Economic Models of Knowledge

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Abstract

"Research is a series of false starts... of believed insights followed by profound confusion, and in the best cases, some small movement in the frontier of knowledge at the end." Unknown.

1 Cross Country Data

The idea here is to collect a standardized set of facts against which all models can be compared.

What I would like is a Table Something like the following one:

Models: $\rightarrow\rightarrow\rightarrow$	Cass-Koopmans	Ak	$A(k,h)$	Romer	Lucas	Other?
Country Heterogeneity $\downarrow\downarrow$						
Initial Conditions						
Technology						
Preferences						
Policy: Taxes						
Policy: Spending						
Policy: Inflation						

Where in each entry of the Table, there is a further Table which looks like:

	Data	Model
$\gamma = ?$		
$\frac{y_t^i}{y_t^j} = ?$		
$\frac{wn}{y} = ?$		
$TFP_t^i = ?$		
y_{1960}^i vs. $\bar{\gamma}_{60-95} = ?$		
y_{1960}^i vs. $R_t^i = ?$		
$\frac{x}{y}$ vs. $y = ?$		
$\frac{x}{y}$ vs. $\gamma = ?$		
$Educ.$ vs. $y = ?$		
γ_n vs. $y = ?$		
γ_n vs. $\gamma = ?$		

This list is VERY incomplete, and just includes some of the things that I, personally find interesting, and some things that everyone seems to include when they do empirical studies of growth and development on cross sectional, country, data.

1.1 Data Table

I think that for the Data version, we would have:

	Data	Model
$\gamma = ?$	1.02 varied	
$\frac{y_t^i}{y_t^j} = ?$	1 to 50	
$\frac{wn}{y} = ?$	≈ 0.67?	
$TFP_t^i = ?$	varied	
y_{1960}^i vs. $\bar{\gamma}_{60-95} = ?$	no relationship	
y_{1960}^i vs. $R_t^i = ?$	no relationship? (risk)	
$\frac{x}{y}$ vs. $y = ?$	$corr(\frac{x}{y}, y) > 0$?	
$\frac{x}{y}$ vs. $\gamma = ?$	$corr(\frac{x}{y}, \gamma) > 0$?	
$Educ.$ vs. $y = ?$	$corr(Ed, y) > 0$?	
γ_n vs. $y = ?$	$corr(\gamma_n, y) < 0$?	
γ_n vs. $\gamma = ?$	$corr(\gamma_n, \gamma) < 0$?	

2 The Cass-Koopmans Model and Implications for Cross Country Heterogeneity

Here I outline the basics of the standard single sector growth model with exogenous technological change and try to develop some simple predictions that the model has for cross country observations for different sources of heterogeneity.

2.1 The Cass-Koopmans Model

Here the equilibrium version of the model is:

An equilibrium is:

a sequence of prices: $\{(p_t, r_t, w_t)\}_{t=0}^{\infty}$

Quantity decisions for the households: $\{(c_t, k_t, x_{kt})\}_{t=0}^{\infty} = z^{HH}$

Quantity decisions for the output firms: $\{(c_t^f, x_{kt}^f, k_t^f, n_t^f)\}_{t=0}^{\infty} = z^f$, $j = 1, ..., J$

SUCH THAT:

1) z^{HH} is the solution to:

$Max_{\{(c_t, k_t, x_{kt}, n_t, \ell_t)\}_{t=0}^{\infty}} U((c_t, \ell_t)_{t=0}^{\infty})$

subject to:

$\sum_{t=0}^{\infty} p_t [c_t + x_{kt}] \leq \sum_{t=0}^{\infty} [r_t k_t + w_t n_t] + \Pi$

$k_{t+1} \leq (1 - \delta_k) k_t + x_{kt}$

$n_t + \ell_t \leq 1,$

k_0 fixed.

2) z^f is the solution to:

$Max_{\{(c_t^f, x_{kt}^f, k_t^f, n_t^f)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_{kt}^f) - r_t k_t^f - w_t n_t^f\right]$

subject to: $c_t^f + x_{kt}^f \leq B \left(k_t^f\right)^{\alpha} (A_t n_t^f)^{1-\alpha}.$

AND

$c_t = c_t^f$

$x_t = x_{kt}^f$

$k_t = k_t^f$

$\Pi = \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_{kt}^f) - r_t k_t^f - w_t n_t^f \right]$

As is standard, assuming that F is CRS and weakly concave, this equilibrium can be found by following the planner's problem:

$Max_{\{(c_t, k_t, x_{kt}, n_t, \ell_t)\}_{t=0}^{\infty}} U((c_t, \ell_t)_{t=0}^{\infty})$

subject to:

$c_t + x_{kt} \leq B k_t^{\alpha} (A_t n_t)^{1-\alpha}$

$k_{t+1} \leq (1-\delta_k) k_t + x_{kt}$

$n_t + \ell_t \leq 1,$

k_0 fixed.

Assuming that $U(c, \ell) = \sum_t \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} v(\ell)$

the first order conditions for a solution are:

$$\frac{U_{ct}}{1} = \frac{U_{\ell t}}{F_{\ell t}}$$

$$U_{ct} = U_{ct+1} \left[1 - \delta_k + F_{kt+1} \right]$$

or, given the specific functional forms,

$$\beta^t c_t^{-\sigma} v(\ell_t) (1-\alpha) B A_t^{1-\alpha} \left[\frac{k_t}{1-\ell_t} \right]^{\alpha} = \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} v'(\ell_t)$$

and,

$$\beta^t c_t^{-\sigma} v(\ell_t) = \beta^{t+1} c_{t+1}^{-\sigma} v(\ell_{t+1}) \left[1 - \delta_k + \alpha A_{t+1}^{1-\alpha} B \left[\frac{1-\ell_{t+1}}{k_{t+1}} \right]^{1-\alpha} \right]$$

OR

$$(1-\alpha)(1-\sigma) B \left[\frac{k_t}{A_t} \right]^{\alpha} = \frac{c_t}{A_t} \frac{(1-\ell_t) v'(\ell_t)}{v(\ell_t)}$$

and,

$$\left[\frac{c_{t+1}}{c_t}\right]^\sigma \frac{v(\ell_t)}{v(\ell_{t+1})} = \beta\left[1-\delta_k+\alpha B(1-\ell_{t+1})^{1-\alpha}\left[\frac{k_{t+1}}{A_{t+1}}\right]^{\alpha-1}\right]$$

These, coupled with the feasibility constraints, renormalized, and at equality:

$$\frac{c_t}{A_t}+\frac{x_{kt}}{A_t} = B\left[\frac{k_t}{A_t}\right]^\alpha (1-\ell_t)^{1-\alpha}$$

$$\frac{k_{t+1}}{A_{t+1}}\frac{A_{t+1}}{A_t} = (1-\delta_k)\frac{k_t}{A_t}+\frac{x_{kt}}{A_t}$$

Are the necessary conditions that a solution must satisfy (and are sufficient with the Transversality Condition).

2.1.1 Balanced Growth

In some cases, the solution to the equations have a stationarity propery. Assume that $A_t = (1+g)^t$ and define new variables, $\hat{k}_t = k_t/(1+g)^t$, $\hat{c}_t = c_t/(1+g)^t$, and $\hat{x}_{kt} = x_{kt}/(1+g)^t$. Then, the equations above can be rewritten as:

$$(1-\alpha)(1-\sigma)B\left[\hat{k}_t\right]^\alpha = \hat{c}_t\frac{(1-\ell_t)v'(\ell_t)}{v(\ell_t)}$$

and,

$$\left[\frac{\hat{c}_{t+1}}{\hat{c}_t}(1+g)^{-1}\right]^\sigma \frac{v(\ell_t)}{v(\ell_{t+1})} = \beta\left[1-\delta_k+\alpha B(1-\ell_{t+1})^{1-\alpha}\left[\hat{k}_{t+1}\right]^{\alpha-1}\right]$$

These, coupled with the feasibility constraints, renormalized, and at equality:

$$\hat{c}_t+\hat{x}_{kt} = B\left[\hat{k}_t\right]^\alpha(1-\ell_t)^{1-\alpha}$$

$$(1+g)\hat{k}_{t+1} = (1-\delta_k)\hat{k}_t+\hat{x}_{kt}$$

A Balanced Growth Path is when, in the solution to the maxiziation problem, the variables $\hat{k}_t$, $\hat{c}_t$, $\hat{x}_{kt}$, and ℓ_t, are constant over time.

In this case then, we have

$$\hat{x} = (g+\delta_k)\hat{k},$$

This allows us to completely eliminate $\hat{x}$ from the system (and drop an equation):

$$(1+g)^{-\sigma} = \beta\left[1-\delta_k+\alpha B\left[\frac{1-\ell}{\hat{k}}\right]^{1-\alpha}\right]$$

$$\hat{c}+(g+\delta_k)\hat{k} = B\hat{k}^\alpha(1-\ell)^{1-\alpha}$$

$$(1-\alpha)(1-\sigma)B\hat{k}^\alpha = \hat{c}\frac{(1-\ell)v'(\ell)}{v(\ell)}$$

It can be seen that the first equation uniquely pins down an optimal capital (detrended) labor ratio which depends only on the deep parameters of the model.

Typically, this system will uniquely pin down a $\hat{k}$ as a function of the parameters of the model and it follows that IF $k_0 = \hat{k}$, then the system is on the BGP for all t (show this by guess and verify?). (Subject to the TC holding!) Again typically (when exactly?) even if this restriction on initial conditions does NOT hold, the solution converges to the BGP at $t \to \infty$.

The system simplifes when we also assume that leisure does not enter the utility function. In this case, we have:

$$(1+g)^{-\sigma} = \beta\left[1 - \delta_k + \alpha B\hat{k}^{\alpha-1}\right]$$

$$\hat{c} + (g+\delta_k)\hat{k} = B\hat{k}^{\alpha}$$

Again, the first equation can be viewed as a restriction on initial conditions, and if $k_0 = \hat{k}^*$ satisfies this equation, it follows that the solution to the problem is given by $\hat{k}_t = k_0$ for all t, with $\hat{c}$ given by the second equation.

Even if this does not hold, this is a standard time stationary single sector growth model and hence it follows that $\hat{k}_t \to \hat{k}^*$ in any case. Moreover, this convergence is monotone. That is:

1. If $k_0 < \hat{k}^*$, then $k_0 < \hat{k}_1 < \hat{k}_2 < ... \to \hat{k}^*$ and
2. If $k_0 > \hat{k}^*$, then $k_0 > \hat{k}_1 > \hat{k}_2 > ... \to \hat{k}^*$

Thus, $\frac{\hat{k}_t}{\hat{k}^*} \to 1$, or $\frac{k_t/(1+g)^t}{\hat{k}^*} \to 1$, or

$$\log(k_t/(1+g)^t) - \log(\hat{k}^*) \to 0, \text{ or,}$$

$$\log(k_t) - t\log(1+g) \to \log(\hat{k}^*)$$

This convergence is 'from above' if $k_0 > \hat{k}^*$, and 'from below' if $k_0 < \hat{k}^*$

Define $\bar{\gamma}(k_0, t) = \frac{1}{t}\left[\log(k_t) - \log(k_0)\right]$. This is the average rate of growth (in logs) for the first t periods given that you start at initial condition k_0.

Because of the pattern of convergence to the BGP described above, it is typically the case that

$\bar{\gamma}(k_0, t) > \log(1+g)$ if $k_0 < \hat{k}^*$

and

$\bar{\gamma}(k_0, t) < \log(1+g)$ if $k_0 > \hat{k}^*$

3 Heterogeneity in Cass-Koopmans

Here we study what the data would like if the economic data for each country in the world was a Cass-Koopmans model. Of course, since different countries have different data, the particular versions of the Cass-Koopmans model will have to be different for each country. And what the data will look like will naturally depend on just exactly how that heterogeneity is introduced.

3.1 Initial Condition Heterogeneity in C-K

We'll restrict our attention to the inelastic labor supply case.

That is we assume that the data from country i is generated from a C-K model in which all of the parameters $(\sigma, \beta, \alpha, \delta, B, (1+g), v)$ are all the same, but k_0^i may differ across $i's$.

1. What is $\frac{y_t^i}{y_t^j}$?

2. What is the pattern between y_t^i and $\bar{\gamma}_{t,t+s} = \frac{1}{s}\left[\log(y_{t+s}^i) - \log(y_t^i)\right]$, the average growth rate between periods t and $t+s$.

3. What is the relationship between interest rates, $1 + R_t^i = 1 - \delta_k + F_k^i(t)$ and y_t^i?

OTHER QUESTIONS?

Here we find that $\frac{y_t^i}{y_t^j} = \frac{BA_t^{1-\alpha}k_t^{i\alpha}}{BA_t^{1-\alpha}k_t^{j\alpha}} = \frac{k_t^{i\alpha}}{k_t^{j\alpha}} = \left[\frac{k_t^i}{k_t^j}\right]^\alpha$, or equivalently, $\frac{k_t^i}{k_t^j} = \left[\frac{y_t^i}{y_t^j}\right]^{1/\alpha}$. So, you can have large output differences, but these are accompanied by large capital differences too. Below I discuss how large these must be.

We also find that there is a negative relationship between y_t^i and $\bar{\gamma}_{t,t+s} = \frac{1}{s}\left[\log(y_{t+s}^i) - \log(y_t^i)\right]$, the average growth rate between periods t and $t+s$ because of the description above about convergence to the BGP from initial conditions. And we find that the relationship between interest rates, $1 + R_t^i = 1 - \delta_k + F_k^i(t)$ and y_t^i is also negative.

Moreover, for quantitatively interesting versions of the model, this relationships are quite siginificant.

Thus, if we see, for example that two countries have differences in income of say, 10, then they would have to have differences in capital stocks of $(10)^{1/\alpha}$. Assuming that $\alpha = .33$, this means that

$$\frac{k_t^i}{k_t^j} = \left[\frac{y_t^i}{y_t^j}\right]^{1/\alpha} = [10]^3 = 1,000$$

and if it is 30, we have:

$$\frac{k_t^i}{k_t^j} = \left[\frac{y_t^i}{y_t^j}\right]^{1/\alpha} = [30]^3 = 27,000, \text{ etc.}$$

This depends on alpha, and is less outrageous when α is larger, but even at $\alpha = 1$, i.e., the Ak model, it is still pretty large for large income gaps.

Similar reasoning holds for rates of return. Note that

$$r_t^i = F_k^i(k_t^i, 1) = \alpha \frac{F^i(k_t^i,1)}{k_t^i} = \alpha \frac{A(k_t^i)^\alpha}{k_t^i} = \alpha(k_t^i)^{\alpha-1},$$

and hence,

$\frac{r_t^i}{r_t^j} = \frac{\alpha(k_t^i)^{\alpha-1}}{\alpha(k_t^j)^{\alpha-1}} = \left[\frac{k_t^i}{k_t^j}\right]^{\alpha-1}$. In terms of income ratios, using what we did above, we see that

$$\frac{r_t^i}{r_t^j} = \left[\frac{k_t^i}{k_t^j}\right]^{\alpha-1} = \left[\left[\frac{y_t^i}{y_t^j}\right]^{1/\alpha}\right]^{\alpha-1} = \left[\frac{y_t^i}{y_t^j}\right]^{(\alpha-1)/\alpha} = \left[\frac{y_t^j}{y_t^i}\right]^{(1-\alpha)/\alpha}$$

Thus, with $\alpha = 0.33$ this becomes:

$$\frac{r_t^i}{r_t^j} = \left[\frac{y_t^j}{y_t^i}\right]^{(1-\alpha)/\alpha} = \left[\frac{y_t^j}{y_t^i}\right]^{(1-.33)/.33} = \left[\frac{y_t^j}{y_t^i}\right]^2.$$

Hence, an income ratio of 10 corresponds to a ratio of rental rates on capital of 100 (with the low income country having the higher rental rate on capital, an income ratio of 30 corresponds to a ratio of rental rates on capital of 900, etc.

Recalling that the interest rate is given by:

$$1 + R_t^i = 1 - \delta_k + r_t^i,$$

we see that for a country like the US, with $R_t^i = .04$ (4%), and $\delta_k = .08$, we must have $r_t^{US} = .12$. For any other country, i, we have

$$1+R_t^i = 1-\delta_k+r_t^i = 1-\delta_k+r_t^{US}\left[\frac{y_t^{US}}{y_t^i}\right]^2 = 1-.08+.12\left[\frac{y_t^{US}}{y_t^i}\right]^2 = 0.92+.12\left[\frac{y_t^{US}}{y_t^i}\right]^2$$

For example, for $\left[\frac{y_t^{US}}{y_t^i}\right] = 10$ and 30, respectively we get:

$$1 + R_t^i = 0.92 + .12\left[\frac{y_t^{US}}{y_t^i}\right]^2 = 0.92 + .12 * 100 = 12.92$$

Or, $R_t^i = 11.92 * 100 = 1192\%$ per year.

$$1 + R_t^i = 0.92 + .12 \left[\frac{y_t^{US}}{y_t^i}\right]^2 = 0.92 + .12 * 900 = 108.92$$

Or, $R_t^i = 107.92 * 100 = 10,792\%$ per year.

Again, these calculations are sensitive to what α is assumed to be. See Table.

3.1.1 Cass-Koopmans Initial Conditions

	Data	Model
$\gamma =?$	1.02 varied	$(1+g)$ by assumption
$\frac{y_t^i}{y_t^j} =?$	1 to 50	convergence implies small if t is large
$\frac{wn}{y} =?$	$\approx 0.67?$	$\approx 0.67?$
$TFP_t^i =?$	varied	no differences
y_{1960}^i vs. $\bar{\gamma}_{60-95} =?$	no relationship	decreasing relationship
y_{1960}^i vs. $R_t^i =?$	no relationship? (risk)	no relationship?
$\frac{x}{y}$ vs. $y =?$	$corr(\frac{x}{y}, y) > 0?$	?
$\frac{x}{y}$ vs. $\gamma =?$	$corr(\frac{x}{y}, \gamma) > 0?$	?
$Educ.$ vs. $y =?$	$corr(Ed, y) > 0?$	not defined
γ_n vs. $y =?$	$corr(\gamma_n, y) < 0?$	not defined
γ_n vs. $\gamma =?$	$corr(\gamma_n, \gamma) < 0?$	not defined

3.1.2 Differences in A in C-K

Next, we'll look at what the data would look like if the only differences in countries was in A. Thus, we will assume that $k_0^i = k_0^j$ for all i, j.

