell_1, x_1; g_1), \dots, (\ell_I, x_I; g_I); (n^f; x^f, g^f))$, is PO if there is not another feasible allocation, $((\ell'_1, x'_1; g'_1), \dots, (\ell'_I, x'_I; g'_I); (n^f; x^f, g'^f))$, (note that g has been changed to g') with $U_i(\ell'_i, x'_i; g'_i) \geq U_i(\ell_i, x_i; g_i)$ for all i and strict for at least one i .*

Lemma 3 *If $((\ell_1, x_1; g_1), \dots, (\ell_I, x_I; g_I); (n^f; x^f; g^f))$ is P.O., then it is C.P.O. given $g = g^f$.*

Proof: Obvious.

Lemma 4 *Under some assumptions, an allocation, $((\ell_1, x_1; g_1), \dots, (\ell_I, x_I; g_I); (n^f; x^f; g^f))$, is CPO if and only if it solves:*

$$\max_{(\ell_i, x_i), (n^f, x^f)} \sum_i \lambda_i U_i(\ell_i, x_i; g_i)$$

s.t. Feasibility

for some choices of λ_i . Note that g is held fixed in this problem.

Problem: Fill in the Assumptions.

Lemma 5 *Under some assumptions, an allocation is PO if and only if it solves:*

$$\max_{(\ell_i, x_i; g_i), (n^f, x^f; g^f)} \sum_i \lambda_i U_i(\ell_i, x_i; g_i)$$

s.t. Feasibility

for some choice of λ_i . Note that g is freely chosen in this problem.

Problem: Fill in the Assumptions.

2 Implementation

How can CPO and PO allocations be 'implemented'? I.e., how can they be reached via 'decentralized' means?

2.1 Implementation with Linear Taxes – TDCE

General Implementation: Given τ'_j 's, and a τ_n and a g , a TDCE 'implements' its equilibrium allocation.

Question: Can you tax all consumption goods and labor income too?

Notation: Prices are w for labor and p_j for good j and p_j^g for the j -th public good.

Consumer's Problem becomes:

$$\begin{aligned} \text{CP} \quad & \text{Max}_{(\ell, x)} \quad U_i(\ell, x; g) \\ \text{s.t.} \quad & \sum_{j=1}^k p_j(1 + \tau_j)x_j \leq (1 - \tau_n)(\bar{n}_i - \ell) \end{aligned}$$

Note that g is held fixed in this maximization problem.

Firm's Problem is:

$$\begin{aligned} \text{FP} \quad & \text{Max}_{(n, x; g)} \quad \sum_j p_j x_j + \sum_j p_j^g g_j - wn \\ \text{s.t.} \quad & \sum_j x_j + \sum_j g_j \leq n. \end{aligned}$$

So, the TDCE is $(w, p) \in R_+^{1+k}$, an allocation, $((\ell_1, x_1; g_1), \dots, (\ell_I, x_I; g_I); (n^f, x^f; g^f)) \in R_+$

so that:

- 1) $(\ell_i, x_i; g_i)$ solves CP;
- 2) $(n^f, x^f; g^f)$ solves FP;
- 3) Feasibility
- 4) Gov't BB:

$$\sum_j p_j^g g_j = \sum_j \sum_i p_j \tau_j x_{ij} + \sum_i (\bar{n}_i - \ell_i) \tau_n w$$

Notice that because of the special functional form of the production function, it follows from the FP that $p_j = p_{j'}^g = w$ for all j, j' . We can normalize all of these to $w = 1$ if we want.

2.2 Implementation with Transfers Only

An ADT equilibrium (Arrow-Debreu equilibrium with Transfers) is given by prices for private goods, $p \in R_+^k$, prices for public goods, $p \in R_+^m$, wages, w and an allocation, $((\ell_1, x_1; g_1), \dots, (\ell_I, x_I; g_I); (n^f, x^f; g^f))$ and a set of transfers, (T_1, \dots, T_I) , such that:

1) For all i $(\ell_i, x_i; g_i)$ solves the Consumer's Problem:

$$\begin{aligned} \text{CPT} \quad & \text{Max}_{(\ell_i, x_i)} \quad U_i(\ell, x; g) \\ \text{s.t.} \quad & \sum_j p_j x_j \leq w(\bar{n}_i - \ell_i) + T_i \\ \text{.....} \quad & \text{non-negativity of all variables.} \end{aligned}$$

2) $(n^f, x^f; g^f)$ solves the Firm's Problem:

$$\begin{aligned} \text{FPT} \quad & \text{Max}_{(n, x; g)} \quad \sum_j p_j x_j + \sum_j p_j^g g_j - wn \\ \text{s.t.} \quad & \sum_j x_j + \sum_j g_j \leq n. \end{aligned}$$

3) Feasibility

$$\begin{aligned} \sum_{ij} x_{ij} &= x_i^f \text{ for all } j \\ g_j &= g_{j'} = g^f \text{ for all } j, j' \\ \sum(\bar{n}_i - \ell_i) &= n^f \end{aligned}$$

4) Government Budget Balance

$$\sum_i T_i + \sum_j p_j^g g_j^f = 0.$$

3 Results: Lump Sum Taxes are Awesome!

3.1 Part I: Analogs of the First Welfare Theorem

We're after two things here:

1) Everything allocation implemented in an ADT equilibrium is CPO

2) No allocation implemented in a TDCE is CPO unless it is equivalent to a Transfer Equilibrium – This is a statement about how either: i) the taxes are related to each other, or, ii) labor supply is inelastic – see below.

