Problem Set #4

Econ 8105-8106
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1. Consider a two-period pure-exchange economy with 2 households, where there is *one type* of consumption good in the first period, and *two types* of consumption good in the second period. Each household has utility function:

$$U^i(c^i, r^i, s^i) = \log c^i + \alpha \log r^i + (1-\alpha)\log s^i$$

where c denotes consumption in the first period, and r, s denote the two different types of consumption in the second period. The households have endowments:

$$\begin{aligned} e_c^1 &= e_c^2 = 1; \\ e_r^1 &= \frac{3}{2},\ e_r^2 = \frac{1}{2}; \\ e_s^1 &= \frac{1}{2},\ e_s^2 = \frac{3}{2}. \end{aligned}$$

(a) Define an Arrow-Debreu equilibrium for this economy, and calculate the equilibrium.

(b) Show that the utility function is homothetic.

(c) Recall the aggregation result that holds when households have the same homothetic utility function. Consider the above economy with only one "aggregate consumer." What are the endowments of this consumer? Calculate the Arrow-Debreu equilibrium with this single consumer.

(d) Find the wealth of each original consumer as a fraction of total wealth, at the equilibrium prices in (c). Split the equilibrium allocation of (c) according to these fractions and compare to your answer in (a).

(e) Finally, consider the following alternative economy:

There are still two periods and two households, but any of N "states of the world," denoted by $s_1, s_2, \ldots, s_N$ may occur in the second period, each with probability $q_1, q_2, \ldots q_N$, respectively. There is only one type of consumption good in each period, but households can make contracts in the first period to buy some amount of the consumption good, c_n^i, in the second period if state s_n occurs, for each n, and they care about expected utility. Endowments of consumption in the first period are still $e_0^1 = e_0^2 = 1$, but endowments of the good in the second period depend on the state: each household gets e_n^i, where $n = 1, 2, \ldots, N$, and for every n, $e_n^1 + e_n^2 = 2$. How can you relate this to (a) - (d) above?

2. Please do problems 3.5, 3.6 (b), and 4.4 from the Stokey, Lucas and Prescott (SLP) book.

3. Consider the infinite-horizon sequence problem (SP) and the functional equation (FE) (you know what they are by now). Suppose that, in (SP),

$$\text{given any } x_0\text{, there is a } \underset{\sim}{x}^* \in \Pi(x_0) \text{ such that } \sum_{t=0}^{\infty} \beta^t F(x_t^*, x_{t+1}^*) = V^*(x_0)$$

and in (FE),

$$\text{given any } x\text{, there is a } y^* \in \Gamma(x) \text{ such that } v(x) = F(x, y^*) + \beta v(y^*).$$

Prove Theorems 4.2 and 4.3 from SLP with these additional assumptions.

4. Consider the following *finite*-horizon sequence problem:

$$\max_{\underset{\sim}{x} \in \Pi(x_0)} \sum_{t=0}^{T} \beta^t F\left(x_t, x_{t+1}\right)$$

where $\underset{\sim}{x} = (x_0, x_1, \ldots, x_{T+1})$, and $\Pi(x_0) = \left\{\underset{\sim}{x} : x_{t+1} \in \Gamma(x_t)\ \forall\, t = 1, 2, \ldots, T\right\}$, for a fixed correspondence Γ that describes feasibility in each period.

Let $\Pi_n(x_n) = \left\{\underset{\sim}{x}^n = (x_n, x_{n+1}, \ldots, x_{T+1}) : x_{t+1} \in \Gamma(x_t)\ \forall\, t = n, n+1, \ldots T\right\}$ be the set of choices that are feasible from x_n, for all $n = 0, 1, \ldots, T$,

and for all $\underset{\sim}{x}^n \in \Pi_n(x_n)$,

let $\qquad u_n(\underset{\sim}{x}^n) = \sum_{t=n}^{T} \beta^{t-n} F\left(x_t, x_{t+1}\right),$

and let $\quad V_n^*(x_n) = \sup_{\underset{\sim}{x}^n \in \Pi_n(x_n)} u_n(\underset{\sim}{x}^n)$, for all $n = 0, 1, \ldots, T$.

Prove analogues of Theorems 4.2 and 4.3 from SLP in this case - that is, define the appropriate functional equation problem corresponding to this sequence problem, and relate the functions V_n^* and the solutions of these equations.

(Note 1: since there are $T+1$ functions V_n^* (one for each $n = 0, 1, \ldots, T$), you will need $T+1$ value functions v_n.)

(Note 2 - regarding assumptions: do Assumption 4.2 and condition (8) in Theorem 4.3 have any use in the finite-horizon case? Should they be replaced with something else?)

(Note 3: you may assume what was assumed in Problem 3 above, i.e. that the sup's in each problem are attained by some element of the feasible set.)

Correction made to Problem 4: in the definition of u_n, use β^{t-n} instead of β^t. Why? We are trying to get an expression for the utility (discounted sums of F) from any period until the end of time. With β^{t-n}, this utility is evaluated as if starting from period n. The relation to the original (infinite-horizon) functional equation is more clear.