In this case,

$$\frac{y_t^i}{y_t^j} = \frac{A_t^i (k_t^i)^\alpha}{A_t^j (k_t^j)^\alpha} = \frac{A_t^i}{A_t^j} \left[\frac{k_t^i}{k_t^j}\right]^\alpha$$

Assume that $A_t^i = A_0^i (1+g)^t$, that is the trend growth rate is the same in all countries, we see that the balanced growth equations are:

$$(1+g)^{-\sigma} = \beta \left[1 - \delta_k + \alpha A_0^i \left(\hat{k}^i\right)^{\alpha - 1}\right]$$

$$\hat{c} + (g + \delta_k)\hat{k} = A_0^i \left(\hat{k}^i\right)^\alpha$$

Where, as above, $\hat{k}^i = \frac{k_t^i}{(1+g)^t}$.

Recall that the equilibrium path will correspond to the BGP if $k_0^i = \hat{k}^i$. In this case, $k_t^i = (1+g)^t \hat{k}^i$ for all t. Even if $k_0^i \neq \hat{k}^i$, this should be approximately true?

Using the BGP equations for the two countries and noting that all other parameters are assumed to be the same, we see that

$$\beta \left[1 - \delta_k + \alpha A_0^i \left(\hat{k}^i\right)^{\alpha-1}\right] = (1+g)^{-\sigma} = \beta \left[1 - \delta_k + \alpha A_0^j \left(\hat{k}^j\right)^{\alpha-1}\right]$$

or,

$$\beta \left[1 - \delta_k + \alpha A_0^i \left(\hat{k}^i\right)^{\alpha-1}\right] = \beta \left[1 - \delta_k + \alpha A_0^j \left(\hat{k}^j\right)^{\alpha-1}\right]$$

or,

$$A_0^i \left(\hat{k}^i\right)^{\alpha-1} = A_0^j \left(\hat{k}^j\right)^{\alpha-1}$$

or,

$$\left[\frac{\hat{k}^i}{\hat{k}^j}\right]^{\alpha-1} = \frac{A_0^j}{A_0^i}, \text{ and so,}$$

$$\left[\frac{\hat{k}^i}{\hat{k}^j}\right] = \left[\frac{A_0^j}{A_0^i}\right]^{1/(\alpha-1)}$$

Thus, in this case,

$$\frac{y_t^i}{y_t^j} = \frac{A_t^i \left(k_t^i\right)^{\alpha}}{A_t^j \left(k_t^j\right)^{\alpha}} = \frac{A_t^i}{A_t^j} \left[\frac{k_t^i}{k_t^j}\right]^{\alpha} = \frac{A_t^i}{A_t^j} \left[\frac{k_t^i}{k_t^j}\right]^{\alpha} = \frac{(1+g)^t A_0^i}{(1+g)^t A_0^j} \left[\frac{(1+g)^t k_0^i}{(1+g)^t k_0^j}\right]^{\alpha}$$

$$= \frac{A_0^i}{A_0^j} \left[\frac{k_0^i}{k_0^j}\right]^{\alpha} = \frac{A_0^i}{A_0^j} \left[\frac{\hat{k}^i}{\hat{k}^j}\right]^{\alpha} = \frac{A_0^i}{A_0^j} \left[\left[\frac{A_0^j}{A_0^i}\right]^{1/(\alpha-1)}\right]^{\alpha} = \frac{A_0^i}{A_0^j} \left[\frac{A_0^j}{A_0^i}\right]^{\alpha/(\alpha-1)}$$

$$= \frac{A_0^i}{A_0^j} \left[\frac{A_0^i}{A_0^j}\right]^{\alpha/(1-\alpha)} = \left[\frac{A_0^i}{A_0^j}\right]^{\frac{1-\alpha+\alpha}{1-\alpha}} = \left[\frac{A_0^i}{A_0^j}\right]^{\frac{\alpha}{1-\alpha}}$$

From this, we can see that we need:

$$\frac{A_0^i}{A_0^j} = \left[\frac{y_t^i}{y_t^j}\right]^{\frac{1-\alpha}{\alpha}}$$

Assuming that $\alpha = 0.33$, it follows that $\frac{1-\alpha}{\alpha} = \frac{0.67}{0.33} = 2$, and thus, if $\frac{y_t^i}{y_t^j} = 10$, we need

$\frac{A_0^i}{A_0^j} = 100$, i.e., technology is 100 times more productive in the rich country versus the poor one.

NEXT QUESTION:

What is the relationship between y_{60}^i and γ? It can be seen that if all countries are on the BGP, then $\gamma = (1+g)$ and hence there should be NO relationship between them, i.e., the relationship should be FLAT.

NEXT QUESTION:

Interest rates? Assuming that all countries are on their BGP's, we have that

$$1 + R_t^i = \frac{p_t}{p_{t+1}} = \frac{u_c(t)}{\beta u_c(t+1)} = \frac{1}{\beta}(1+g)^\sigma$$

Thus, there should be no relationship between y_{60}^i and R_t^i.

NEXT QUESTION:

Measured TFP?

$$\log(TFP_t^i) = \log(y_t^i) - .33\log(k_t^i) - .67\log(n_t^i) = \log\left[\frac{A_t^i(k_t^i)^{.33}(n_t^i)^{.67}}{(k_t^i)^{.33}(n_t^i)^{.67}}\right] = \log(A_t^i)$$

So, this delivers $d\gamma/dy = 0$ and $dR/dy = 0$ and big y^i/y^j, but you need pretty big differences in $A's$.

3.1.3 Cass-Koopmans A^i Differences

	Data	Model
$\gamma =?$	1.02 varied	$(1+g)$
$\frac{y_t^i}{y_t^j} =?$	1 to 50	$\left[\frac{y_t^i}{y_t^j}\right]^{\frac{1-\alpha}{\alpha}} = \frac{A_0^i}{A_0^j}$
$\frac{wn}{y} =?$	$\approx 0.67?$	$\approx 0.67?$
$TFP_t^i =?$	varied	$TFP_t^i : \left[\frac{y_t^i}{y_t^j}\right]^2$
y_{1960}^i vs. $\bar{\gamma}_{60-95} =?$	no relationship	no relationship
y_{1960}^i vs. $R_t^i =?$	no relationship? (risk)	no relationship
$\frac{x}{y}$ vs. $y =?$	$corr(\frac{x}{y}, y) > 0?$	?
$\frac{x}{y}$ vs. $\gamma =?$	$corr(\frac{x}{y}, \gamma) > 0?$	?
$Educ.$ vs. $y =?$	$corr(Ed, y) > 0?$	not defined
γ_n vs. $y =?$	$corr(\gamma_n, y) < 0?$	not defined
γ_n vs. $\gamma =?$	$corr(\gamma_n, \gamma) < 0?$	not defined

3.1.4 Differences in δ_k in C-K

3.1.5 Cass-Koopmans δ_k Differences

	Data	Model
$\gamma =?$	1.02 varied	$(1+g)$
$\frac{y_t^i}{y_t^j} =?$	1 to 50	?
$\frac{wn}{y} =?$	$\approx 0.67?$	$\approx 0.67?$
$TFP_t^i =?$	varied	?
y_{1960}^i vs. $\bar{\gamma}_{60-95} =?$	no relationship	no relationship
y_{1960}^i vs. $R_t^i =?$	no relationship? (risk)	no relationship?
$\frac{x}{y}$ vs. $y =?$	$corr(\frac{x}{y}, y) > 0?$	?
$\frac{x}{y}$ vs. $\gamma =?$	$corr(\frac{x}{y}, \gamma) > 0?$	?
$Educ.$ vs. $y =?$	$corr(Ed, y) > 0?$	not defined
γ_n vs. $y =?$	$corr(\gamma_n, y) < 0?$	not defined
γ_n vs. $\gamma =?$	$corr(\gamma_n, \gamma) < 0?$	not defined

3.1.6 Differences in β in C-K

The BGP equations are:

$$(1+g)^{-\sigma} = \beta^i \left[1 - \delta_k + \alpha A_0 \left(\hat{k}^i\right)^{\alpha-1}\right]$$

$$\hat{c} + (g+\delta_k)\hat{k} = A_0 \left(\hat{k}^i\right)^{\alpha}$$

1. Thus, a higher value of β^i will be associated with a higher value of $\hat{k}^i$ and thus a higher value of y_t^i for all t. What does this relationship look like?

$$\beta^i \left[1 - \delta_k + \alpha A_0 \left(\hat{k}^i\right)^{\alpha-1}\right] = \beta^j \left[1 - \delta_k + \alpha A_0 \left(\hat{k}^j\right)^{\alpha-1}\right]$$

Assume that $\delta_k = 1$?

Then,

$$\left[\left(\frac{\hat{k}^j}{\hat{k}^i}\right)^{1-\alpha}\right] = \frac{\beta^j}{\beta^i}$$

or,

$$\frac{\hat{k}^j}{\hat{k}^i} = \left[\frac{\beta^j}{\beta^i}\right]^{1/(1-\alpha)}$$

Thus,

$$\frac{y_t^i}{y_t^j} = \frac{(1+g)^t A_0 \left(\hat{k}^i\right)^\alpha}{(1+g)^t A_0 \left(\hat{k}^j\right)^\alpha} = \left[\frac{\hat{k}^i}{\hat{k}^j}\right]^\alpha = \left[\frac{\beta^i}{\beta^j}\right]^{\frac{\alpha}{(1-\alpha)}} = \left[\frac{\beta^i}{\beta^j}\right]^{0.5}$$

Thus, it would be impossibly difficult to get the observed differences in y_t^i from only differences across countries in $\beta's$.

y vs. γ? there should be no relationship on BGP's.

Interest rates? Assuming that all countries are on their BGP's, we have that

$$1 + R_t^i = \frac{p_t}{p_{t+1}} = \frac{u_c(t)}{\beta^i u_c(t+1)} = \frac{1}{\beta^i}(1+g)^\sigma$$

So, those countries with higher $\beta's$ should have lower interest rates.

TFP? Is equal to A_t in all countries.

3.1.7 Cass-Koopmans β differences

	Data	Model
$\gamma =?$	1.02 varied	$(1+g)$
$\frac{y_t^i}{y_t^j} =?$	1 to 50	$\frac{y_t^i}{y_t^j} = \left[\frac{\beta^i}{\beta^j}\right]^{0.5}$
$\frac{wn}{y} =?$	$\approx 0.67?$	$\approx 0.67?$
$TFP_t^i =?$	varied	equal in all countries
y_{1960}^i vs. $\bar{\gamma}_{60-95} =?$	no relationship	no relationship
y_{1960}^i vs. $R_t^i =?$	no relationship? (risk)	$1 + R_t^i = \frac{1}{\beta^i}(1+g)^\sigma$
$\frac{x}{y}$ vs. $y =?$	$corr(\frac{x}{y}, y) > 0?$	?
$\frac{x}{y}$ vs. $\gamma =?$	$corr(\frac{x}{y}, \gamma) > 0?$	?
$Educ.$ vs. $y =?$	$corr(Ed, y) > 0?$	not defined
γ_n vs. $y =?$	$corr(\gamma_n, y) < 0?$	not defined
γ_n vs. $\gamma =?$	$corr(\gamma_n, \gamma) < 0?$	not defined

3.1.8 Differences in σ in C-K

3.1.9 Cass-Koopmans σ Differences

	Data	Model
γ =?	1.02 varied	$(1+g)$
$\frac{y_t^i}{y_t^j}$ =?	1 to 50	?
$\frac{wn}{y}$ =?	≈ 0.67?	≈ 0.67?
TFP_t^i =?	varied	?
y_{1960}^i vs. $\bar{\gamma}_{60-95}$ =?	no relationship	no relationship
y_{1960}^i vs. R_t^i =?	no relationship? (risk)	no relationship?
$\frac{x}{y}$ vs. y =?	$corr(\frac{x}{y}, y) > 0$?	?
$\frac{x}{y}$ vs. γ =?	$corr(\frac{x}{y}, \gamma) > 0$?	?
$Educ.$ vs. y =?	$corr(Ed, y) > 0$?	not defined
γ_n vs. y =?	$corr(\gamma_n, y) < 0$?	not defined
γ_n vs. γ =?	$corr(\gamma_n, \gamma) < 0$?	not defined

4 Adding Fiscal Policy to Cass-Koopmans

Here the equilibrium version of the model is:

An equilibrium is:

a sequence of prices: $\{(p_t, r_t, w_t)\}_{t=0}^{\infty}$

Quantity decisions for the households: $\{(c_t, k_t, x_{kt})\}_{t=0}^{\infty} = z^{HH}$

Quantity decisions for the output firms: $\{(c_t^f, x_{kt}^f, k_t^f, n_t^f)\}_{t=0}^{\infty} = z^f$,

SUCH THAT:

1) z^{HH} is the solution to:

$Max_{\{(c_t, k_t, x_{kt}, n_t, \ell_t)\}_{t=0}^{\infty}} U((c_t, \ell_t)_{t=0}^{\infty})$

subject to:

$\sum_{t=0}^{\infty} p_t [c_t + x_{kt}] \leq \sum_{t=0}^{\infty} [(1-\tau_{kt}) r_t k_t + (1-\tau_{nt}) w_t n_t + T_t] + \Pi$

$k_{t+1} \leq (1-\delta_k)k_t + x_{kt}$

$\ell_t + n_t \leq 1$

k_0 fixed.

2) z^f is the solution to:

$Max_{\{(c_t^f, x_{kt}^f, k_t^f, n_t^f)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_{kt}^f) - r_t k_t^f - w_t n_t^f\right]$

subject to: $c_t^f + x_{kt}^f \leq B\left(k_t^f\right)^{\alpha}(A_t n_t^f)^{1-\alpha}$.

AND

$g_t + c_t = c_t^f$

$x_t = x_{kt}^f$

$x_{ht} = x_{ht}^f$

$\Pi = \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_{kt}^f + x_{ht}^f) - r_t k_t^f - w_t n_t^f\right]$

$\sum_t [p_t g_t + T_t] = \sum_t [\tau_{kt} r_t k_t + \tau_{nt} w_t n_t]$

the first order conditions for a solution are:

$$\frac{U_{ct}}{1} = \frac{U_{\ell t}}{(1-\tau_{nt})F_{\ell t}}$$

$$U_{ct} = U_{ct+1}\left[1 - \delta_k + (1-\tau_{kt+1})F_{kt+1}\right]$$

or, given the specific functional forms,

$$\beta^t c_t^{-\sigma} v\left(\ell_t\right)(1-\alpha)BA_t^{1-\alpha}\left[\frac{k_t}{1-\ell_t}\right]^{\alpha}(1-\tau_{nt}) = \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} v'(\ell_t)$$

and,

$$\beta^t c_t^{-\sigma} v\left(\ell_t\right) = \beta^{t+1} c_{t+1}^{-\sigma} v\left(\ell_{t+1}\right)\left[1 - \delta_k + (1-\tau_{kt+1})\alpha A_{t+1}^{1-\alpha} B\left[\frac{1-\ell_{t+1}}{k_{t+1}}\right]^{1-\alpha}\right]$$

OR

$$(1-\alpha)(1-\sigma)B\left[\frac{k_t}{A_t}\right]^{\alpha}(1-\tau_{nt}) = \frac{c_t}{A_t}\frac{(1-\ell_t)v'(\ell_t)}{v(\ell_t)}$$

and,

$$\left[\frac{c_{t+1}}{c_t}\right]^{\sigma}\frac{v(\ell_t)}{v(\ell_{t+1})}=\beta\left[1-\delta_k+(1-\tau_{kt+1})\alpha B(1-\ell_{t+1})^{1-\alpha}\left[\frac{k_{t+1}}{A_{t+1}}\right]^{\alpha-1}\right]$$

These, coupled with the feasibility constraints, renormalized, and at equality:

$$\frac{c_t}{A_t}+\frac{x_{kt}}{A_t}+\frac{g_t}{A_t}=B\left[\frac{k_t}{A_t}\right]^{\alpha}(1-\ell_t)^{1-\alpha}$$

$$\frac{k_{t+1}}{A_{t+1}}\frac{A_{t+1}}{A_t}=(1-\delta_k)\frac{k_t}{A_t}+\frac{x_{kt}}{A_t}$$

The government budget constraint is redundant and can be dropped.

Are the necessary conditions that a solution must satisfy (and are sufficient with the Transversality Condition).

4.1 Balanced Growth

In some cases, the solution to the equations have a stationarity propery. Typically, this will require that the policy variables also satisfy some sort of stationarity assumption. We will assume that $\tau_{nt}=\tau_n$, $\tau_{kt}=\tau_k$ and will assume that $A_t=(1+g)^t$ and define new variables, $\hat{k}_t=k_t/(1+g)^t$, $\hat{c}_t=c_t/(1+g)^t$, $\hat{x}_{kt}=x_{kt}/(1+g)^t$, $\hat{g}_t=g_t/(1+g)^t$ Then, the equations above can be rewritten as:

$$(1-\alpha)(1-\sigma)B\left[\hat{k}_t\right]^{\alpha}(1-\tau_n)=\hat{c}_t\frac{(1-\ell_t)v'(\ell_t)}{v(\ell_t)}$$

and,

$$\left[\frac{\hat{c}_{t+1}}{\hat{c}_t}(1+g)^{-1}\right]^{\sigma}\frac{v(\ell_t)}{v(\ell_{t+1})}=\beta\left[1-\delta_k+(1-\tau_k)\alpha B(1-\ell_{t+1})^{1-\alpha}\left[\hat{k}_{t+1}\right]^{\alpha-1}\right]$$

These, coupled with the feasibility constraints, renormalized, and at equality:

$$\hat{c}_t+\hat{x}_{kt}+\hat{g}_t=B\left[\hat{k}_t\right]^{\alpha}(1-\ell_t)^{1-\alpha}$$

$$(1+g)\hat{k}_{t+1}=(1-\delta_k)\hat{k}_t+\hat{x}_{kt}$$

A Balanced Growth Path is when, in the solution to the maxiziation problem, the variables $\hat{k}_t$, $\hat{c}_t$, $\hat{x}_{kt}$, and ℓ_t, are constant over time. This will typically require that $\hat{g}_t$ is constant, i.e., g_t grows at rate $(1+g)^t$.

