Problem Set #6

Econ 8105-8106
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1. Consider the following optimal growth problem:

$$\max_{\{c_t, k_{t+1}, x_t\}_{t=0,1,\ldots}} \sum_{t=0}^{\infty} \beta^t \log c_t$$
subject to:
$$c_t + x_t \leq A k_t^{\alpha}$$
$$k_{t+1} \leq k_t(1-\delta) + x_t$$
$$c_t,\ k_{t+1} \geq 0$$
$$k_0 \textit{ given.}$$

where $0 < \alpha < 1$, $0 < \beta < 1$, and $0 < \delta < 1$.

(a) Use the first-order conditions of this problem to write an equation that a solution to the problem must satisfy, relating the loss and benefit from giving up a small amount of consumption in period t for consumption in period $t+1$.(The *Euler equation*). Write the transversality condition for this problem.

(b) Define a steady-state for this problem. Use (a) to calculate the steady-state values of all variables.

Now suppose $\delta = 1$.

(c) Write the functional equation (Bellman's equation) for this problem. Guess that the value function has the form $v(k) = B_0 + B_1 \log k$, and calculate B_0, B_1, and the optimal policy functions $k' = g(k)$ and $c = c(k)$. Show that the sequence defined recursively by $k_{t+1} = g(k_t)$ satisfies the Euler equation and transversality condition, and that, given any positive value for k_0, this sequence converges to the steady-state you found in part (c).

(d) Log-linearize the Euler equation and the feasibility constraint of this problem around the steady state. Guess that the optimal decisions for c_t and k_{t+1} take the form

$$\log c_t = \gamma_c + \psi_c \log k_t$$
$$\log k_{t+1} = \gamma_k + \psi_k \log k_t$$

and solve for the coefficients γ_c, ψ_c, γ_k, and ψ_k.

(e) Consider now the neoclassical growth model with elastic labor supply. I.e., the problem is:

$$\max_{\{c_t, k_{t+1}, x_t\}_{t=0,1,\ldots}} \sum_{t=0}^{\infty} \beta^t \left[\log c_t + \gamma \log(1-l_t)\right]$$

$$\text{subject to:}$$

$$c_t + k_{t+1} \leq A k_t^{\alpha} l_t^{1-\alpha}$$

$$c_t,\ k_{t+1}, l_t, (1-l_t) \geq 0$$

$$k_0 \ given.$$

Guess that the value function is of the form: $v(k) = B_0 + B_1 \log k$ and calculate the policy functions $k' = g(k)$, $c = c(k)$ and $l = l(k)$.[Hint: the policy function for l doesn't depend on k].

2. Consider the following optimal growth problem:

$$\max \sum_{t=0}^{\infty} \beta^t \log\left(\frac{C_t}{N_t}\right)$$

$$\text{subject to :}$$

$$C_t + K_{t+1} - (1-\delta)K_t \leq \left(\gamma^{1-\alpha}\right)^t A K_t^{\alpha} N_t^{1-\alpha}$$

$$C_t, K_t \geq 0$$

$$K_0 \ given,\ N_0 \ given$$

$$N_t = \eta^t N_0$$

Where C_t is aggregate consumption, N_t is population, which grows at the constant rate η, K_t is aggregate capital stock.

The parameters are β, the discount rate, δ, the depreciation rate, γ and η, the exogenously specified growth rates, and A and α, the parameters on the production function.

(a) Write the Euler equation for this problem.

(b) A *balanced growth path* for this problem is a solution in which the per capita variables ($\frac{C_t}{N_t}$, $\frac{K_t}{N_t}$ and $\frac{Y_t}{N_t}$) all grow at constant rates (though they may not be the same). Show that in a balanced growth path for this problem, the following are true:

1. $\frac{C_t}{N_t}$, $\frac{K_t}{N_t}$, and $\frac{Y_t}{N_t}$, all grow at the constant (gross) rate γ - that is, for example, $\frac{C_{t+1}/N_{t+1}}{C_t/N_t} = \gamma$ for all t.
2. the real interest rate $r_t - \delta$, the factor shares of output, $\frac{w_t N_t}{Y_t}$ and $\frac{r_t K_t}{Y_t}$, and the capital-output ratio, $\frac{K_t}{Y_t}$ are all constant. (r_t is the marginal product of capital and w_t is the marginal product of labor in the production function.)
3. the variable $\frac{K_t}{Y_t}$ (capital-output ratio) is constant.

(c) Calibrate the parameters of this model by using data for a country other than the US, from the past 30 or 40 years, assuming that country was in a balanced growth path of this model during that time.

3. **Vintage Capital:** Consider a standard one sector optimal growth model with only one difference: The k_{t+1} new units of capital saved at time t remain fully productive until time $t+2$ (i.e. they do not depreciate), and then they disappear. In this model, the technology is given by:

$$c_t + k_{t+1} \leq AF(k_{t-1}, k_t)$$

(a) Formulate the optimal growth problem.

(b) Define a steady state for this economy. Show that a unique steady state exits and state the necessary assumptions on u, F to guarantee it.

(c) Is it true that this economy will converge to a cyclical sequence of consumption and capital? I.e., the optimal sequences of consumption and capital are $(c_e, c_o, c_e, c_o, ..)$ and $(k_e, k_o, k_e, k_o, ..)$, where c_e, k_e and c_o, k_o are the optimal policy in even and odd periods, respectively.

4. Consider the following economy. A representative household has preferences given by:

$$\sum_{t=0}^{\infty} \beta^t \log c_t$$

The production side is given by:

$$\begin{aligned} c_t + x_t &\leq Ak_t^\alpha \\ k_{t+1} &\leq (1-\delta)k_t + \xi_t x_t \\ \xi_t &= \gamma^t, \quad 1 < \gamma < \frac{1}{\beta} \end{aligned}$$

where ξ_t is a parameter that measures how costly it is to produce investment goods.

(a) Formulate the planning problem for this economy.

(b) Show that in a steady state of this economy c_t and y_t grow at the common (gross) growth rate $g = \gamma^{\frac{\alpha}{1-\alpha}}$ and k_t grows at the rate $g_k = \gamma^{\frac{1}{1-\alpha}}$ [Hint: Steady state relationships can be deduced entirely from the feasibility conditions]

(c) Define $\hat{c}_t = \frac{c_t}{g^t}$, $\hat{y}_t = \frac{y_t}{g^t}$, $\hat{k}_t = \frac{k_t}{g_k^t}$. Write the planner's problem of these transformed variables.

(d) Formulate the Bellman's equations for this problem. Carefully explain why this transformed problem can be written in such way. Describe briefly how you would solve for the optimal policy functions when $\delta = 1$. How would relate these policy functions to the optimal solution to the problem in (a)?