Proposition 6 *ADT Equilibrium allocations are CPO*

Sketch of Proof: The proof of this mimics the normal 1st Welfare Theorem. That is, suppose it is false. Then, there is another allocation that is better (holding g fixed) for at least one household and no worse for any household. But, as in the usual FWT proof, this implies that this new allocation must cost at least as much for all households (at the ADT Equilibrium prices) and strictly more for the household that is strictly better off. But, then, use feasibility with equality, to obtain a contradiction.

Problem: Make sure this is right by going through the details.

Proposition 7 *Typically, TDCE allocations are not CPO.*

Sketch of Proof: This argument involves the use of the FOC's of the household problem. Perhaps there is a more general way of showing this?

From the FOC's of the household problem at the TDCE allocation, we have:

$$\frac{\frac{\partial U_i}{\partial x_j}}{p_j(1+\tau_j)} = \frac{\frac{\partial U_i}{\partial \ell}}{(1-\tau_n)}.$$

Similarly, the FOC from the condition that an allocation is CPO, we have:

$$\frac{\frac{\partial U_i}{\partial x_j}}{1} = \frac{\frac{\partial U_i}{\partial \ell}}{1}.$$

Notice that, from the FOC from the firm's problem, it follows that $p_j = w = 1$ (remember the normalization that $w = 1$). This holds in both the ADT Equilibrium and in the TDCE.

Thus, these FOC's will agree if and only iff:

$$(1 + \tau_j) = (1 + \tau_{j'}) = (1 - \tau_n).$$

I.e., ALL goods, including leisure, should be taxed at a common rate. It can be shown that this implies that the TDCE allocation is also an ADT equilibrium allocation.

Problem: Fix the details of this argment – add the necessary assumptions and prove the result completely. I.e., interiority, differentiability, etc.

Problem: Can you give a proof that doesn't depend on interiority/differentiability? (Larry doesn't know the answer to this.)

Notice that, because of this result, if you tried to solve a Ramsey Problem in this formulation without any restrictions on taxes, it would either say that you should use only lump sum taxes/transfers, OR, equivalently, you should set

$$(1 + \tau_j) = (1 + \tau_{j'}) = (1 - \tau_n).$$

3.2 Part II: Analogs of the Second Welfare Theorem

We're after two things:

- 1) Every CPO allocation can be implemented as an ADTE;
- 2) No CPO allocation can be implemented as a TDCE – unless they are equivalent to ADTE.

As we saw above, this requires that $\tau_j = \tau_{j'}$ for all j, j' and $(1 + \tau_j) = (1 - \tau_n)$ – i.e., either a subsidy on working, or a tax on leisure.

Theorem 8 Proposition 9 *If a allocation is CPO, there is a choice of transfers, (T_1, \dots, T_I) and prices as an ADT equilibrium allocation.*

Sketch of Proof:

Of course, just set the prices and wage to $1 - p_j = p_{j'}^g = w = 1$, and set T_i , by:

$$\sum_j p_j x_j^i = w(\bar{n}_i - \ell_i) + T_i.$$

Then, use the fact that the allocation is CPO to show that the FOC's to the Consumer's Problem at these prices, given this choice of g , are satisfied. That the Firm's Problem is solved follows from our choice of prices and feasibility. Thus, this implements the allocation as an ADT equilibrium.

Outline of Alternative Proof without using differentiability.

Step 1. Notice that the restricted utility U_i holding g fixed satisfies the assumptions of the 2nd Welfare Theorem where the relevant aggregate endowment of labor is given by $\sum_i \bar{n}_i - \sum_j g_j$.

Step 2. Depends on exactly what you think the 2nd Welfare Theorem says. For some versions, they automatically give the result given Step 1.

If you think that the 2nd Welfare Theorem says: "If (x_1, \dots, x_I) is PO, then there is a price system p , such that (x_1, \dots, x_I) with p is a CE equilibrium with no trade."

Then, you still need to show that you can implement this with transfers and a different budget constraint.

In either case, it's step 1 that is the critical step.

So, this shows that the optimal way of financing government spending is through the use of lump sum taxes. I.e., it's not just the best way to do redistribution, but also paying for spending.

You should use lump sum taxes to finance g even if g is not chosen 'correctly'. I.e., you can't do any better than this.

4 Discussion

So, you should use Lump Sum Taxes. But they are never actually used. Why not?

1) Do you know enough about people to do this? I.e., These are typically implemented through a system of individual specific transfers. How does the Planner know which households to give which transfers? If there is private information about 'type,' perhaps these individual specific transfers are no longer feasible – They don't satisfy 'Incentive Compatibility.'

2) Of course, this is the essence of the Mirrlees approach to income taxation, our next topic of study. This approach allows for individual specific transfers, but requires that agents will 'reveal' what their true 'type' is so that the system of transfers can be 'implemented.'