In this case then, we have

$$\hat{x}=(g+\delta_k)\hat{k},$$

This allows us to completely eliminate $\hat{x}$ from the system (and drop an equation):

$$(1+g)^{-\sigma} = \beta\left[1-\delta_k+(1-\tau_k)\alpha B\left[\tfrac{1-\ell}{\hat{k}}\right]^{1-\alpha}\right]$$

$$\hat{c}+(g+\delta_k)\hat{k}+\hat{g} = B\hat{k}^{\alpha}(1-\ell)^{1-\alpha}$$

$$(1-\alpha)(1-\sigma)B\hat{k}^{\alpha}(1-\tau_n) = \hat{c}\tfrac{(1-\ell)v'(\ell)}{v(\ell)}$$

It can be seen that the first equation uniquely pins down an optimal capital (detrended) labor ratio which depends only on the deep parameters of the model and the policy variables.

Typically, this system will uniquely pin down a $\hat{k}$ as a function of the parameters of the model and it follows that IF $k_0 = \hat{k}$, then the system is on the BGP for all t (show this by guess and verify?). (Subject to the TC holding!) Again typically (when exactly?) even if this restriction on initial conditions does NOT hold, the solution converges to the BGP at $t \to \infty$.

The system simplifes when we also assume that leisure does not enter the utility function. In this case, we have:

$$(1+g)^{-\sigma} = \beta\left[1-\delta_k+(1-\tau_k)\alpha B\hat{k}^{\alpha-1}\right]$$

$$\hat{c}+(g+\delta_k)\hat{k} = B\hat{k}^{\alpha}$$

Again, the first equation can be viewed as a restriction on initial conditions, and if $k_0 = \hat{k}^*$ satisfies this equation, it follows that the solution to the problem is given by $\hat{k}_t = k_0$ for all t, with $\hat{c}$ given by the second equation.

Even if this does not hold, in some cases, even with active fiscal policy, it can be shown that this model behaves as does a standard single sector growth model under certain additional restrictions on the fiscal policy variables. For example, if $T_t = 0$ for all t, $\tau_{kt} = \tau_{nt} = \tau$ for all t, and $p_t g_t = \tau w_t n_t + \tau r_t k_t$ for all t (recall that $n_t = 1$). Thus, if there is a proportional income tax, and period by period balanced budget holds (and there are no transfers or lump sum taxes), then this model has the same equilibrium path as one without taxes but instead has a different feasibility constraint given by:

$$c_t + x_{kt} \leq (1-\tau)BF(k_t, A_t n_t)$$

Since this is now a standard time stationary single sector growth model and hence it follows that $\hat{k}_t \to \hat{k}^*$ in any case. Moreover, this convergence is monotone. That is:

1. If $k_0 < \hat{k}^*$, then $k_0 < \hat{k}_1 < \hat{k}_2 < ... \to \hat{k}^*$ and

2. If $k_0 > \hat{k}^*$, then $k_0 > \hat{k}_1 > \hat{k}_2 > ... \to \hat{k}^*$

Thus, $\frac{\hat{k}_t}{\hat{k}^*} \to 1$, or $\frac{k_t/(1+g)^t}{\hat{k}^*} \to 1$, or

$$\log(k_t/(1+g)^t) - \log(\hat{k}^*) \to 0, \text{ or,}$$

$$\log(k_t) - t\log(1+g) \to \log(\hat{k}^*)$$

This convergence is 'from above' if $k_0 > \hat{k}^*$, and 'from below' if $k_0 < \hat{k}^*$

Define $\bar{\gamma}(k_0, t) = \frac{1}{t}[\log(k_t) - \log(k_0)]$. This is the average rate of growth (in logs) for the first t periods given that you start at initial condition k_0.

Because of the pattern of convergence to the BGP described above, it is typically the case that

$\bar{\gamma}(k_0, t) > \log(1+g)$ if $k_0 < \hat{k}^*$

and

$\bar{\gamma}(k_0, t) < \log(1+g)$ if $k_0 > \hat{k}^*$

4.1.1 Cass-Koopmans Tax Policy Differences

	Data	Model
$\gamma =?$	1.02 varied	$(1+g)$
$\frac{y_t^i}{y_t^j} =?$	1 to 50	?
$\frac{wn}{y} =?$	≈ 0.67?	≈ 0.67?
$TFP_t^i =?$	varied	?
y_{1960}^i vs. $\bar{\gamma}_{60-95} =?$	no relationship	no relationship
y_{1960}^i vs. $R_t^i =?$	no relationship? (risk)	no relationship?
$\frac{x}{y}$ vs. $y =?$	$corr(\frac{x}{y}, y) > 0$?	?
$\frac{x}{y}$ vs. $\gamma =?$	$corr(\frac{x}{y}, \gamma) > 0$?	?
$Educ.$ vs. $y =?$	$corr(Ed, y) > 0$?	not defined
γ_n vs. $y =?$	$corr(\gamma_n, y) < 0$?	not defined
γ_n vs. $\gamma =?$	$corr(\gamma_n, \gamma) < 0$?	not defined

4.1.2 Cass-Koopmans Government Spending Differences

	Data	Model
$\gamma =?$	1.02 varied	$(1+g)$
$\frac{y_t^i}{y_t^j} =?$	1 to 50	?
$\frac{wn}{y} =?$	$\approx 0.67?$	$\approx 0.67?$
$TFP_t^i =?$	varied	?
y_{1960}^i vs. $\bar{\gamma}_{60-95} =?$	no relationship	no relationship
y_{1960}^i vs. $R_t^i =?$	no relationship? (risk)	no relationship?
$\frac{x}{y}$ vs. $y =?$	$corr(\frac{x}{y}, y) > 0?$	?
$\frac{x}{y}$ vs. $\gamma =?$	$corr(\frac{x}{y}, \gamma) > 0?$	?
$Educ.$ vs. $y =?$	$corr(Ed, y) > 0?$	not defined
γ_n vs. $y =?$	$corr(\gamma_n, y) < 0?$	not defined
γ_n vs. $\gamma =?$	$corr(\gamma_n, \gamma) < 0?$	not defined

4.1.3 Cass-Koopmans Monetary Policy Differences

	Data	Model
$\gamma =?$	1.02 varied	$(1+g)$
$\frac{y_t^i}{y_t^j} =?$	1 to 50	?
$\frac{wn}{y} =?$	$\approx 0.67?$	$\approx 0.67?$
$TFP_t^i =?$	varied	?
y_{1960}^i vs. $\bar{\gamma}_{60-95} =?$	no relationship	no relationship
y_{1960}^i vs. $R_t^i =?$	no relationship? (risk)	no relationship?
$\frac{x}{y}$ vs. $y =?$	$corr(\frac{x}{y}, y) > 0?$	?
$\frac{x}{y}$ vs. $\gamma =?$	$corr(\frac{x}{y}, \gamma) > 0?$	?
$Educ.$ vs. $y =?$	$corr(Ed, y) > 0?$	not defined
γ_n vs. $y =?$	$corr(\gamma_n, y) < 0?$	not defined
γ_n vs. $\gamma =?$	$corr(\gamma_n, \gamma) < 0?$	not defined

4.1.4 What would have happened with n endogenous?

This could have been asked in the context of any of the experiments above.

4.2 Summary

Like all models, there are some successes and failures here. And some things that the modelling just gives up on from the start (e.g., Education and Income, or Population and Income). This is not a weakness. Models are meant to be abstractions, and hence, necessarily they will not be perfect. If they were, they would have to be WAY too complicated to get anything useful out of them!

I think that the biggest weaknesses is the difficulty that the model(s) has (have) in generating sufficiently large differences in income per capita across countries. Initial conditions seem to require outrageous differences in k and R. But, absent this, it only gives A_t or TFP differences as the way to go. These need to be large, and in addition, no one knows what it is. Thus, income differences are 'explained' as differences in the 'unexplained residual.' Not a very satisfactory set of affairs!

This is why people who use the single sector growth model are naturally led to cry out for 'theories of TFP.' Although it's not exactly clear what this would mean.

In the next sections of these notes, we'll try and do something like that... they can all be thought of as 'endogenizing' the technological parameters of the single sector model, although they do it in VERY different ways!

5 Including Technological Knowledge in a Competitive Model: Some False Starts

How might we 'endogenize' the technology, and it's change in the one sector growth model?

What follows here is some notes of a loose discussion that was carried out in Econ 8311. Some of the tries were kind of stupid, mostly the ones I suggested. The students tend to have more common sense than I do.. so their suggestions were not in the category of 'obviously totally flawed from the beginning.'

5.1 First Try:

An equilibrium is:

a sequence of prices: $\{(p_t, r_t, w_t, r_t^A)\}_{t=0}^{\infty}$

Quantity decisions for the households: $\{(c_t, k_t, x_t, \ell_t, n_t)\}_{t=0}^{\infty} = z^{HH}$

Quantity decisions for the output firms: $\{(c_t^f, x_t^f, k_t^f, n_t^f)\}_{t=0}^{\infty} = z^f$

Quantity decisions for the R&D firms: $\{(A_t^{RD}, k_t^{RD}, n_t^{RD})\}_{t=0}^{\infty} = z^{RD}$

SUCH THAT:

1) z^{HH} is the solution to:

$Max_{\{(c_t,k_t,x_t,\ell_t,n_t)\}_{t=0}^{\infty}} U((c_t, \ell_t)_{t=0}^{\infty})$

subject to:

$\sum_{t=0}^{\infty} p_t [c_t + x_t] \leq \sum_{t=0}^{\infty} [r_t k_t + w_t n_t] + \Pi$

$k_{t+1} \leq (1-\delta)k_t + x_t$

$n_t + \ell_t \leq 1$

k_0 fixed.

2) z^f is the solution to:

$Max_{\{(c_t^f,x_t^f,k_t^f,n_t^f)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_t^f) - r_t k_t^f - w_t n_t^f\right]$

subject to: $c_t^f + x_t^f \leq A_t^f F(k_t^f, n_t^f)$

3) z^{RD} is the solution to:

$Max_{\{(A_t^{RD},k_t^{RD},n_t^{RD})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[r_t^A A_t^{RD} - r_t k_t^{RD} - w_t n_t^{RD}\right]$

subject to: $A_t^{RD} \leq G(A_t^{RD}, k_t^{RD}, n_t^{RD})$

AND

$c_t = c_t^f$

$x_t = x_t^f$

$k_t = k_t^f + k_t^{RD}$

$n_t = n_t^f + n_t^{RD}$

$A_t^f = A_t^{RD}$

$\Pi = \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_t^f) - r_t k_t^f - w_t n_t^f\right] + \sum_{t=0}^{\infty} \left[r_t^A A_t^{RD} - r_t k_t^{RD} - w_t n_t^{RD}\right]$

$= \Pi^f + \Pi^{RD}$

where

$\Pi^f = \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_t^f) - r_t k_t^f - w_t n_t^f\right]$

and,

$\Pi^{RD} = \sum_{t=0}^{\infty} \left[r_t^A A_t^{RD} - r_t k_t^{RD} - w_t n_t^{RD}\right].$

Problems:

1. Although the firm seems to be using the output of the R&D sector, since we have the feasibility equation: $A_t^f = A_{t-1}^{RD}$ included, they don't seem to be either choosing it, or buying it. It is not in the list of their choice variables!

2. The R&D firm seems to be using it too. Since it appears both on the RHS of the firm's constraint, and on both sides of the R&D firms constraint!

3. Doesn't it follow that the R&D firm isn't getting any net revenue or something?

4. Does the R&D firm 'own' A_0^{RD}? That's what it looks like. This is not the usual way to model factors. We could probably give the households ownership of A_0^{RD} and have them sell it once and for all to the RD firms in period 0, etc etc.

5. There is no 'source' for revenue for the R&D firm!

Because of this:

Claim 1 *There is no equilibrium of this form.*

Formally: Assume that $AF(k, n)$ is CRS in (k, n). (I guess this would be the idea...) Then, it follows as usual that in any equilibrium, $\Pi^f = 0$.

Then, it follows that in any equilibrium if one should exist, that

$p_t(c_t^f + x_t^f) = r_t k_t^f + w_t n_t^f.$

¿From feasibility, we have that $c_t^f = c_t$, $x_t^f = x_t$, $k_t = k_t^f + k_t^{RD}$ and $n_t = n_t^f + n_t^{RD}$, and hence, using the HH budget constraint at equality (from monotonicity of preferences), we get that:

$\sum_{t=0}^{\infty} p_t [c_t + x_t] = \sum_{t=0}^{\infty} [r_t k_t + w_t n_t] + \Pi$

$\sum_{t=0}^{\infty} p_t \left[c_t^f + x_t^f\right] = \sum_{t=0}^{\infty} \left[r_t k_t^f + w_t n_t^f\right] + \sum_{t=0}^{\infty} \left[r_t k_t^{RD} + w_t n_t^{RD}\right] + \Pi$

$0 = \sum_{t=0}^{\infty} \left[r_t k_t^{RD} + w_t n_t^{RD}\right] + \Pi^f + \Pi^{RD}$

$0 = \sum_{t=0}^{\infty} \left[r_t k_t^{RD} + w_t n_t^{RD}\right] + \Pi^{RD}.$

But, $\sum_{t=0}^{\infty} \left[r_t k_t^{RD} + w_t n_t^{RD}\right] > 0$ and hence $\Pi^{RD} < 0$ a contradiction of profit maximization on the part of the R&D firm. We are assuming that the RD firm can get non-negative profits by setting $k_t^{RD} = n_t^{RD} = 0$ for all t.

Thus, there can be no equilibrium of this sort.

We assumed:

1) U is strictly monotone

2) $AF(k, n)$ is CRS in (k, n).

3) $\Pi^{RD}(0, 0) \geq 0$.

Basic Problem Was: No one was paying for RD.

5.2 Try # 2: Making f pay for A^f:

An equilibrium is:

a sequence of prices: $\{(p_t, r_t, w_t, r_t^A)\}_{t=0}^{\infty}$

Quantity decisions for the households: $\{(c_t, k_t, x_t, \ell_t, n_t)\}_{t=0}^{\infty} = z^{HH}$

Quantity decisions for the output firms: $\{(c_t^f, x_t^f, k_t^f, n_t^f, A_t^f)\}_{t=0}^{\infty} = z^f$

Quantity decisions for the R&D firms: $\{(A_t^{RD}, k_t^{RD}, n_t^{RD})\}_{t=0}^{\infty} = z^{RD}$

SUCH THAT:

1) z^{HH} is the solution to:

$Max_{\{(c_t,k_t,x_t,\ell_t,n_t)\}_{t=0}^{\infty}} U((c_t, \ell_t)_{t=0}^{\infty})$

subject to:

$\sum_{t=0}^{\infty} p_t \left[c_t + x_t\right] \leq \sum_{t=0}^{\infty} \left[r_t k_t + w_t n_t\right] + \Pi$

$k_{t+1} \leq (1-\delta)k_t + x_t$

$n_t + \ell_t \leq 1$

k_0 fixed.

2) z^f is the solution to:

$Max_{\{(c_t^f, x_t^f, k_t^f, n_t^f, A_t^f)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_t^f) - r_t k_t^f - w_t n_t^f - r_t^A A_t^f\right]$

subject to: $c_t^f + x_t^f \leq A_t^f F(k_t^f, n_t^f)$

3) z^{RD} is the solution to:

$Max_{\{(A_t^{RD}, k_t^{RD}, n_t^{RD})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[r_t^A A_t^{RD} - r_t k_t^{RD} - w_t n_t^{RD}\right]$

subject to: $A_t^{RD} \leq G(A_t^{RD}, k_t^{RD}, n_t^{RD})$

A_0^{RD} is fixed.

AND

$c_t = c_t^f$

$x_t = x_t^f$

$k_t = k_t^f + k_t^{RD}$

$n_t = n_t^f + n_t^{RD}$

$A_t^f = A_t^{RD}$

$\Pi = \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_t^f) - r_t k_t^f - w_t n_t^f - r_t^A A_t^f\right] + \sum_{t=0}^{\infty} \left[r_t^A A_t^{RD} - r_t k_t^{RD} - w_t n_t^{RD}\right]$

Problems:

1. This seems to have fixed some of the problems.... At least there is now a source of revenue for the R&D firm. But, A^{RD} still seems to be being used twice, once by the R&D firm, once by the output firm. Is this a problem?

2. $AF(k,n)$ is IRS in (A,k,n)

Claim 2 *There is no equilibrium of this form.*

Formally: Assume that $AF(k,n)$ is CRS in (k,n).

¿From the maximization problem of the firm:

i. Suppose that $p_\tau(c_\tau^f + x_\tau^f) > r_\tau k_\tau^f + w_\tau n_\tau^f$ for some τ.

Then notice that by changing plans to $(2c_\tau^f, 2x_\tau^f, 2k_\tau^f, 2n_\tau^f, A_\tau^f)$, leaving $(c_t^f, x_t^f, k_t^f, n_t^f, A_t^f)$ unchanged for other t's, profits are higher, a contradiction.

ii. Suppose that $p_\tau(c_\tau^f + x_\tau^f) < r_\tau k_\tau^f + w_\tau n_\tau^f$ for some τ.

Then notice that by changing plans to $(0,0,0,0,A_\tau^f)$, in period τ leaving $(c_t^f, x_t^f, k_t^f, n_t^f, A_t^f)$ unchanged for other t's, profits are higher, a contradiction. (This uses the assumption that $F(0,0) = 0$.)

iii. Suppose that $p_\tau(c_\tau^f + x_\tau^f) = r_\tau k_\tau^f + w_\tau n_\tau^f$ for all τ.

Then,

$$\Pi^f = \sum_{t=0}^{\infty}\left[p_t(c_t^f + x_t^f) - r_t k_t^f - w_t n_t^f - r_t^A A_t^f\right] = \sum_{t=0}^{\infty} r_t^A A_t^f < 0.$$

Thus, again, there can be no equilibrium of this sort.

This came from the fact that the true production function of the firm is now IRS, the constraint is:

$y_t^f \leq A_t^f F(k_t^f, n_t^f)$, which is strictly IRS in (A,k,n).

We assumed:

1) $AF(k,n)$ is CRS in (k,n).

That's it?

