Problem Set #5

Econ 8105-8106
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1. In this exercise you are asked to compute the optimal policy in the one sector growth model using a method called Howard's policy improvement algorithm. This method consists in starting from an arbitrary feasible police function and iterating to convergence following the steps:

   (a) From a feasible policy, $y = g_0(x)$ compute the value of following this policy forever, i.e.:

   $$V_{g_j}(x) = \sum_{t=0}^{\infty} \beta^t F\left(x_t, g_j(x_t)\right)$$

   for $j = 0$

   (b) Generate a new policy $y = g_{j+1}(x)$ that solves the problem:

   $$\max_{y \in \Gamma(x)} \left\{F(x_t, y) + \beta V_{g_j}(y)\right\}$$

   for each x

   (c) Iterate over j to convergence on steps (a) and (b).

   Using this method in the one sector growth model:

   $$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t \log(c_t) \\ & s.t. c_t + k_{t+1} \leq Ak_t^\alpha \\ k_0 given, c_t \geq & \; 0, k_{t+1} \geq 0 \end{aligned}$$

   Guess any feasible optimal policy, i.e., $k_{t+1} = h_0(Ak_t^\alpha)$ for some constant $h_0 \in (0, 1)$.Define:

   $$V_0(k_0) = \sum_{t=0}^{\infty} \beta^t \log(Ak_t^\alpha - h_0 Ak_t^\alpha)$$

   Choose $h_1$to solve

   $$\max\{\log(Ak^\alpha - k') + \beta V_0(k')\} \tag{1}$$

   where $k' = h_1 Ak_t^\alpha$.

   Then define $V_1(k_0) = \sum_{t=0}^{\infty} \beta^t \log(Ak_t^\alpha - h_1 Ak_t^\alpha)$. Continue iterating untill h_j converges.

   Show that the optimal policy function converges in one step.

2. Consider the following optimal growth problem:

$$\max_{\{c_t,k_{t+1},x_t\}_{t=0,1,\dots}} \sum_{t=0}^{\infty} \beta^t \log c_t$$

$$\begin{aligned} &\text{subject to:} \\ c_t + x_t &\le Ak_t^\alpha \\ k_{t+1} &\le k_t(1-\delta) + x_t \\ c_t,\ k_{t+1} &\ge 0 \\ k_0 &\ \textit{given}. \end{aligned}$$

where $0 < \alpha < 1$, $0 < \beta < 1$, and $0 < \delta < 1$.

Consider the following parameters: $\alpha = 0.3,\ \beta = 0.6,\ \delta = 1,\ A = 20$.

(a) Take the discrete grid $X = \{2, 4, 6, 8, 10\}$ for the values of the capital stock. Consider the procedure of directly iterating on the functional equation,

$$v_{n+1}(k) = \max_{k' \in X} \left\{ \log(20k^{0.3} - k') + (0.6)v_n(k') \right\}$$

with the initial guess $v_0(k) = 0$ for all k. Calculate v_1 and v_2 by hand.

(b) Now consider the grid $X = \{0.05, 0.10, 0.15, \dots, 9.90, 9.95, 10\}$. Use a computer to perform value function iteration. Perform the iteration until $\max_{k \in X} |v_{n+1}(k) - v_n(k)| < 10^{-5}$, and report the resulting value function and policy function. Use the policy function to calculate k_t and c_t for the first 25 periods, for the given initial condition $k_0 = 1.00$. Do the same for $k_0 = 9.00$. (Note that with the approximate policy function as calculated, k_t must be in the grid X, but not c_t.) For the above parameter values, calculate the policy function you derived in exercise 1 (i.e. calculate h_1). What can you say about the convergence of the policy function in both exercises?

(c) Now, let $\delta = 0.5$. Repeat the computation in part (b)