Problem Set #5

Econ 8105-8106
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1. Consider the following optimal growth problem:

$$\max_{\{c_t, k_{t+1}, x_t\}_{t=0,1,\dots}} \sum_{t=0}^{\infty} \beta^t \log c_t$$

$$\begin{aligned} &\text{subject to:} \\ &c_t + x_t \leq A k_t^{\alpha} \\ &k_{t+1} \leq k_t(1-\delta) + x_t \\ &c_t,\ k_{t+1} \geq 0 \\ &k_0 \textit{ given}. \end{aligned}$$

where $0 < \alpha < 1$, $0 < \beta < 1$, and $0 < \delta < 1$.

(a) Use the first-order conditions of this problem to write an equation that a solution to the problem must satisfy, relating the loss and benefit from giving up a small amount of consumption in period t for consumption in period $t+1$. (The *Euler equation*).

(b) Write a condition stating that, as time goes to infinity, the "value" of the capital stock, in terms of the discounted utility of current consumption, goes to zero. (The *transversality condition*).

(c) Define a steady-state for this problem. Use (a) to calculate the steady-state values of all variables.

Now suppose $\delta = 1$.

(d) Write the functional equation (Bellman's equation) for this problem. Guess that the value function has the form $v(k) = B_0 + B_1 \log k$, and calculate B_0, B_1, and the optimal policy function $g(k)$. Show that the sequence defined recursively by $k_{t+1} = g(k_t)$ satisfies the Euler equation and transversality condition, and that, given any positive value for k_0, this sequence converges to the steady-state you found in part (c).

(e) Log-linearize the Euler equation and the feasibility constraint of this problem around the steady state. Guess that the optimal decisions for c_t and k_{t+1} take the form

$$\log c_t = \gamma_c + \psi_c \log k_t$$

$$\log k_{t+1} = \gamma_k + \psi_k \log k_t$$

and solve for the coefficients γ_c, ψ_c, γ_k, and ψ_k.

2. Consider the problem in 1. with the following parameters:

   $\alpha = 0.3,\ \beta = 0.6,\ \delta = 1,\ A = 20.$

   (a) Take the discrete grid $X = \{2, 4, 6, 8, 10\}$ for the values of the capital stock. Consider the procedure of directly iterating on the functional equation,

   $$v_{n+1}(k) = \max_{k' \in X} \left\{ \log(20k^{0.3} - k') + (0.6)v_n(k') \right\}$$

   with the initial guess $v_0(k) = 0$ for all k.

   Calculate v_1 and v_2 by hand.

   (b) Now consider the (much finer) grid $X = \{0.05, 0.10, 0.15, \ldots, 9.90, 9.95, 10\}$. Use a computer to perform value function iteration. Perform the iteration until $\max_{k \in X} |v_{n+1}(k) - v_n(k)| < 10^{-5}$, and report the resulting value function and policy function. Compare these functions to your answer in 1.(d) above. Use the policy function to calculate k_t and c_t for the first 25 periods, for the given initial condition $k_0 = 1.00$. Do the same for $k_0 = 9.00$. (Note that with the approximate policy function as calculated, k_t must be in the grid X, but not c_t.)

   (c) Now, let $\delta = 0.5$. Repeat the computation in part (b)

   (Note: there is now no analytical solution for comparison)

3. Consider the following optimal growth problem:

   $$\max \sum_{t=0}^{\infty} \beta^t \log\left(\frac{C_t}{N_t}\right)$$

   $$\text{subject to :}$$

   $$C_t + K_{t+1} - (1-\delta)K_t \leq \left(\gamma^{1-\alpha}\right)^t A K_t^{\alpha} N_t^{1-\alpha}$$

   $$C_t, K_t \geq 0$$

   $$K_0 \ given,\ N_0 \ given$$

   $$N_t = \eta^t N_0$$

   Where C_t is aggregate consumption, N_t is population, which grows constantly at the rate η, K_t is aggregate capital stock.

   The parameters of this problem are β, the discount rate, δ, the depreciation rate, γ and η, the exogenously specified growth rates, and A and α, the parameters on the production function.

   (a) Write the Euler equation for this problem.

(b) A *balanced growth path* for this problem is defined as a solution in which the variables $\frac{C_t}{N_t}$ (consumption per capita), $\frac{K_t}{N_t}$ (capital stock per capita), and $\frac{Y_t}{N_t}$ (output per capita), all grow at constant rates (though they may not be the same). Show that in a balanced growth path for this problem, the following are true:

1. $\frac{C_t}{N_t}$, $\frac{K_t}{N_t}$, and $\frac{Y_t}{N_t}$, all grow at the constant (gross) rate γ - that is, for example, $\frac{C_{t+1}/N_{t+1}}{C_t/N_t} = \gamma$ for all t.
2. the real interest rate $r_t - \delta$, the factor shares of output, $\frac{w_t N_t}{Y_t}$ and $\frac{r_t K_t}{Y_t}$, and the capital-output ratio, $\frac{K_t}{Y_t}$ are all constant. (r_t is the marginal product of capital and w_t is the marginal product of labor in the production function.)
3. the variable $\frac{K_t}{Y_t}$ (capital-output ratio) is constant.

(c) Calibrate the parameters of this model by using data for a country other than the Netherlands, from the past 30 or 40 years, assuming that country was in a balanced growth path of this model during that time.