5.3 Try 3: Making F and G CRS:

Getting rid of the IRS in the output sector (and also the A sector):

Can we fix this by changing the form of the production function for the output sector? I.e., suppose that the constraint is of the form $y_t^f \leq F(A_t^f, k_t^f, n_t^f)$ with $F(A, k, n)$ CRS in the three factors.

An equilibrium is:

a sequence of prices: $\{(p_t, r_t, w_t, r_t^A)\}_{t=0}^{\infty}$

Quantity decisions for the households: $\{(c_t, k_t, x_t, \ell_t, n_t)\}_{t=0}^{\infty} = z^{HH}$

Quantity decisions for the output firms: $\{(c_t^f, x_t^f, k_t^f, n_t^f, A_t^f)\}_{t=0}^{\infty} = z^f$

Quantity decisions for the R&D firms: $\{(A_t^{RD}, k_t^{RD}, n_t^{RD})\}_{t=0}^{\infty} = z^{RD}$

SUCH THAT:

1) z^{HH} is the solution to:

$Max_{\{(c_t,k_t,x_t,\ell_t,n_t)\}_{t=0}^{\infty}} U((c_t, \ell_t)_{t=0}^{\infty})$

subject to:

$\sum_{t=0}^{\infty} p_t [c_t + x_t] \leq \sum_{t=0}^{\infty} [r_t k_t + w_t n_t] + \Pi$

$k_{t+1} \leq (1 - \delta)k_t + x_t$

$n_t + \ell_t \leq 1$

k_0 fixed.

2) z^f is the solution to:

$Max_{\{(c_t^f,x_t^f,k_t^f,n_t^f,A_t^f)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_t^f) - r_t k_t^f - w_t n_t^f - p_t^A A_t^f\right]$

subject to: $c_t^f + x_t^f \leq F(A_t^f, k_t^f, n_t^f)$

3) z^{RD} is the solution to:

$Max_{\{(A_t^{RD},k_t^{RD},n_t^{RD})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[r_t^A A_t^{RD} - r_t k_t^{RD} - w_t n_t^{RD}\right]$

subject to: $A_{t+1}^{RD} \leq G(A_t^{RD}, k_t^{RD}, n_t^{RD})$

A_0^{RD} is fixed.

AND

$c_t = c_t^f$

$x_t = x_t^f$

$k_t = k_t^f + k_t^{RD}$

$n_t = n_t^f + n_t^{RD}$

$A_t^f = A_t^{RD}$

$\Pi = \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_t^f) - r_t k_t^f - w_t n_t^f - r_t^A A_t^f\right] + \sum_{t=0}^{\infty} \left[r_t^A A_t^{RD} - r_t k_t^{RD} - w_t n_t^{RD}\right]$

Problems:

1. Still, it seems like A_t^{RD} is getting used 'twice' once by the output firm, once by the R&D firm. It looks to me like it is internally consistent, in that an equilibrium might exist, but since A_t^{RD} is a public good between the y firm and the RD firms, there is no guarantee that the equilibrium will be efficient. Try this... see what goes wrong with the First Welfare Theorem if you try and prove it! *****

***** Pricila conjectured that this IS efficient? Is she right? ******

***** Roozbeh conjectured that there is no equilibrium. Is he Right? ******

Is this correct? Is this model equivalent to one in which the R&D firm has a convex technology but produces two outputs, (A^f, A^{RD}) and these are perfect complements in output? I.e., the constraints are:

$\max(A_{t+1}^f, A_{t+1}^{RD}) \leq G(A_t^{RD}, k_t^{RD}, n_t^{RD})$????

And feasibility is still $A_{t+1}^f = A_{t+1}^{RD}$?

2. Because of this, the equilibrium outcome will depend on the industrial structure too. I.e., it will be different if the R&D firm is merged with the y firm or if it is separate?

3. How would we adapt this to multiple firms in each sector?

5.4 Try #4: Merging the Firms

****** Pricila conjectured that this is the same as Try #3, is she right? Francesca also seemed to think that this was true in a conversation after class. *****

Just like Try #3, except the two firms are merged into one with the aggregate feasibilty constraint on A no longer applicable, but having it appear in the constraints of the firm instead:

R&D inside the output firms:

An equilibrium is:

a sequence of prices: $\{(p_t, r_t, w_t)\}_{t=0}^{\infty}$

Quantity decisions for the households: $\{(c_t, k_t, x_t, \ell_t, n_t)\}_{t=0}^{\infty} = z^{HH}$

Quantity decisions for the output firms: $\{(c_t^f, x_t^f, k_t^f, k_t^{RD}, n_t^f, n_t^{RD}, A_t^f)\}_{t=0}^{\infty} = z^f$

SUCH THAT:

1) z^{HH} is the solution to:

$Max_{\{(c_t,k_t,x_t,\ell_t,n_t)\}_{t=0}^{\infty}} U((c_t, \ell_t)_{t=0}^{\infty})$

subject to:

$\sum_{t=0}^{\infty} p_t [c_t + x_t] \leq \sum_{t=0}^{\infty} [r_t k_t + w_t n_t] + \Pi$

$k_{t+1} \leq (1-\delta)k_t + x_t$

$n_t + \ell_t \leq 1$

k_0 fixed.

2) z^f is the solution to:

$Max_{\{(c_t^f,x_t^f,k_t^f,k_t^{RD},n_t^f,n_t^{RD},A_t^f)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_t^f) - r_t(k_t^f + k_f^{RD}) - w_t(n_t^f + n_t^{RD})\right]$

subject to: $c_t^f + x_t^f \leq F(A_t^f, k_t^f, n_t^f)$, and,

$A_{t+1}^{RD} \leq G(A_t^{rd}, k_t^{RD}, n_t^{RD})$

$A_t^f + A_t^{rd} \leq A_t^{RD}$

A_0^{RD} is fixed.

AND

$c_t = c_t^f$

$x_t = x_t^f$

$k_t = k_t^f + k_t^{RD}$

$n_t = n_t^f + n_t^{RD}$

$\Pi = \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_t^f) - r_t(k_t^f + k_t^{RD}) - w_t(n_t^f + n_t^{RD}) \right] +$

This is an internally consistent model as long as the standard assumptions are satisified. In particular, this requires that $F(A, k, n)$ and $G(A, k, n)$ are CRS, concave, etc., etc.

Problems:

1. We still have the bounded output problem that we identified in Try #4.

2. What if there is more than one firm? See Try #6 below.

5.5 Try # 5: Making A purely private

An equilibrium is:

a sequence of prices: $\{(p_t, r_t, w_t, r_t^A)\}_{t=0}^{\infty}$

Quantity decisions for the households: $\{(c_t, k_t, x_t, \ell_t, n_t)\}_{t=0}^{\infty} = z^{HH}$

Quantity decisions for the output firms: $\{(c_t^f, x_t^f, k_t^f, n_t^f, A_t^f)\}_{t=0}^{\infty} = z^f$

Quantity decisions for the R&D firms: $\{(A_t^{RD}, k_t^{RD}, n_t^{RD})\}_{t=0}^{\infty} = z^{RD}$

SUCH THAT:

1) z^{HH} is the solution to:

$Max_{\{(c_t, k_t, x_t, \ell_t, n_t)\}_{t=0}^{\infty}} U((c_t, \ell_t)_{t=0}^{\infty})$

subject to:

$\sum_{t=0}^{\infty} p_t\left[c_t + x_t\right] \le \sum_{t=0}^{\infty}\left[r_t k_t + w_t n_t\right] + \Pi$

$k_{t+1} \le (1-\delta)k_t + x_t$

$n_t + \ell_t \le 1$

k_0 fixed.

2) z^f is the solution to:

$Max_{\{(c_t^f, x_t^f, k_t^f, n_t^f, A_t^f)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\left[p_t(c_t^f + x_t^f) - r_t k_t^f - w_t n_t^f - p_t^A A_t^f\right]$

subject to: $c_t^f + x_t^f \le F(A_t^f, k_t^f, n_t^f)$

3) z^{RD} is the solution to:

$Max_{\{(A_t^{RD}, A_t^{rd}, k_t^{RD}, n_t^{RD})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\left[r_t^A A_t^{RD} - r_t k_t^{RD} - w_t n_t^{RD} - r_t^A A_t^{rd}\right]$

subject to: $A_{t+1}^{RD} \le G(A_t^{rd}, k_t^{RD}, n_t^{RD})$

A_0^{RD} is fixed.

AND

$c_t = c_t^f$

$x_t = x_t^f$

$k_t = k_t^f + k_t^{RD}$

$n_t = n_t^f + n_t^{RD}$

$A_t^f + A_t^{rd} = A_t^{RD}$

$\Pi = \sum_{t=0}^{\infty}\left[p_t(c_t^f + x_t^f) - r_t k_t^f - w_t n_t^f - r_t^A A_t^f\right] + \sum_{t=0}^{\infty}\left[r_t^A A_t^{RD} - r_t k_t^{RD} - w_t n_t^{RD} - r_t^A A_t^{rd}\right]$

IF both F and G satisfy the standard requirements, this looks like a standard competitive model. So, this would be one way to do it. (I think.) Other Alternatives are Below!

This looks a lot like that $A(k,h)$ model, where the role of h is played here by A^{RD}, the role of k is played by k and the role of A is played by B, where $G(A,k,n) = (1-\delta_A)A + BA^{\eta_A}k^{\eta_k}n^{\eta_n}$ and $\eta_A + \eta_k + \eta_n = 1$. See below.

5.6 Try Number 6

What happens with multiple firms in Try Number 5?

R&D inside the output firms:

An equilibrium is:

a sequence of prices: $\{(p_t, r_t, w_t)\}_{t=0}^{\infty}$

Quantity decisions for the households: $\{(c_t, k_t, x_t, \ell_t, n_t)\}_{t=0}^{\infty} = z^{HH}$

Quantity decisions for the output firms: $\{(c_t^{jf}, x_t^{jf}, k_t^{jf}, k_t^{jRD}, n_t^{jf}, n_t^{jRD}, A_t^{jf})\}_{t=0}^{\infty} = z^{jf}$, $j = 1, ..., J$

SUCH THAT:

1) z^{HH} is the solution to:

$Max_{\{(c_t,k_t,x_t,\ell_t,n_t)\}_{t=0}^{\infty}} U((c_t, \ell_t)_{t=0}^{\infty})$

subject to:

$\sum_{t=0}^{\infty} p_t [c_t + x_t] \leq \sum_{t=0}^{\infty} [r_t k_t + w_t n_t] + \Pi$

$k_{t+1} \leq (1 - \delta) k_t + x_t$

$n_t + \ell_t \leq 1$

k_0 fixed.

2) z^f is the solution to:

$$Max_{\{(c_t^{jf}, x_t^{jf}, k_t^{jf}, k_t^{jRD}, n_t^{jf}, n_t^{jRD}, A_t^{jf}, A_t^{jrd} A_t^{jRD})\}_{t=0}^{\infty}}$$

$$\sum_{t=0}^{\infty} \left[p_t(c_t^{jf} + x_t^{jf}) - r_t(k_t^{jf} + k_f^{jRD}) - w_t(n_t^{jf} + n_t^{jRD})\right]$$

subject to: $c_t^{jf} + x_t^{jf} \leq F(A_t^{jf}, k_t^{jf}, n_t^{jf})$, and,

$$A_{t+1}^{jRD} \leq G(A_t^{jrd}, k_t^{jRD}, n_t^{jRD})$$

$$A_t^{jf} + A_t^{jrd} \leq A_t^{jRD}$$

A_0^{jRD} is fixed.

AND

$c_t = \sum_j c_t^{jf}$

$x_t = \sum_j x_t^{jf}$

$k_t = \sum_j \left[k_t^{jf} + k_t^{jRD}\right]$

$n_t = \sum_j \left[n_t^{jf} + n_t^{jRD}\right]$

$\Pi = \sum_j \Pi^j$

$\Pi^j = \sum_{t=0}^{\infty} \left[p_t(c_t^{jf} + x_t^{jf}) - r_t(k_t^{jf} + k_t^{jRD}) - w_t(n_t^{jf} + n_t^{jRD})\right]$

So, in this formulation, there is no trade in knowledge across firms. It is purely 'knowledge in a firm.'

How about if we change it to:

a sequence of prices: $\{(p_t, r_t, w_t, r_t^A)\}_{t=0}^{\infty}$

2) z^f is the solution to:

$$Max_{\{(c_t^{jf}, x_t^{jf}, k_t^{jf}, k_t^{jRD}, n_t^{jf}, n_t^{jRD}, A_t^{jf}, A_t^{jrd}, A_t^{jRD})\}_{t=0}^{\infty}}$$

$$\sum_{t=0}^{\infty} \left[p_t(c_t^{jf} + x_t^{jf}) - r_t(k_t^{jf} + k_f^{jRD}) - w_t(n_t^{jf} + n_t^{jRD}) - r_t^A(A_t^{jf} + A_t^{jrd})\right]$$

subject to: $c_t^{jf} + x_t^{jf} \leq F(A_t^{jf}, k_t^{jf}, n_t^{jf})$, and,

$$A_{t+1}^{jRD} \leq G(A_t^{jf}, k_t^{jRD}, n_t^{jRD})$$

A_0^{jRD} is fixed.

And in feasibility make it:

$$\sum_j (A_t^{jf} + A_t^{jrd}) = \sum_j A_t^{jRD} \text{ for all } t.$$

Both of these are internally consistent models as long as the standard assumptions are satisified (I think!!!). In particular, this requires that $F(A, k, n)$ and $G(A, k, n)$ are CRS, concave, etc., etc.

Issues:

1. Boundedness again.

2. Is R&D private or public? We'll return to this below.

5.7 A General Problem with Feasibility

This section discusses a General Problem arising with making $F(A, k, n)$ and $G(A, k, n)$ CRS in (A, k, n). This is the problem that, if the technology does NOT exhibit

'LINEARITY IN THE REPRODUCIBLE FACTORS'

Then output is bounded along any feasible sequence.

To see this:

Suppose that $F(A, k, n) = A^{\alpha_A} k^{\alpha_k} n^{\alpha_n}$ with $\alpha_A + \alpha_k + \alpha_n = 1$, $\alpha_i > 0$ for $i = A, k, n$, and that $G(A, k, n) = (1 - \delta_A)A + BA^{\eta_A} k^{\eta_k} n^{\eta_n}$ with $\eta_A + \eta_k + \eta\alpha_n = 1$, $\eta_i > 0$ for $i = A, k, n$,

Claim 3 *Output is bounded along any feasible sequence.*

To see this, simply note that $y_t = F(A_t^f, k_t^f, n_t^f) \leq F(A_t^f, k_t^f, 1) = A_t^{f\alpha_A} k_t^{f\alpha_k}$ with $\alpha_A + \alpha_k < 1$. Note that $\delta k \to \infty$ if $k \to \infty$

I'll show this assuming $\delta_k = \delta_A = 1$. It's easy in the one sector model? Why is it so hard here?

By successive substitution, note that we have

$A_{t+1} \leq BA_t^{\eta_A} k_t^{\eta_k} \leq BA_t^{\eta_A} [A_{t-1}^{\alpha_A} k_{t-1}^{\alpha_k}]^{\eta_k} = BA_t^{\eta_A} A_{t-1}^{\alpha_A \eta_k} k_{t-1}^{\alpha_k \eta_k}$

$\leq BA_t^{\eta_A} A_{t-1}^{\alpha_A \eta_k} [A_{t-2}^{\alpha_A} k_{t-2}^{\alpha_k}]^{\alpha_k \eta_k} = BA_t^{\eta_A} A_{t-1}^{\alpha_A \eta_k} A_{t-2}^{\alpha_A \alpha_k \eta_k} k_{t-2}^{\alpha_k^2 \eta_k} \leq ...$

Let $\bar{A}_t = \max_{0 \leq s \leq t} A_s$.

Then, we have

$A_{t+1} \leq \bar{A}_t^{\eta_A} \bar{A}_{t-1}^{\alpha_A \eta_k} \bar{A}_{t-2}^{\alpha_A \alpha_k \eta_k} k_{t-2}^{\alpha_k^2 \eta_k} \leq ...$

$\leq k_0 \Pi_{s=0}^{t} A_{t-s}^{\eta_A \nu^s}$

where $\nu = \max(\alpha_k, \eta_k)$.

*********This looks wrong! Why does η_A appear in all of them???*****

This assumes that $\bar{A}_t \geq 1$. We've also assumed that $k_0 \geq 1$, since the term that would appear in the inequality is $k_0^{\alpha_k^{t/2} \eta_k^{t/2}}$ if t is even, etc. If either of these does not hold, they can be replaced by 1 in the inequalities and we can go on from there.

Then,

$A_{t+1} \leq k_0 \Pi_{s=0}^{t} A_{t-s}^{\eta_A \nu^s} \leq k_0 \Pi_{s=0}^{t} \bar{A}_t^{\eta_A \nu^s} \leq k_0 \bar{A}_t^{\eta_A/(1-\nu)}$.

Notice that by construction, $1 - \nu = \min(1 - \alpha_k, 1 - \eta_k) \geq 1 - \alpha_k > \alpha_A$ since $\alpha_k + \alpha_A < 1$.

Thus, $\alpha_A/(1 - \nu) < 1$, and because of this, it follows that for large enough A, $k_0 A^{\eta_A/(1-\nu)} < A$.

Now suppose that $A_t \to \infty$, moreover, assume that this is monotone (this keeps us from having to carefully construct a subsequence which is minimally monotonically increasing below)– $A_{t+1} > A_t$ for all t.

Then

$A_{t+1} \leq k_0 \bar{A}_t^{\eta_A/(1-\nu)} = k_0 A_t^{\eta_A/(1-\nu)} \leq A_t$ a contradiction. This completes the proof of the claim.

What about if $\delta < 1$????? ***Majorize the functions by increasing the exponents perhaps?****

This is simple to see in the one sector model— INTUITION and GRAPH!!!!

Again, we've run into a fundamental Problem with the approach. And it seems to depend on almost nothing, only that there is the strictest sort of diminishing returns in the reproducible factors. Note that it goes away if $\eta_n = 0$.

IN THE ONE SECTOR MODEL:

Here the constraint is:

$c_t + x_{kt} + x_{At} \leq F(A_t, k_t, n_t) = B A_t^{\alpha_A} k_t^{\alpha_k} n_t^{\alpha_n}$ with $\alpha_A + \alpha_k + \alpha_n = 1$, all positive. With laws of motion given by:

$k_{t+1} \leq (1-\delta)k_t + x_{kt}$

$A_{t+1} \leq (1-\delta_A)A_t + x_{At}$

In this case,

$k_{t+1} - k_t \leq BA_t^{\alpha_A} k_t^{\alpha_k} - \delta k_t$

$A_{t+1} - A_t \leq BA_t^{\alpha_A} k_t^{\alpha_k} - \delta_A A_t$

Can show that $BA_t^{\alpha_A} k_t^{\alpha_k}$ is maximized at $A_t = \phi k_t$. That is, the solution to the problem:

$$Max_{\{A,k\}} BA^{\alpha_A} k^{\alpha_k} \qquad s.t. \qquad A + k \leq z$$

has $\frac{A}{k} = \phi$, where ϕ depends on $\alpha_A, \alpha_k, \delta$ and δ_A.

In this case,

$$BA_t^{\alpha_A} k_t^{\alpha_k} \leq B\left(\phi k_t\right)^{\alpha_A} k_t^{\alpha_k} = B\phi^{\alpha_k} k_t^{\alpha_A+\alpha_k},$$

where, by assumption, $\alpha_A + \alpha_k = 1 - \alpha_n < 1$.

Thus, from above, we have that

$k_{t+1} - k_t \leq BA_t^{\alpha_A} k_t^{\alpha_k} - \delta k_t \leq B\phi^{\alpha_k} k_t^{\alpha_A+\alpha_k} - \delta k_t.$

But, since $\alpha_A + \alpha_k < 1$, it follows that for large k,

$B\phi^{\alpha_k} k_t^{\alpha_A+\alpha_k} < \delta k_t$ and hence, $k_{t+1} < k_t$.

A similar arguement holds for A if $\delta_A > 0$. Thus, if both $\delta > 0$ and $\delta_A > 0$, output is bounded.

If $\delta_A = 0$, the argument is more subtle, since it is, in this case FEASIBLE to have $A_t \to \infty$, and hence, have $y_t \to \infty$, but it can be shown that it is never optimal to do this. (Right????)

*****Need to show that $\phi < \infty$, is this true even if $\delta_A = 0$????

This is NOT a problem if:

$\alpha_n = \eta_n = 0$ (maybe you only need $\eta_n = 0$?)

However, in this case, $w_t = 0$, and hence $w_t n_t = 0$ and hence, $w_t n_t / y_t = 0$, not the .67 or so that we observe in the data.

One 'out' of this is the $A(k,h)$ model, where 'labor supply' is a mix of human capital and hours. In this case, each 'R&D firm' is a household where the output is privately owned human capital. Thus, A in all of the notes above corresponds to h in the $A(k,h)$ model.

Even in this case, there are some issues about how you model things.

If $z = nh$, this is IRS in (n,h), although this is at the level of the individual household. Some non-convexities at the 'micro-level' are okay– e.g., indivisibilities in the purchase of cars, etc. ****LOTTERIES????*****

Another issue that arises is the form that preferences/utility should take in this case. Should it be:

$U(c,\ell) = \sum_t \beta^t u(c_t,\ell_t) = \sum_t \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} v(\ell)$?

Or, should it be:

$U(c,\ell,h) = \sum_t \beta^t u(c_t,\ell_t h_t) = \sum_t \beta^t \frac{1}{1-\sigma} \left[v(c_t,\ell_t h_t)\right]^{1-\sigma}$

where v is homothetic – $v(c_t,\ell_t h_t) = \left[\theta c_t^\rho + (1-\theta)(\ell_t h_t)^\rho\right]^{1/\rho}$?

The second turns it into a concave problem.

******WRITE OUT THE HH PROBLEM HERE******

6 Alternatives:

What did we learn from our series of False Starts?

Factors to consider:

1. Is it an internally consistent specification of equilibrium? (I.e., does an equilibrium even exist?)

2. Will it Grow? (This requires linearity in the reproducible factors.)

3. How sensitive is it to the specification of Industrial Structure? If it is VERY sensitive, what is the 'right' Industrial Structure to use?

4. What is the source of differences across countries?

5. Are there any serious couter-factual implications? E.g., Interest rate facts.

6. What is measured TFP?

Approaches that will work/ have been used/ have some shot at success:

1. Drop Muliplicative A, i.e., Ak and $A(k,h)$ models.

a) country differences?
b) still need linearity in the reproducible factors or else there is no growth.

2. IRS with Planner's Problem as the 'positive theory'

3. CRS at the individual level, but IRS at the aggregate level through external effects, knowledge spillovers.

4. IRS at the individual level, but monopoly power/ monopolistic competition.

IRS, how it's handled, and 'size' of a country?

We'll say a bit about each of these.

Planner's Problem
a) country differences?
b) still need linearity in the reproducible factors or else there is no growth.
c) Decentralization?

External Effects

a) country differences?
b) still need linearity in the reproducible factors or else there is no growth.
c) How exactly? Industrial Structure?

Monopolistic Competition

a) country differences?
b) still need linearity in the reproducible factors or else there is no growth.
c) How exactly? Industrial Structure?

7 Convex Models of Endogenous Growth: The Ak Model

In this version of the models, we identify A from the previous discussion with k in the math. I.e., $k = knowledge$.

Here the equilibrium version of the model is:

An equilibrium is:

a sequence of prices: $\{(p_t, r_t, w_t)\}_{t=0}^{\infty}$

Quantity decisions for the households: $\{(c_t, k_t, x_t)\}_{t=0}^{\infty} = z^{HH}$

Quantity decisions for the output firms: $\{(c_t^f, x_t^f, k_t^f)\}_{t=0}^{\infty} = z^f$,

SUCH THAT:

1) z^{HH} is the solution to:

$Max_{\{(c_t,k_t,x_t)\}_{t=0}^{\infty}} U((c_t)_{t=0}^{\infty})$

subject to:

$\sum_{t=0}^{\infty} p_t [c_t + x_t] \leq \sum_{t=0}^{\infty} [r_t k_t] + \Pi$

$k_{t+1} \leq (1-\delta)k_t + x_t$

k_0 fixed.

2) z^f is the solution to:

$Max_{\{(c_t^f,x_t^f,k_t^f)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_t^f) - r_t k_t^f\right]$

subject to: $c_t^f + x_t^f \leq Ak_t^f$.

AND

$c_t = c_t^f$

$x_t = x_t^f$

$k_t = k_t^f$

$\Pi = \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_t^f) - r_t k_t^f \right]$

Since this is a standard convex model with a representative household and a representative firm, the equilibrium is unique (if it exists *****What do you need for this?****) and will solve the planners problem:

$Max_{\{(c_t,k_t,x_t)\}_{t=0}^{\infty}} U((c_t)_{t=0}^{\infty})$

subject to:

$[c_t + x_t] \le Ak_t$

$k_{t+1} \le (1-\delta)k_t + x_t$

k_0 fixed.

Assuming that $U(c) = \sum_t \beta^t u(c_t)$, we have:

$p_t = \beta^t u'(c_t)/\beta^0 u'(c_0)$, and $r_t/p_t = A$ for all t.

As you have seen in the past, and assuming that $u(c) = c^{1-\sigma}/(1-\sigma)$, this model features constant growth at the rate γ where

$$\gamma = [\beta(1-\delta+A)]^{1/\sigma}.$$

7.0.1 Differences in k_0^i in the Ak Model

1. What is $\frac{y_t^i}{y_t^j}$?

$$\frac{y_t^i}{y_t^j} = \frac{Ak_t^i}{Ak_t^j} = \frac{A\gamma^t k_0^i}{A\gamma^t k_0^j} = \frac{k_0^i}{k_0^j}$$

Thus, it's possible to have as wide a dispersion in y as wanted, but it must be exactly matched by an equal dispersion (in relative terms) in $k's$ in any period.

2. Relationship between y_t^i and future γ? The growth rate does not depend on initial conditions. So, if differences in levels in period t are due to differences in $k_t's$ then the future growth rate will not depend on the period t level. Thus, this relationship should be flat.

3. Interest rates on loans (e.g., consumer borrowing) would be given by

$$1 + R_t = 1 - \delta + r_t = 1 - \delta + A$$

This does not depend on either initial conditions, or the current level of k_t. So, the observed interest rates should be the same for all observations.

4. What is measured TFP?

$$\log(TFP_t^i) = \log(y_t^i) - .33\log(k_t^i) - .67\log(0) = -\infty$$

Not a very interesting answer! (This is just because we have assumed that labor doesn't enter the production function, we will get a more interesting answer when we look at the $A(k,h)$ model below.

7.0.2 Ak: Initial Conditions

	Data	Model
$\gamma =?$	1.02 varied	$\gamma^\sigma = \beta\left[1 - \delta_k + A\right]$
$\frac{y_t^i}{y_t^j} =?$	1 to 50	$\frac{y_t^i}{y_t^j} = \frac{k_0^i}{k_0^j}$
$\frac{wn}{y} =?$	≈ 0.67?	not defined
$TFP_t^i =?$	varied	not defined
y_{1960}^i vs. $\bar{\gamma}_{60-95} =?$	no relationship	no relationship
y_{1960}^i vs. $R_t^i =?$	no relationship? (risk)	no relationship
$\frac{x}{y}$ vs. $y =?$	$corr(\frac{x}{y}, y) > 0$?	no relationship
$\frac{x}{y}$ vs. $\gamma =?$	$corr(\frac{x}{y}, \gamma) > 0$?	all countries have the same $\frac{x}{y}$ and γ
$Educ.$ vs. $y =?$	$corr(Ed, y) > 0$?	not defined
γ_n vs. $y =?$	$corr(\gamma_n, y) < 0$?	not defined
γ_n vs. $\gamma =?$	$corr(\gamma_n, \gamma) < 0$?	not defined

7.0.3 Differences in A in the Ak Model

1. What is $\frac{y_t^i}{y_t^j}$?

$$\frac{y_t^i}{y_t^j} = \frac{A^i k_t^i}{A^j k_t^j} = \frac{A^i \gamma_i^t k_0^i}{A^j \gamma_j^t k_0^j} = \frac{A^i \gamma_i^t}{A^j \gamma_j^t}$$

Where γ_i solves:

$$(\gamma_i)^\sigma = \beta\left[1 - \delta_k + A^i\right]$$

Thus, countries with larger $A^{i\prime}s$ will have larger $y^{i\prime}s$ and this difference will grow larger over time.

Thus, it's possible to have as wide a dispersion in y as wanted, and this is even true without any initial differences in $k's$, but it must be almost (because of the A^i/A^j term) exactly matched by an equal dispersion (in relative terms) in $k's$ in any period. Moreover, this dispersion grows over time.

2. Relationship between y_t^i and future γ? Countries with high y^i are those countries with high A^i. These countries also have higher γ_i and so there should be an increasing relationship between y_t^i and γ.

3. Relationship between y_t^i and interest rates? Interest rates on loans (e.g., consumer borrowing) would be given by

$$1 + R_t^i = 1 - \delta + r_t^i = 1 - \delta + A^i$$

Countries with high y^i are those countries with high A^i. These countries also have higher R^i and so there should be an increasing relationship between y_t^i and R^i.

4. What is measured TFP?

$$\log(TFP_t^i) = \log(y_t^i) - .33\log(k_t^i) - .67\log(0) = -\infty$$

Not a very interesting answer! (This is just because we have assumed that labor doesn't enter the production function, we will get a more interesting answer when we look at the $A(k,h)$ model below.

7.0.4 Ak: A^i Differences

	Data	Model
$\gamma = ?$	1.02 varied	$\gamma^{i\sigma} = \beta \left[1 - \delta_k + A^i\right]$
$\frac{y_t^i}{y_t^j} = ?$	1 to 50	$\frac{y_t^i}{y_t^j} = \frac{\gamma_i^t}{\gamma_j^t} \frac{y_t^i}{y_t^j} \to \infty$
$\frac{wn}{y} = ?$	$\approx 0.67?$	not defined
$TFP_t^i = ?$	varied	not defined
y_{1960}^i vs. $\bar{\gamma}_{60-95} = ?$	no relationship	high $y_{60}^i \Longrightarrow$ high $\bar{\gamma}_{60-05}$
y_{1960}^i vs. $R_t^i = ?$	no relationship? (risk)	high $y_{60}^i \Longrightarrow$ high R_t^i
$\frac{x}{y}$ vs. $y = ?$	$corr(\frac{x}{y}, y) > 0?$	$corr(\frac{x}{y}, y) > 0$
$\frac{x}{y}$ vs. $\gamma = ?$	$corr(\frac{x}{y}, \gamma) > 0?$	$corr(\frac{x}{y}, \gamma) > 0$
$Educ.$ vs. $y = ?$	$corr(Ed, y) > 0?$	not defined
γ_n vs. $y = ?$	$corr(\gamma_n, y) < 0?$	not defined
γ_n vs. $\gamma = ?$	$corr(\gamma_n, \gamma) < 0?$	not defined

7.0.5 Differences in δ_k in the Ak Model

This is exactly the same as the differences in A section.

7.0.6 Ak: δ_k differences

	Data	Model
$\gamma = ?$	1.02 varied	$\gamma^{i\sigma} = \beta \left[1 - \delta_k^i + A\right]$
$\frac{y_t^i}{y_t^j} = ?$	1 to 50	$\frac{y_t^i}{y_t^j} = \frac{\gamma_i^t}{\gamma_j^t} \frac{y_t^i}{y_t^j} \to \infty$
$\frac{wn}{y} = ?$	$\approx 0.67?$	not defined
$TFP_t^i = ?$	varied	not defined
y_{1960}^i vs. $\bar{\gamma}_{60-95} = ?$	no relationship	high $y_{60}^i \Longrightarrow$ high $\bar{\gamma}_{60-05}$
y_{1960}^i vs. $R_t^i = ?$	no relationship? (risk)	high $y_{60}^i \Longrightarrow$ high R_t^i
$\frac{x}{y}$ vs. $y = ?$	$corr(\frac{x}{y}, y) > 0?$	$corr(\frac{x}{y}, y) > 0$
$\frac{x}{y}$ vs. $\gamma = ?$	$corr(\frac{x}{y}, \gamma) > 0?$	$corr(\frac{x}{y}, \gamma) > 0$
$Educ.$ vs. $y = ?$	$corr(Ed, y) > 0?$	not defined
γ_n vs. $y = ?$	$corr(\gamma_n, y) < 0?$	not defined
γ_n vs. $\gamma = ?$	$corr(\gamma_n, \gamma) < 0?$	not defined

7.0.7 Differences in β in the Ak Model

Countries with high $\beta's$ have higher $\gamma's$.

1. What is $\frac{y_t^i}{y_t^j}$?

$$\frac{y_t^i}{y_t^j} = \frac{Ak_t^i}{Ak_t^j} = \frac{A\gamma_i^t k_0^i}{A\gamma_j^t k_0^j} = \frac{A\gamma_i^t}{A\gamma_j^t}$$

Where γ_i solves:

$$(\gamma_i)^\sigma = \beta^i \left[1 - \delta_k + A\right]$$

Thus, countries with larger $\beta^{i\prime}s$ will have larger $y^{i\prime}s$ and this difference will grow larger over time.

Thus, it's possible to have as wide a dispersion in y as wanted, and this is even true without any initial differences in $k's$, but it must be exactly matched by an equal dispersion (in relative terms) in $k's$ in any period. Moreover, this dispersion grows over time.

2. Relationship between y_t^i and future γ? Countries with high y^i are those countries with high β^i. These countries also have higher γ_i and so there should be an increasing relationship between y_t^i and γ.

3. Relationship between y_t^i and interest rates? Interest rates on loans (e.g., consumer borrowing) would be given by

$$1 + R_t^i = 1 - \delta + r_t^i = 1 - \delta + A$$

Countries with high y^i are those countries with high β^i. But, there is no relationship between y_t^i and R^i.

4. What is measured TFP?

$$\log(TFP_t^i) = \log(y_t^i) - .33\log(k_t^i) - .67\log(0) = -\infty$$

Not a very interesting answer! (This is just because we have assumed that labor doesn't enter the production function, we will get a more interesting answer when we look at the $A(k,h)$ model below.

7.0.8 Ak: β Differences

	Data	Model
$\gamma =?$	1.02 varied	$\gamma^{i\sigma} = \beta^i [1 - \delta_k + A]$
$\frac{y_t^i}{y_t^j} =?$	1 to 50	$\frac{y_t^i}{y_t^j} = \frac{\gamma_i^t}{\gamma_j^t} \frac{y_t^i}{y_t^j} \to \infty$
$\frac{wn}{y} =?$	$\approx 0.67?$	not defined
$TFP_t^i =?$	varied	not defined
y_{1960}^i vs. $\bar{\gamma}_{60-95} =?$	no relationship	high $y_{60}^i \Longrightarrow$ high $\bar{\gamma}_{60-05}$
y_{1960}^i vs. $R_t^i =?$	no relationship? (risk)	high $y_{60}^i \Longrightarrow$ high R_t^i
$\frac{x}{y}$ vs. $y =?$	$corr(\frac{x}{y}, y) > 0?$	$corr(\frac{x}{y}, y) > 0$
$\frac{x}{y}$ vs. $\gamma =?$	$corr(\frac{x}{y}, \gamma) > 0?$	$corr(\frac{x}{y}, \gamma) > 0$
$Educ.$ vs. $y =?$	$corr(Ed, y) > 0?$	not defined
γ_n vs. $y =?$	$corr(\gamma_n, y) < 0?$	not defined
γ_n vs. $\gamma =?$	$corr(\gamma_n, \gamma) < 0?$	not defined

7.0.9 Differences in σ in the Ak Model

7.0.10 Ak: σ Differences

	Data	Model
$\gamma =?$	1.02 varied	$\gamma^{i\sigma_i} = \beta [1 - \delta_k + A]$
$\frac{y_t^i}{y_t^j} =?$	1 to 50	$\frac{y_t^i}{y_t^j} : \frac{\gamma_i^t}{\gamma_i^t} \frac{y_t^i}{y_t^j} \to \infty$
$\frac{wn}{y} =?$	$\approx 0.67?$	not defined
$TFP_t^i =?$	varied	not defined
y_{1960}^i vs. $\bar{\gamma}_{60-95} =?$	no relationship	high $y_{60}^i \Longrightarrow$ high $\bar{\gamma}_{60-05}$
y_{1960}^i vs. $R_t^i =?$	no relationship? (risk)	high $y_{60}^i \Longrightarrow$ high R_t^i
$\frac{x}{y}$ vs. $y =?$	$corr(\frac{x}{y}, y) > 0?$	$corr(\frac{x}{y}, y) > 0$
$\frac{x}{y}$ vs. $\gamma =?$	$corr(\frac{x}{y}, \gamma) > 0?$	$corr(\frac{x}{y}, \gamma) > 0$
$Educ.$ vs. $y =?$	$corr(Ed, y) > 0?$	not defined
γ_n vs. $y =?$	$corr(\gamma_n, y) < 0?$	not defined
γ_n vs. $\gamma =?$	$corr(\gamma_n, \gamma) < 0?$	not defined

7.1 Adding Policy in the Ak Model

New version of the model, TDCE only:

Here the equilibrium version of the model is:

An equilibrium is:

a sequence of prices: $\{(p_t, r_t, w_t)\}_{t=0}^{\infty}$

Quantity decisions for the households: $\{(c_t, k_t, x_t)\}_{t=0}^{\infty} = z^{HH}$

Quantity decisions for the output firms: $\{(c_t^f, x_t^f, k_t^f)\}_{t=0}^{\infty} = z^f$,

SUCH THAT:

1) z^{HH} is the solution to:

$Max_{\{(c_t,k_t,x_t)\}_{t=0}^{\infty}} U((c_t)_{t=0}^{\infty})$

subject to:

$\sum_{t=0}^{\infty} p_t [c_t + x_t] \leq \sum_{t=0}^{\infty} [(1-\tau_k) r_t k_t + T_t] + \Pi$

$k_{t+1} \leq (1-\delta)k_t + x_t$

k_0 fixed.

2) z^f is the solution to:

$Max_{\{(c_t^f,x_t^f,k_t^f)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_t^f) - r_t k_t^f\right]$

subject to: $c_t^f + x_t^f \leq A k_t^f$.

AND

$g_t + c_t = c_t^f$

$x_t = x_t^f$

$k_t = k_t^f$

$\Pi = \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_t^f) - r_t k_t^f\right]$

$\sum_t [p_t g_t + T_t] = \sum_t \tau_t r_t k_t$

NOTE: What if taxes were on UNDEPRECIATED capital instead? I.e., $Tax_t = \tau_t(r_t - \delta)k_t$?

Assuming that $U(c) = \sum_t \beta^t u(c_t)$, we have:

$p_t = \beta^t u'(c_t)/\beta^0 u'(c_0)$, and $r_t/p_t = A$ for all t.

As you have seen in the past, and assuming that $\tau_t = \tau$ for all t, this model features constant growth at the rate γ_τ where

$$\gamma_\tau = [\beta(1 - \delta + (1-\tau)A)]^{1/\sigma}.$$

7.1.1 Differences in τ in the Ak Model

Countries with high $\tau's$ have lower $\gamma's$.

1. What is $\frac{y_t^i}{y_t^j}$?

$$\frac{y_t^i}{y_t^j} = \frac{Ak_t^i}{Ak_t^j} = \frac{A\gamma_i^t k_0^i}{A\gamma_j^t k_0^j} = \frac{\gamma_i^t}{\gamma_j^t}$$

Where γ_i solves:

$$(\gamma_i)^\sigma = \beta\left[1 - \delta_k + (1-\tau_i)A\right]$$

Thus, countries with larger $\tau^{i\prime}s$ will have smaller $y^{i\prime}s$ and this difference will grow larger over time.

Thus, it's possible to have as wide a dispersion in y as wanted, and this is even true without any initial differences in $k's$, but it must be exactly matched by an equal dispersion (in relative terms) in $k's$ in any period. Moreover, this dispersion grows over time.

2. Relationship between y_t^i and future γ? Countries with high y^i are those countries with low τ^i. These countries also have higher γ_i and so there should be an increasing relationship between y_t^i and γ.

3. Relationship between y_t^i and interest rates? Interest rates on loans (e.g., consumer borrowing) would be given by

$$1 + R_t^i = 1 - \delta + r_t^i = 1 - \delta + A$$

or is this $1 - \delta + (1 - \tau^i)A$

******** I think that

$$100 * [(1 - \delta + A) - 1]$$

is the interest rate paid by firms on loans, but

$$100 * [(1 - \delta + (1 - \tau^i)A) - 1]$$

is the amount received (after tax) by consumers.

Countries with high y^i are those countries with low τ^i. There no relationship between y_t^i and and the interest rates paid by firms, but those countries with high y^i should be those countries with a low τ^i and hence a high R^i.

4. What is measured TFP?

$$\log(TFP_t^i) = \log(y_t^i) - .33 \log(k_t^i) - .67 \log(0) = -\infty$$

Not a very interesting answer! (This is just because we have assumed that labor doesn't enter the production function, we will get a more interesting answer when we look at the $A(k, h)$ model below.

7.1.2 Ak: Tax Policy Differences

	Data	Model
$\gamma = ?$	1.02 varied	$\gamma_i^\sigma = \beta\,[1 - \delta_k + (1 - \tau_i)A]$
$\frac{y_t^i}{y_t^j} = ?$	1 to 50	$\frac{y_t^i}{y_t^j} : \frac{\gamma_i^t}{\gamma_j^t}\frac{y_t^i}{y_t^j} \to \infty$
$\frac{wn}{y} = ?$	≈ 0.67?	not defined
$TFP_t^i = ?$	varied	not defined
y_{1960}^i vs. $\bar{\gamma}_{60-95} = ?$	no relationship	high $y_{60}^i \Longrightarrow$ high $\bar{\gamma}_{60-05}$
y_{1960}^i vs. $R_t^i = ?$	no relationship? (risk)	high $y_{60}^i \Longrightarrow$ high R_t^i
$\frac{x}{y}$ vs. $y = ?$	$corr(\frac{x}{y}, y) > 0$?	$corr(\frac{x}{y}, y) > 0$
$\frac{x}{y}$ vs. $\gamma = ?$	$corr(\frac{x}{y}, \gamma) > 0$?	$corr(\frac{x}{y}, \gamma) > 0$
$Educ.$ vs. $y = ?$	$corr(Ed, y) > 0$?	not defined
γ_n vs. $y = ?$	$corr(\gamma_n, y) < 0$?	not defined
γ_n vs. $\gamma = ?$	$corr(\gamma_n, \gamma) < 0$?	not defined

7.1.3 Ak: Government Spending Differences

	Data	Model
$\gamma = ?$	1.02 varied	
$\frac{y_t^i}{y_t^j} = ?$	1 to 50	
$\frac{wn}{y} = ?$	$\approx 0.67?$	
$TFP_t^i = ?$	varied	
y_{1960}^i vs. $\bar{\gamma}_{60-95} = ?$	no relationship	
y_{1960}^i vs. $R_t^i = ?$	no relationship? (risk)	
$\frac{x}{y}$ vs. $y = ?$	$corr(\frac{x}{y}, y) > 0?$	
$\frac{x}{y}$ vs. $\gamma = ?$	$corr(\frac{x}{y}, \gamma) > 0?$	
$Educ.$ vs. $y = ?$	$corr(Ed, y) > 0?$	
γ_n vs. $y = ?$	$corr(\gamma_n, y) < 0?$	
γ_n vs. $\gamma = ?$	$corr(\gamma_n, \gamma) < 0?$	

7.1.4 Differences in μ in the Ak Model

The rate of inflation is different. For concreteness, have 2 goods, one with a CIA constraint. Feasibility is:

$$c_{1t} + c_{2t} + x_{kt} \leq Ak_t$$

$$k_{t+1} \leq (1 - \delta_k)k_t + x_{kt}.$$

HH problem is:

$$\max \quad \sum_t \beta^t u(c_{1t}, c_{2t})$$

Where $u(c_{1t}, c_{2t}) = \{\theta c_{1t}^{\rho} + (1-\theta)c_{2t}^{\rho}\}^{1/\rho}$.

There is no relationship between $\gamma' s$ and μ^i.

1. What is $\frac{y_t^i}{y_t^j}$?

2. Relationship between y_t^i and future γ? none
3. Relationship between y_t^i and interest rates? none for real rates at least.

4. What is measured TFP?

$$\log(TFP_t^i) = \log(y_t^i) - .33\log(k_t^i) - .67\log(0) = -\infty$$

Not a very interesting answer! (This is just because we have assumed that labor doesn't enter the production function, we will get a more interesting answer when we look at the $A(k, h)$ model below.

7.1.5 Ak: Monetary Policy Differences

Cash/Credit model

	Data	Model
$\gamma =?$	1.02 varied	$\gamma_i^\sigma = \beta\left[1-\delta_k+(1-\tau)A\right]$
$\frac{y_t^i}{y_t^j} =?$	1 to 50	?
$\frac{wn}{y} =?$	$\approx 0.67?$	not defined
$TFP_t^i =?$	varied	not defined
y_{1960}^i vs. $\bar{\gamma}_{60-95} =?$	no relationship	?
y_{1960}^i vs. $R_t^i =?$	no relationship? (risk)	?
$\frac{x}{y}$ vs. $y =?$	$corr(\frac{x}{y}, y) > 0?$	?
$\frac{x}{y}$ vs. $\gamma =?$	$corr(\frac{x}{y}, \gamma) > 0?$	?
$Educ.$ vs. $y =?$	$corr(Ed, y) > 0?$	not defined
γ_n vs. $y =?$	$corr(\gamma_n, y) < 0?$	not defined
γ_n vs. $\gamma =?$	$corr(\gamma_n, \gamma) < 0?$	not defined

8 Convex Models of Endogenous Growth: The $A(k,h)$ Model

8.1 $A(k,h)$ Models, Inelastic Labor Supply

Here, the equilibrium version of the model is:

In this version of the models, we identify A from the previous discussion with k in the math. I.e., $k = knowledge.$

Here the equilibrium version of the model is:

An equilibrium is:

a sequence of prices: $\{(p_t, r_t, w_t)\}_{t=0}^{\infty}$

Quantity decisions for the households: $\{(c_t, k_t, h_t, x_{kt}, x_{ht}, z_t)\}_{t=0}^{\infty} = z^{HH}$

Quantity decisions for the output firms: $\{(c_t^f, x_{ht}^f, x_{kt}^f, k_t^f, z_t^f)\}_{t=0}^{\infty} = z^f$,

SUCH THAT:

1) z^{HH} is the solution to:

$Max_{\{(c_t,k_t,x_t)\}_{t=0}^{\infty}} U((c_t)_{t=0}^{\infty})$

subject to:

$\sum_{t=0}^{\infty} p_t [c_t + x_{kt} + x_{ht}] \leq \sum_{t=0}^{\infty} [r_t k_t + w_t z_t] + \Pi$

$k_{t+1} \leq (1-\delta_k)k_t + x_{kt}$

$h_{t+1} \leq (1-\delta_h)h_t + x_{ht}$

$z_t \leq n_t h_t$, $n_t \leq 1$

(h_0, k_0) fixed.

2) z^f is the solution to:

$Max_{\{(c_t^f, x_t^f, k_t^f)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_{kt}^f + x_{ht}^f) - r_t k_t^f - w_t z_t^f\right]$

subject to: $c_t^f + x_{kt}^f + x_{ht}^f \leq F(k_t^f, z_t^f)$.

AND

$c_t = c_t^f$

$x_{kt} = x_{kt}^f$

$x_{ht} = x_{ht}^f$

$k_t = k_t^f$

$z_t = z_t^f$

$\Pi = \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_{kt}^f + x_{ht}^f) - r_t k_t^f - w_t z_t^f\right]$

Since this is a standard convex model with a representative household and a representative firm, the equilibrium is unique (if it exists *****What do you need for this?****) and will solve the planners problem:

$Max_{\{(c_t,k_t,h_t,x_{kt},x_{ht})\}_{t=0}^{\infty}} U((c_t)_{t=0}^{\infty})$

subject to:

$[c_t + x_{kt} + x_{ht}] \leq F(k_t, z_t)$

$k_{t+1} \leq (1-\delta_k)k_t + x_{kt}$

$h_{t+1} \leq (1-\delta_h)h_t + x_{ht}$

$z_t = h_t$

(h_0, k_0) fixed.

Assuming that $U(c) = \sum_t \beta^t u(c_t)$, we have:

$p_t = \beta^t u'(c_t)/\beta^0 u'(c_0)$, and $r_t/p_t = F_k(k_t, z_t)$, $w_t = F_z(k_t, z_t)$ for all t.

As you have seen in the past, and assuming that $u(c) = c^{1-\sigma}/(1-\sigma)$, this model features the two Euler equations:

(EEK) $\left[\frac{c_{t+1}}{c_t}\right]^\sigma = \beta\,[1 - \delta_k + F_k(k_t, z_t)]$, and

(EEH) $\left[\frac{c_{t+1}}{c_t}\right]^\sigma = \beta\,[1 - \delta_h + F_z(k_t, z_t)]$.

Thus,

$$\beta\,[1 - \delta_k + F_k(k_t, z_t)] = \beta\,[1 - \delta_h + F_z(k_t, z_t)], \text{ or,}$$

$$F_k(k_t, z_t) - \delta_k = F_z(k_t, z_t) - \delta_h.$$

There is typically a single k/z ratio satisfying this constraint since the LHS is a decreasing function of k/z and the RHS is an increasing function of k/z.

For example, assuming that $F(k, z) = F(k, h) = Ak^\alpha h^{1-\alpha}$ and that $\delta_h = \delta_k$, we see that in this case $F_k(k_t, z_t) = F_z(k_t, z_t)$ for all t which reduces to:

$$\frac{h_t}{k_t} = \frac{1-\alpha}{\alpha} \text{ for all } t$$

on any interior path. We'll assume that there is no issue about non-negativity of either x_{kt} and x_{ht}. (This is an assumption about (h_0, k_0).) Given this, we have that F_k is given by:

$$F_k(k_t, h_t) = F_z(k_t, h_t) = \alpha F(k_t, h_t)/k_t$$

$$= \alpha A k_t^{\alpha-1} h_t^{1-\alpha} = \alpha A \left[\frac{h_t}{k_t}\right]^{1-\alpha} = \alpha A \left[\frac{1-\alpha}{\alpha}\right]^{1-\alpha} = A\alpha^\alpha (1-\alpha)^{1-\alpha}$$

Substituting this end we find that, in any interier equilibrium:

$$\left[\frac{c_{t+1}}{c_t}\right]^\sigma = \beta\,[1 - \delta_k + A\alpha^\alpha (1-\alpha)^{1-\alpha}] \text{ for all } t.$$

Since the RHS does not depend on t, it follows that in any interior equilibrium, there is constant growth in all factors at the rate:

$$\gamma = [\beta(1 - \delta + A\alpha^{\alpha}(1-\alpha)^{1-\alpha})]^{1/\sigma}.$$

8.1.1 Differences in Initial Conditions in the $A(k, h)$ Model

Properties:

1. This does not depend on initial conditions. So, if differences in levels in period t are due to differences in $k_t's$ and/or $h_t's$ then the future growth rate will not depend on the period t levels.

2. Interest rates on loans (e.g., consumer borrowing) would be given by

$$1 + R_t = 1 - \delta + r_t = 1 - \delta + A\alpha^{\alpha}(1-\alpha)^{1-\alpha}$$

This does not depend on either initial conditions, or the current level of k_t. So, the observed interest rates should be the same for all observations.

8.1.2 $A(k, h)$: Initial Conditions

	Data	Model
$\gamma = ?$	1.02 varied	$\gamma = [\beta(1 - \delta + A\alpha^{\alpha}(1-\alpha)^{1-\alpha})]^{1/\sigma}$
$\frac{y_t^i}{y_t^j} = ?$	1 to 50	$\frac{y_t^i}{y_t^j} = \frac{k_0^i}{k_0^j}$
$\frac{wn}{y} = ?$	$\approx 0.67?$	≈ 0.67
$TFP_t^i = ?$	varied	$TFP_t^i = \left[\frac{z_t^i}{n_t^i}\right]^{1-\alpha} = [h_t^i]^{1-\alpha}$
y_{1960}^i vs. $\bar{\gamma}_{60-95} = ?$	no relationship	no relationship
y_{1960}^i vs. $R_t^i = ?$	no relationship? (risk)	$1 + R_t^i = (1 - \delta + A\alpha^{\alpha}(1-\alpha)^{1-\alpha})$
$\frac{x}{y}$ vs. $y = ?$	$corr(\frac{x}{y}, y) > 0?$	$corr(\frac{x}{y}, y) = 0$
$\frac{x}{y}$ vs. $\gamma = ?$	$corr(\frac{x}{y}, \gamma) > 0?$	all countries have the same $\frac{x}{y}$ and γ
$Educ.$ vs. $y = ?$	$corr(Ed, y) > 0?$	$corr(h, y) > 0$
γ_n vs. $y = ?$	$corr(\gamma_n, y) < 0?$	not defined
γ_n vs. $\gamma = ?$	$corr(\gamma_n, \gamma) < 0?$	not defined

8.1.3 Differences in $A's$ (or δ_k or δ_h) in the $A(k,h)$ Model

This gives rise to higher growth rates and higher interest rates in the high A countries. It also gives rise to higher accumulation in higher A countries, of both k and h so it gives a positive correlation between $y's$, $\gamma's$, and h/y, k/y.

$$\gamma_i = \left[\beta(1-\delta_i + A_i\alpha^\alpha(1-\alpha)^{1-\alpha})\right]^{1/\sigma}$$

$$1 + R_t^i = (1-\delta_i + A_i\alpha^\alpha(1-\alpha)^{1-\alpha})$$

8.1.4 $A(k,h)$: A^i Differences

	Data	Model
$\gamma = ?$	1.02 varied	$\gamma_i = \left[\beta(1-\delta + A_i\alpha^\alpha(1-\alpha)^{1-\alpha})\right]^{1/\sigma}$
$\frac{y_t^i}{y_t^j} = ?$	1 to 50	$\frac{y_t^i}{y_t^j} = \frac{\gamma_i^t}{\gamma_j^t}\frac{y_t^i}{y_t^j} \rightarrow \infty$
$\frac{wn}{y} = ?$	$\approx 0.67?$	≈ 0.67
$TFP_t^i = ?$	varied	$TFP_t^i = \left[\frac{z_t^i}{n_t^i}\right]^{1-\alpha} = [h_t^i]^{1-\alpha}$
y_{1960}^i vs. $\bar{\gamma}_{60-95} = ?$	no relationship	high $y_{60}^i \Longrightarrow$ high $\bar{\gamma}_{60-05}$
y_{1960}^i vs. $R_t^i = ?$	no relationship? (risk)	$1 + R_t^i = (1-\delta + A_i\alpha^\alpha(1-\alpha)^{1-\alpha})$
$\frac{x}{y}$ vs. $y = ?$	$corr(\frac{x}{y}, y) > 0?$	$corr(\frac{x}{y}, y) > 0$
$\frac{x}{y}$ vs. $\gamma = ?$	$corr(\frac{x}{y}, \gamma) > 0?$	$corr(\frac{x}{y}, \gamma) > 0$
$Educ.$ vs. $y = ?$	$corr(Ed, y) > 0?$	$corr(h, y) > 0$
γ_n vs. $y = ?$	$corr(\gamma_n, y) < 0?$	not defined
γ_n vs. $\gamma = ?$	$corr(\gamma_n, \gamma) < 0?$	not defined

8.1.5 $A(k,h)$: δ_k Differences

	Data	Model
$\gamma = ?$	1.02 varied	$\gamma_i = [\beta(1-\delta_i + A\alpha^\alpha(1-\alpha)^{1-\alpha})]^{1/\sigma}$
$\frac{y_t^i}{y_t^j} = ?$	1 to 50	$\frac{y_t^i}{y_t^j} = \frac{\gamma_i^t}{\gamma_j^t}\frac{y_t^i}{y_t^j} \to \infty$
$\frac{wn}{y} = ?$	$\approx 0.67?$	≈ 0.67
$TFP_t^i = ?$	varied	$TFP_t^i = \left[\frac{z_t^i}{n_t^i}\right]^{1-\alpha} = [h_t^i]^{1-\alpha}$
y_{1960}^i vs. $\bar{\gamma}_{60-95} = ?$	no relationship	high $y_{60}^i \Longrightarrow$ high $\bar{\gamma}_{60-05}$
y_{1960}^i vs. $R_t^i = ?$	no relationship? (risk)	$1 + R_t^i = (1-\delta_i + A\alpha^\alpha(1-\alpha)^{1-\alpha})$
$\frac{x}{y}$ vs. $y = ?$	$corr(\frac{x}{y}, y) > 0?$	$corr(\frac{x}{y}, y) > 0$
$\frac{x}{y}$ vs. $\gamma = ?$	$corr(\frac{x}{y}, \gamma) > 0?$	$corr(\frac{x}{y}, \gamma) > 0$
$Educ.$ vs. $y = ?$	$corr(Ed, y) > 0?$	$corr(h, y) > 0$
γ_n vs. $y = ?$	$corr(\gamma_n, y) < 0?$	not defined
γ_n vs. $\gamma = ?$	$corr(\gamma_n, \gamma) < 0?$	not defined

8.1.6 Differences in $\beta's$ in the $A(k,h)$ Model

$$\gamma_i = [\beta_i(1-\delta + A\alpha^\alpha(1-\alpha)^{1-\alpha})]^{1/\sigma}$$

$$1 + R_t^i = (1-\delta + A\alpha^\alpha(1-\alpha)^{1-\alpha})$$

8.1.7 $A(k,h)$: β Differences

	Data	Model
$\gamma = ?$	1.02 varied	$\gamma_i = [\beta_i(1-\delta + A\alpha^\alpha(1-\alpha)^{1-\alpha})]^{1/\sigma}$
$\frac{y_t^i}{y_t^j} = ?$	1 to 50	$\frac{y_t^i}{y_t^j} = \frac{\gamma_i^t}{\gamma_j^t}\frac{y_t^i}{y_t^j} \to \infty$
$\frac{wn}{y} = ?$	$\approx 0.67?$	≈ 0.67
$TFP_t^i = ?$	varied	$TFP_t^i = \left[\frac{z_t^i}{n_t^i}\right]^{1-\alpha} = [h_t^i]^{1-\alpha}$
y_{1960}^i vs. $\bar{\gamma}_{60-95} = ?$	no relationship	high $y_{60}^i \Longrightarrow$ high $\bar{\gamma}_{60-05}$
y_{1960}^i vs. $R_t^i = ?$	no relationship? (risk)	$1 + R_t^i = (1-\delta + A\alpha^\alpha(1-\alpha)^{1-\alpha})$
$\frac{x}{y}$ vs. $y = ?$	$corr(\frac{x}{y}, y) > 0?$	$corr(\frac{x}{y}, y) > 0$
$\frac{x}{y}$ vs. $\gamma = ?$	$corr(\frac{x}{y}, \gamma) > 0?$	$corr(\frac{x}{y}, \gamma) > 0$
$Educ.$ vs. $y = ?$	$corr(Ed, y) > 0?$	$corr(h, y) > 0$
γ_n vs. $y = ?$	$corr(\gamma_n, y) < 0?$	not defined
γ_n vs. $\gamma = ?$	$corr(\gamma_n, \gamma) < 0?$	not defined

8.1.8 Differences in $\sigma' s$ in the $A(k,h)$ Model

$$\gamma_i = [\beta(1-\delta + A\alpha^{\alpha}(1-\alpha)^{1-\alpha})]^{1/\sigma_i}$$

$$1 + R_t^i = (1-\delta + A\alpha^{\alpha}(1-\alpha)^{1-\alpha})$$

8.1.9 $A(k,h)$: σ Differences

	Data	Model
$\gamma = ?$	1.02 varied	$\gamma = [\beta(1-\delta + A\alpha^{\alpha}(1-\alpha)^{1-\alpha})]^{1/\sigma_i}$
$\frac{y_t^i}{y_t^j} = ?$	1 to 50	$\frac{y_t^i}{y_t^j} = \frac{\gamma_i^t}{\gamma_j^t}\frac{y_t^i}{y_t^j} \to \infty$
$\frac{wn}{y} = ?$	$\approx 0.67?$	≈ 0.67
$TFP_t^i = ?$	varied	$TFP_t^i = \left[\frac{z_t^i}{n_t^i}\right]^{1-\alpha} = [h_t^i]^{1-\alpha}$
y_{1960}^i vs. $\bar{\gamma}_{60-95} = ?$	no relationship	high $y_{60}^i \Longrightarrow$ high $\bar{\gamma}_{60-05}$
y_{1960}^i vs. $R_t^i = ?$	no relationship? (risk)	$1 + R_t^i = (1-\delta + A\alpha^{\alpha}(1-\alpha)^{1-\alpha})$
$\frac{x}{y}$ vs. $y = ?$	$corr(\frac{x}{y}, y) > 0?$	$corr(\frac{x}{y}, y) > 0$
$\frac{x}{y}$ vs. $\gamma = ?$	$corr(\frac{x}{y}, \gamma) > 0?$	$corr(\frac{x}{y}, \gamma) > 0$
$Educ.$ vs. $y = ?$	$corr(Ed, y) > 0?$	$corr(h, y) > 0$
γ_n vs. $y = ?$	$corr(\gamma_n, y) < 0?$	not defined
γ_n vs. $\gamma = ?$	$corr(\gamma_n, \gamma) < 0?$	not defined

8.2 Adding Fiscal Policy to the $A(k,h)$ Model

New version of the model, TDCE only:

Here the equilibrium version of the model is:

An equilibrium is:

a sequence of prices: $\{(p_t, r_t, w_t)\}_{t=0}^{\infty}$

Quantity decisions for the households: $\{(c_t, k_t, h_t, x_{kt}, x_{ht})\}_{t=0}^{\infty} = z^{HH}$

Quantity decisions for the output firms: $\{(c_t^f, x_{kt}^f, x_{ht}^f, k_t^f, z_t^f)\}_{t=0}^{\infty} = z^f$,

SUCH THAT:

1) z^{HH} is the solution to:

$Max_{\{(c_t,k_t,h_t,x_{kt},x_{ht})\}_{t=0}^{\infty}} U((c_t)_{t=0}^{\infty})$

subject to:

$\sum_{t=0}^{\infty} p_t \left[c_t + x_{kt} + x_{ht}\right] \leq \sum_{t=0}^{\infty} \left[(1-\tau_{kt})r_t k_t + (1-\tau_{nt})w_t h_t + T_t\right] + \Pi$

$k_{t+1} \leq (1-\delta_k)k_t + x_{kt}$

$h_{t+1} \leq (1-\delta_h)h_t + x_{ht}$

(h_0, k_0) fixed.

2) z^f is the solution to:

$Max_{\{(c_t^f,x_{kt}^f,x_{ht}^f,k_t^f,z_t^f)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_{kt}^f + x_{ht}^f) - r_t k_t^f - w_t z_t^f\right]$

subject to: $c_t^f + x_{kt}^f + x_{ht}^f \leq A\left(k_t^f\right)^{\alpha} (z_t^f)^{1-\alpha}$.

AND

$g_t + c_t = c_t^f$

$x_t = x_{kt}^f$

$x_{ht} = x_{ht}^f$

$k_t = k_t^f$

$h_t = z_t^f$

$\Pi = \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_{kt}^f + x_{ht}^f) - r_t k_t^f - w_t z_t^f\right]$

$\sum_t \left[p_t g_t + T_t\right] = \sum_t \left[\tau_{kt} r_t k_t + \tau_{nt} w_t h_t\right]$

NOTE: What if taxes were on UNDEPRECIATED capital instead? I.e., $Tax_t = \tau_t(r_t - \delta)k_t$?

Assuming that $U(c) = \sum_t \beta^t u(c_t)$, we have:

$p_t = \beta^t u'(c_t)/\beta^0 u'(c_0)$, and $r_t/p_t = A$ for all t.

As you have seen in the past, and assuming that $u(c) = c^{1-\sigma}/(1-\sigma)$, this model features the two Euler equations:

(EEK) $\left[\frac{c_{t+1}}{c_t}\right]^\sigma = \beta\,[1-\delta_k + (1-\tau_k)F_k(k_t, z_t)]$, and

(EEH) $\left[\frac{c_{t+1}}{c_t}\right]^\sigma = \beta\,[1-\delta_h + (1-\tau_n)F_z(k_t, z_t)]$.

Thus,

$$\beta\,[1-\delta_k + (1-\tau_k)F_k(k_t, z_t)] = \beta\,[1-\delta_h + (1-\tau_n)F_z(k_t, z_t)], \text{ or,}$$

$$(1-\tau_k)F_k(k_t, z_t) - \delta_k = (1-\tau_n)F_z(k_t, z_t) - \delta_h.$$

There is typically a single k/z ratio satisfying this constraint since the LHS is a decreasing function of k/z and the RHS is an increasing function of k/z.

For example, assuming that $F(k,z) = F(k,h) = Ak^\alpha h^{1-\alpha}$ and that $\delta_h = \delta_k$, we see that in this case $(1-\tau_k)F_k(k_t, z_t) = (1-\tau_n)F_z(k_t, z_t)$ for all t which reduces to:

$$\frac{h_t}{k_t} = \frac{1-\alpha}{\alpha}\frac{1-\tau_n}{1-\tau_k} \text{ for all } t$$

on any interior path. We'll assume that there is no issue about non-negativity of either x_{kt} and x_{ht}. (This is an assumption about (h_0, k_0).) Given this, we have that F_k is given by:

$$F_k(k_t, h_t) = \alpha F(k_t, h_t)/k_t$$

$$= \alpha A k_t^{\alpha-1} h_t^{1-\alpha} = \alpha A \left[\frac{h_t}{k_t}\right]^{1-\alpha} = \alpha A \left[\frac{1-\alpha}{\alpha}\frac{1-\tau_n}{1-\tau_k}\right]^{1-\alpha} = A\alpha^\alpha(1-\alpha)^{1-\alpha}\left[\frac{1-\tau_n}{1-\tau_k}\right]^{1-\alpha}$$

Substituting this end we find that, in any interier equilibrium:

$$\left[\frac{c_{t+1}}{c_t}\right]^\sigma = \beta\left[1-\delta_k + (1-\tau_k)A\alpha^\alpha(1-\alpha)^{1-\alpha}\left[\frac{1-\tau_n}{1-\tau_k}\right]^{1-\alpha}\right]$$

$$= \beta\,[1-\delta_k + (1-\tau_k)^\alpha(1-\tau_n)^{1-\alpha}A\alpha^\alpha(1-\alpha)^{1-\alpha}] \text{ for all } t.$$

Since the RHS does not depend on t, it follows that in any interior equilibrium, there is constant growth in all factors at the rate:

$$\gamma = [\beta\,[1-\delta_k + (1-\tau_k)^\alpha(1-\tau_n)^{1-\alpha}A\alpha^\alpha(1-\alpha)^{1-\alpha}]]^{1/\sigma}.$$

This simplifies to:

$$\gamma = [\beta\,[1-\delta_k + (1-\tau)A\alpha^\alpha(1-\alpha)^{1-\alpha}]]^{1/\sigma}$$

when $\tau_n = \tau_k$.

This is decreasing in each tax rate. Alternatively, it is decreasing in τ_k and is also lower (higher) when $\frac{1-\tau_n}{1-\tau_k}$ moves below (above) 1. So, there are effects of both the 'level' of taxes, and the composition of taxes.

8.2.1 The Effects of Taxes in the $A(k, h)$ model

Properties:

1. This does not depend on initial conditions. So, if differences in levels in period t are due to differences in $k_t's$ and/or $h_t's$ then the future growth rate will not depend on the period t levels.

2. Interest rates on loans (e.g., consumer borrowing) would be given by

$$1 + R_t = 1 - \delta_k + (1 - \tau_k)^{\alpha}(1 - \tau_n)^{1-\alpha} A\alpha^{\alpha}(1 - \alpha)^{1-\alpha}$$

This does not depend on either initial conditions, or the current level of (h_t, k_t). So, the observed interest rates should be the same for all observations.

This reduces to:

$$1 + R_t = 1 - \delta_k + (1 - \tau) A\alpha^{\alpha}(1 - \alpha)^{1-\alpha}$$

if $\tau_k = \tau_n = \tau$.

3. However, if $h_0^i = h_0^{i'}$, $k_0^i = k_0^{i'}$ and $\tau_n^i = \tau_k^i = \tau^i < \tau^{i'} = \tau_n^{i'} = \tau_k^{i'}$ and $1960 \neq 0$ (i.e., time doesn't start in 1960), then $y_{1960}^i > y_{1960}^{i'}$ and $\gamma^i > \gamma^{i'}$. So, it the only difference in countries is in tax rates and these are PERMANENT, we should see a pattern between y_{1960}^i and γ^i.

4.********* There is not a pattern to measured, before tax, interest rates however, since these are given by

$$1 + R_t^i = 1 - \delta + (1 - \tau_k)^{\alpha}(1 - \tau_n)^{1-\alpha} A\alpha^{\alpha}(1 - \alpha)^{1-\alpha}$$

Is this right?

8.2.2 $A(k, h)$: Tax Policy Differences

$\tau_k = \tau_h = \tau$

	Data	Model
$\gamma =?$	1.02 varied	$\gamma_i = [\beta(1-\delta+(1-\tau_i)A\alpha^{\alpha}(1-\alpha)^{1-\alpha})]^{1/\sigma_i}$
$\frac{y_t^i}{y_t^j} =?$	1 to 50	$\frac{y_t^i}{y_t^j} = \frac{\gamma_i^t}{\gamma_j^t}\frac{y_t^i}{y_t^j} \to \infty$
$\frac{wn}{y} =?$	$\approx 0.67?$	≈ 0.67
$TFP_t^i =?$	varied	$TFP_t^i = \left[\frac{z_t^i}{n_t^i}\right]^{1-\alpha} = [h_t^i]^{1-\alpha}$
y_{1960}^i vs. $\bar{\gamma}_{60-95} =?$	no relationship	high $y_{60}^i \Longrightarrow$ high $\bar{\gamma}_{60-05}$
y_{1960}^i vs. $R_t^i =?$	no relationship? (risk)	$1+R_t^i = (1-\delta+(1-\tau_i)A\alpha^{\alpha}(1-\alpha)^{1-\alpha})$
$\frac{x}{y}$ vs. $y =?$	$corr(\frac{x}{y}, y) > 0?$	$corr(\frac{x}{y}, y) > 0$
$\frac{x}{y}$ vs. $\gamma =?$	$corr(\frac{x}{y}, \gamma) > 0?$	$corr(\frac{x}{y}, \gamma) > 0$
$Educ.$ vs. $y =?$	$corr(Ed, y) > 0?$	$corr(h, y) > 0$
γ_n vs. $y =?$	$corr(\gamma_n, y) < 0?$	not defined
γ_n vs. $\gamma =?$	$corr(\gamma_n, \gamma) < 0?$	not defined

8.2.3 The Effects of Productive Government Spending in the $A(k,h)$ model

what exercise?

1. g_t to h? i.e., $h_{t+1} \leq (1-\delta_h) + x_{ht} + g_t$, with $p_t g_t = T_t$ for all periods?

Is there a neutrality result for this one?

or

2. subsidize h? i.e., $\tau_{xht} < 0$ with $T_t = p_t \tau_{xht} x_{ht}$?

Here, we should see h go up. This should increase growth, but perhaps decrease welfare.

What is the definition of equilibrium in this setting?

Set it up to allow for both:

New version of the model, TDCE only:

Here the equilibrium version of the model is:

An equilibrium is:

a sequence of prices: $\{(p_t, r_t, w_t)\}_{t=0}^{\infty}$

Quantity decisions for the households: $\{(c_t, k_t, h_t, x_{kt}, x_{ht})\}_{t=0}^{\infty} = z^{HH}$

Quantity decisions for the output firms: $\{(c_t^f, x_{kt}^f, x_{ht}^f, k_t^f, z_t^f)\}_{t=0}^{\infty} = z^f$,

SUCH THAT:

1) z^{HH} is the solution to:

$Max_{\{(c_t,k_t,h_t,x_{kt},x_{ht})\}_{t=0}^{\infty}} U((c_t)_{t=0}^{\infty})$

subject to:

$\sum_{t=0}^{\infty} p_t\,[c_t + x_{kt} + (1+\tau_{xht})x_{ht}] \leq \sum_{t=0}^{\infty} [r_t k_t + w_t h_t + T_t] + \Pi$

$k_{t+1} \leq (1-\delta_k)k_t + x_{kt}$

$h_{t+1} \leq (1-\delta_h)h_t + x_{ht} + g_t$

(h_0, k_0) fixed.

2) z^f is the solution to:

$Max_{\{(c_t^f,x_{kt}^f,x_{ht}^f,k_t^f,z_t^f)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_{kt}^f + x_{ht}^f) - r_t k_t^f - w_t z_t^f\right]$

subject to: $c_t^f + x_{kt}^f + x_{ht}^f \leq A\left(k_t^f\right)^{\alpha} (z_t^f)^{1-\alpha}$.

AND

$c_t = c_t^f$

$x_t = x_{kt}^f$

$g_t + x_{ht} = x_{ht}^f$

$k_t = k_t^f$

$h_t = z_t^f$

$\Pi = \sum_{t=0}^{\infty} \left[p_t(c_t^f + x_{kt}^f + x_{ht}^f) - r_t k_t^f - w_t z_t^f\right]$

$\sum_t [p_t g_t + T_t] = 0$

Assuming that $U(c) = \sum_t \beta^t u(c_t)$, with $u(c) = c^{1-\sigma}/(1-\sigma)$, the FOC's for the HH problem are:

$$c_t: \qquad \beta^t u_c(t) = \lambda$$

$$k_{t+1}: \qquad \lambda\left[r_{t+1} - p_t + (1-\delta_k)p_{t+1}\right] = 0$$

$$h_{t+1}: \qquad \lambda\left[w_{t+1} - (1+\tau_{xh})p_t + (1+\tau_{xh})(1-\delta_h)p_{t+1}\right] = 0$$

OR

$$k_{t+1}: \qquad p_t = \left[r_{t+1} + (1-\delta_k)p_{t+1}\right]$$

$$h_{t+1}: \qquad p_t(1+\tau_{xh}) = \left[w_{t+1} + (1+\tau_{xh})(1-\delta_h)p_{t+1}\right]$$

OR

$$k_{t+1}: \qquad p_t = p_{t+1}\left[F_k(t+1) + (1-\delta_k)\right]$$

$$h_{t+1}: \qquad p_t(1+\tau_{xh}) = p_{t+1}\left[F_h(t+1) + (1+\tau_{xh})(1-\delta_h)\right]$$

$$h_{t+1}: \qquad p_t = p_{t+1}\left[\tfrac{1}{(1+\tau_{xh})}F_h(t+1) + (1-\delta_h)\right]$$

Substituting for p_t from above gives:

$$k_{t+1}: \qquad \left[\tfrac{c_{t+1}}{c_t}\right]^\sigma = \beta\left[F_k(t+1) + (1-\delta_k)\right]$$
$$h_{t+1}: \qquad \left[\tfrac{c_{t+1}}{c_t}\right]^\sigma = \beta\left[\tfrac{1}{(1+\tau_{xh})}F_h(t+1) + (1-\delta_h)\right]$$

Notice that this is just EXACTLY the same as what we had in the section above on taxes where $\tau_k = 0$ and $(1-\tau_n) = \frac{1}{(1+\tau_{xh})}$.

Thus, assuming that $\delta_k = \delta_h$ and that the production function is Cobb-Douglas, we see that we get:

$$\gamma = \left[\beta\left[1 - \delta_k + \left[\tfrac{1}{(1+\tau_{xh})}\right]^{1-\alpha} A\alpha^\alpha(1-\alpha)^{1-\alpha}\right]\right]^{1/\sigma}$$

Note that γ is decreasing in τ_{xh} and hence a subsidy increases γ.

8.2.4 $A(k,h)$: Government Spending Differences

1. g_t to h? i.e., $h_{t+1} \leq (1-\delta_h) + x_{ht} + g_t$, with $p_t g_t = T_t$ for all periods? This should be completely neutral I believe, that is, just lower x_{ht} by g_t.

This one should be COMPLETELY NEUTRAL as long as it is not above the privately chosen rate of x_{ht}. If it is, it will have effects of increasing growth and income.

or

2. subsidize h? i.e., $\tau_{xht} < 0$ with $T_t = p_t \tau_{xht} x_{ht}$?

	Data	Model
$\gamma =?$	1.02 varied	$\gamma_i = \left[\beta\left[1-\delta_k + \left[\frac{1}{(1+\tau_{xhi})}\right]^{1-\alpha} A\alpha^{\alpha}(1-\alpha)^{1-\alpha}\right]\right]^{1}$
$\frac{y_t^i}{y_t^j} =?$	1 to 50	$\frac{y_t^i}{y_t^j} = \frac{\gamma_i^t}{\gamma_j^t}\frac{y_t^i}{y_t^j} \to \infty$
$\frac{wn}{y} =?$	≈ 0.67?	≈ 0.67
$TFP_t^i =?$	varied	$TFP_t^i = \left[\frac{z_t^i}{n_t^i}\right]^{1-\alpha} = [h_t^i]^{1-\alpha}$
y_{1960}^i vs. $\bar{\gamma}_{60-95} =?$	no relationship	high $y_{60}^i \Longrightarrow$ high $\bar{\gamma}_{60-05}$
y_{1960}^i vs. $R_t^i =?$	no relationship? (risk)	$1 + R_t^i = \left[1-\delta_k + \left[\frac{1}{(1+\tau_{xhi})}\right]^{1-\alpha} A\alpha^{\alpha}(1-\alpha)^{1-\alpha}\right.$
$\frac{x}{y}$ vs. $y =?$	$corr(\frac{x}{y}, y) > 0$?	$corr(\frac{x}{y}, y) > 0$?
$\frac{x}{y}$ vs. $\gamma =?$	$corr(\frac{x}{y}, \gamma) > 0$?	$corr(\frac{x}{y}, \gamma) > 0$
$Educ.$ vs. $y =?$	$corr(Ed, y) > 0$?	$corr(h, y) > 0$
γ_n vs. $y =?$	$corr(\gamma_n, y) < 0$?	not defined
γ_n vs. $\gamma =?$	$corr(\gamma_n, \gamma) < 0$?	not defined

8.2.5 $A(k,h)$: Monetary Policy Differences

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8.2.6 The Effects of Monetary Policy in the $A(k,h)$ model

8.3 Alternative Technologies for h Accumulation

Learning by doing, public/private, firms and IRS

LBD as perfect complements, standard $A(k,h)$ as perfect substitutes, things in between?

It's easy to imagine alternative, reasonable models of the formation of human capital:

1. Suppose that there is learning on the job, i.e., other things equal, you get more h if you work more. This can be captured by:

$$h_{t+1} \leq (1-\delta_h)h_t + G(x_{ht}, z_t)$$

where $z_t = n_t h_t$ as an example. If $G(x,z) = A_G x^{\alpha_G} z^{1-\alpha_G}$ then $\alpha_G = 1$ is the example above, and $\alpha_G = 0$ says that x_h does not affect the accumulation of h, only work experience does.

Alternatively, one could adapt the output sector instead:

$$\max(c_t + x_{kt}, \theta x_{ht}) \leq F(k_t, z_t)$$

and

$$h_{t+1} \leq (1-\delta_h)h_t + x_{ht}$$

would give knowledge accumulation proportional to output by assumption. This is the opposite extreme to our base case. In the base case, we assumed that output for $c + x_k$ and x_h are perfect substitutes. This makes them perfect complements. But, once we've said that other alternatives come to mind:

$$[\theta(c_t + x_{kt})^\rho + (1-\theta)x_{ht}^\rho]^{1/\rho} \leq F(k_t, z_t)$$

$$h_{t+1} \leq (1-\delta_h)h_t + x_{ht}$$

with $\rho \geq 1$.

is the most obvious generalization that nests both of these extreme cases. And then sensible discussions can be had about the alternatives: Is new worker ability a substitute for, or a complement to the production of output?

2. The second obvious kind of qualitative change that one might make is to formally include a separate sector for the production of Education. In this case, the feasibility restrictions would look something like:

$$c_t + x_{kt} \leq F(k_t^y, z_t^y)$$

$$x_{ht} \leq G(k_t^h, z_t^h)$$

$$k_t^y + k_t^h \leq k_t$$

$$z_t^y + z_t^h \leq z_t = n_t h_t$$

and

$$h_{t+1} \leq (1 - \delta_h)h_t + x_{ht}$$

Obviously, our base care corresponds to the aggregation assumption that $F = G$ but other possibilities could be explored. Of particular interest is allowing for the possibility that G is more labor intensive that F is, with the idea being that one of the key inputs into G is the time of the students. Thus, something like $F(k,z) = Ak^{\alpha}z^{1-\alpha}$ and $G(k,z) = A^h k^{\eta} z^{1-\eta}$ with $\eta < \alpha$ would be called for.

One has to be careful with this. This is not the same as having 2 kinds of labor in the production function, one 'skilled' i.e., teachers, and one 'unskilled', i.e., students, and to do this would require some careful thinking about the relative productivity of students and teachers with their time aggregated according to opportunity costs (i.e., output sector wage rates). It also is not the same as doing a more detailed model with people of different ages in which the young are students and the old are teachers.

Each of these alternatives (or both) brings with it some special considerations with respect to the tax code. For example, in the first: Is x_h part of the compensation of workers? I.e., one of the things that workers take away from the job at the end of the day is the new knowledge that they have accumulated from working. Should this be modelled as a market transaction with the income flows taxed at income tax rates? Or is this a non-market transaction?

Similarly with the second approach, is the second sector taxed at all? Is the time of students taxed? Etc.

8.4 $A(k,h)$ Models, Elastic Labor Supply

Here, the equilibrium version of the model is:

What should the utility function be?

$$\max \quad \sum_t \beta^t u(c_t, \ell_t)$$

OR

$$\max \quad \sum_t \beta^t u(c_t, \ell_t h_t)?????$$

9 Individual Heterogeneity Within a Country

There are a continuum of households indexed by $i \in [0,1]$ and a continuum of firms indexed by $j \in [0,1]$. They are all identical, the households have the same utility functions, initial endowments and labor supplies. The firms all have the same technology. For simplicity, we will assume that each consumer has an equal share in each of the firms.

An equilibrium is:

a sequence of prices: $\{(p_t, r_t, w_t)\}_{t=0}^{\infty}$

Quantity decisions for the households: $\{(c_{it}, k_{it}, x_{ikt}, \ell_{it}, n_{it})\}_{t=0}^{\infty} = z_i^{HH}$

Quantity decisions for the output firms: $\{(c_{jt}^f, x_{jkt}^f, k_{jt}^f, n_{jt}^f)\}_{t=0}^{\infty} = z_j^f$,

SUCH THAT:

1) For each $i \in [0,1]$, z_i^{HH} is the solution to:

$$Max_{\{(c_{it},k_{it},x_{ikt},x_{iht},n_{mit},n_{hit},\ell_{it})\}_{t=0}^{\infty}} \sum_t \beta^t u(c_{it}, \ell_{it}) \qquad or \qquad \sum_t \beta^t u(c_{it}, \ell_{it} h_{it})$$

subject to:

$$\sum_{t=0}^{\infty} p_t \left[c_{it} + x_{ikt} + x_{iht}\right] \leq \sum_{t=0}^{\infty} \left[r_t k_{it} + w_t z_{mit}\right] + \Pi_i$$

$$k_{it+1} \leq (1-\delta_k) k_{it} + x_{ikt}$$

$$h_{it+1} \leq (1-\delta_h) h_{it} + G_i(x_{iht}, h_{it}, n_{hit})$$

$$z_{it}^m = n_{mit} h_{it}$$

$$n_{mit} + n_{hit} + \ell_{it} \leq 1,$$

(h_{i0}, k_{i0}) fixed.

Here, n_{mit} is the amount of time working in the market, while, n_{hit} is the amount of time devoted to 'learning' or, augmenting one's own stock of human capital. It's natural to assume that $G_i(x_{iht}, h_{it}, n_{hit}) = G_i(x_{iht}, z_{hit})$, where $z_{hit} = n_{hit} h_{it}$ is the quantity of 'effective' labor used in learning. This would make it symmetric with the formulation above with respect to effective labor in the market activity. Similarly, putting $\ell_{it} h_{it}$ in the utility function makes this quality adjusted hours affecting utility. This has some technical advantages over the alternative of ℓ_{it} entering the utility function, since then the problem can be reformalized as a standard concave maximization problem. This is, or course, irrelevant in the inelastic labor supply version.

2) For each $j \in [0,1]$, z_j^f is the solution to:

$$Max_{\{(c_{jt}^f, x_{jkt}^f, x_{jht}^f, k_{jt}^f, z_{jt}^f)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[p_t(c_{jt}^f + x_{jkt}^f + x_{jht}^f) - r_t k_{jt}^f - w_t z_{jt}^f\right]$$

subject to: $c_{jt}^f + x_{jkt}^f \leq F(k_{jt}^f, z_{jt}^f)$.

AND

$$\int_0^1 c_{it} di = \int_0^1 c_{jt}^f dj$$

$$\int_0^1 x_{ikt} di = \int_0^1 x_{jkt}^f dj$$

$$\int_0^1 k_{it} di = \int_0^1 k_{jt}^f dj$$

$$\int_0^1 n_{it} di = \int_0^1 n_{jt}^f dj$$

$$\int_0^1 \Pi_i di = \int_0^1 \sum_{t=0}^{\infty} \left[p_t(c_{jt}^f + x_{jkt}^f) - r_t k_{jt}^f - w_t n_{jt}^f\right] dj$$

$$K_t = \int_0^1 k_{jt}^f dj$$

As is standard, in this formulation, we will, for each HH get 2 Euler Equations governing the dynamics of individual savings/investment decisions. For simplicity, we assume that labor is inelastically supplied. In this case, we have:

$$\text{(EEK)} \qquad \left[\frac{c_{it+1}}{c_{it}}\right]^{\sigma} = \beta\left[1 - \delta_k + \frac{r_{t+1}}{p_{t+1}}\right]$$

Since the right hand side of this equation does NOT depend on i, it follows immediately that the growth rate of consumption is the same for all households. Note that this depends on the assumption that both σ and β are the same for all households.

Similarly, there is an EE for the accumulation of h:

$$\text{(EEH)} \qquad \left[\frac{c_{it+1}}{c_{it}}\right]^{\sigma} = \beta\left[*_i\right]$$

FILL IN WHAT $*_i$ IS!

Since the LHS doesn't depend on i, it follows that the right hand side doesn't either. I.e., The natural condition holds, all investments by all individuals in all assets (both physical and human capital) are, in equilibrium equal to a common interest rate, which can be thought of as the implicit interest rate on loans to consumers for consumption loans,

$$1 + R_t = \frac{1}{\beta}\left[\frac{c_{it+1}}{c_{it}}\right]^{\sigma}.$$

MOSTLY CONJECTURES BELOW HERE:

In some simple cases, one can go further. For example, suppose $G_i = x_{hit}$ for all i.

Conjecture 4 *In this case, the system is always on the BGP after period $t = 1$, with $\frac{h_{it+1}}{h_{it}} = \gamma$ where $\gamma^{\sigma} = \beta\left[1 - \delta_k + F_k\right]$. Note that F_k doesn't depend on j (since it is the common $\frac{r}{p}$ for all j). (Does this implicitly ignore non-negativity constraints?)*

If this is right, then not only is it true that consumption growth is equal across all households, productivity growth is also equal across all households. So, the picture is that some households are initially 'richer' than others, they have higher initial levels of c and this 'advantage' persists indefinitely.

What about a more general form of accumulation in which $G_i = B_i x_{hit}^{\eta}(n_{hit}h_{it})^{1-\eta}$?

Conjecture 5 *If $B_i = B$ for all i, the conjecture still holds. Maybe even for CRS $G_i's$ that are identical across households?*

What is TFP in an economy like this?

$$TFP_t = \frac{y_t}{\left[\int_I k_{it}di\right]^{.33}\left[\int_I n_{mit}di\right]^{.67}}.$$

Note that there is implicitly a GNP accounting assumption here. This is that none of the n_{hit} is counted in hours in the data (as it might be if n_{hit} were training received while at the workplace). Making the standard assumptions that there is a representative firm with a Cobb-Douglas productin function, $F(k,z) = Ak^{\alpha}z^{1-\alpha}$ we get:

$$y_t = A\left[\int_I k_{it}di\right]^{\alpha}\left[\int_I z_{mit}di\right]^{1-\alpha} = A\left[\int_I k_{it}di\right]^{\alpha}\left[\int_I n_{mit}h_{it}di\right]^{1-\alpha}$$

Thus, we have:

$$TFP_t = \frac{y_t}{\left[\int_I k_{it}di\right]^{.33}\left[\int_I n_{mit}di\right]^{.67}} = \frac{A\left[\int_I k_{it}di\right]^{\alpha}\left[\int_I n_{mit}h_{it}di\right]^{1-\alpha}}{\left[\int_I k_{it}di\right]^{.33}\left[\int_I n_{mit}di\right]^{.67}} = A\frac{\left[\int_I n_{mit}h_{it}di\right]^{.67}}{\left[\int_I n_{mit}di\right]^{.67}}$$

This is kind of a mess unless n_{mit} doesn't depend on i. I think that would hold under either of the Conjectures given above, but have not tried to show it. Assuming that this is correct we get:

$$TFP_t = A\frac{\left[n_{mt}\int_I h_{it}di\right]^{.67}}{[In_{mt}]^{.67}} = A\frac{\left[In_{mt}\bar{h}_t\right]^{.67}}{[In_{mt}]^{.67}} = A\left[\bar{h}_t\right]^{.67}$$

where $\bar{h}_t = \frac{1}{I}\int_I h_{it}di$.

An interesting question would be how does all of this analysis change if B_i is NOT independent of i. This would be a way to capture the idea that some people are 'better learners' than others. (Or any other kinds of differences across G_i, e.g., $G_i = \theta_i x_{hit}$.) In this case, would it still be true that n_{mit} and n_{hit} are independent of i for example? Or would a planner optimally choose to have good learners do most of the h accumulation? (And hence it would also occur in equilibrium...)

Although these are interesting discussions that allow us to get closer to some of the applied labor literature, it is not clear how (or IF), these considerations affect our tables.