Econ 8105-8106
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1 Two Quantitative Examples

In these notes I describe the details of the two numerical examples I did in class. The first uses the BGP assumption to generate a time series from the growth model that can be compared to US time series. The second is a more general technique, that of approximating the policy function by linearizing optimality conditions.

2 Simple Example of One Sector Model

$$\max_{\{\hat{c}_t, \hat{n}_t, \hat{\ell}_t, \hat{x}_t, \hat{k}_t\}} \quad \sum_{t=0}^{\infty} \beta^t \left[\nu \log(\hat{c}_t) + (1-\nu)\log(\hat{\ell}_t)\right]$$

subject to:

$$\hat{c}_t + \hat{x}_t \leq \hat{F}(\hat{k}_t, A_t \hat{n}_t)$$

$$\hat{k}_{t+1} \leq (1-\delta)\hat{k}_t + \hat{x}_t$$

$$\hat{n}_t + \hat{\ell}_t \leq 1$$

$\hat{k}_0$ given.

Assume that $\hat{F}(k,z) = Ak^{\alpha}z^{1-\alpha}$, $A_t = (1+g)^t$.

Substitute to see that this is equivalent to (monotonicity):

$$\max_{\{\hat{c}_t, \hat{n}_t, \hat{k}_t\}} \quad \sum_{t=0}^{\infty} \beta^t \left[\nu \log(\hat{c}_t) + (1-\nu)\log(1-\hat{n}_t)\right]$$

subject to:

$$\hat{c}_t + \hat{k}_{t+1} \leq A\hat{k}_t^{\alpha}((1+g)^t \hat{n}_t)^{1-\alpha} + (1-\delta)\hat{k}_t.$$

$\hat{k}_0$ given.

Let $c_t = \frac{\hat{c}_t}{(1+g)^t}$, $\quad k_t = \frac{\hat{k}_t}{(1+g)^t}$, $(\hat{n}_t = n_t)$ and substitute to rewrite problem as:

$\max_{\{c_t, n_t, k_t\}} \qquad \sum_{t=0}^{\infty} \beta^t \left[\nu \log((1+g)^t c_t) + (1-\nu) \log(1-n_t)\right]$

subject to:

$$(1+g)^t c_t + (1+g)^{t+1} k_{t+1} \leq A((1+g)^t k_t)^{\alpha}((1+g)^t n_t)^{1-\alpha} + (1-\delta)(1+g)^t k_t.$$

k_0 given.

Or, equivalently:

$\max_{\{c_t, n_t, k_t\}} \qquad \sum_{t=0}^{\infty} \beta^t \left[\nu t \log(1+g) + \nu \log(c_t) + (1-\nu) \log(1-n_t)\right]$

subject to:

$$c_t + (1+g) k_{t+1} \leq A k_t^{\alpha} n_t^{1-\alpha} + (1-\delta) k_t.$$

k_0 given.

Or equivalently, since $\sum_{t=0}^{\infty} \beta^t \left[\nu t \log(1+g)\right]$ is a constant:

$\max_{\{c_t, n_t, k_t\}} \qquad \sum_{t=0}^{\infty} \beta^t \left[\nu \log(c_t) + (1-\nu) \log(1-n_t)\right]$

subject to:

$$c_t + (1+g) k_{t+1} \leq A k_t^{\alpha} n_t^{1-\alpha} + (1-\delta) k_t.$$

k_0 given.

The FOC's for this problem are (with the multiplier being $\beta^t \lambda_t$):

$c_t : \qquad \frac{\beta^t \nu}{c_t} = \beta^t \lambda_t$

$n_t : \qquad \frac{\beta^t (1-\nu)}{(1-n_t)} = \beta^t \lambda_t F_n(t)$

$k_{t+1} : \qquad (1+g)\beta^t \lambda_t = \beta^{t+1} \lambda_{t+1} \left[1 - \delta + F_k(t+1)\right]$

$Feas_t : \qquad c_t + (1+g) k_{t+1} = A k_t^{\alpha} n_t^{1-\alpha} + (1-\delta) k_t.$

Where $F(k, n) = A k^{\alpha} n^{1-\alpha}$.

From Cobb-Douglas, we have that $F_n(t) = \frac{(1-\alpha)F(t)}{n_t}$ and $F_k(t) = \frac{\alpha F(t)}{k_t}$.
Substituting this and $\frac{\beta^t \nu}{c_t} = \beta^t \lambda_t$ everywhere and eliminating that equation gives:

$n_t : \qquad \frac{\beta^t (1-\nu)}{(1-n_t)} = \frac{\beta^t \nu}{c_t} \frac{(1-\alpha)F(t)}{n_t}$

$$k_{t+1}: \qquad (1+g)\frac{\beta^t \nu}{c_t} = \frac{\beta^{t+1}\nu}{c_{t+1}}\left[1-\delta+\frac{\alpha F(t+1)}{k_{t+1}}\right]$$

$$Feas_t: \qquad c_t + (1+g)k_{t+1} = Ak_t^{\alpha} n_t^{1-\alpha} + (1-\delta)k_t.$$

$\frac{(1-\alpha)F(t)}{n_t} = \frac{(1-\alpha)Ak_t^{\alpha}n_t^{1-\alpha}}{n_t} = (1-\alpha)A\left[\frac{k_t}{n_t}\right]^{\alpha}$, and $\frac{\alpha F(t+1)}{k_{t+1}} = \frac{\alpha A k_{t+1}^{\alpha} n_{t+1}^{1-\alpha}}{k_{t+1}} = \alpha A\left[\frac{n_{t+1}}{k_{t+1}}\right]^{1-\alpha}$
Simplify:

$$n_t: \qquad \frac{(1-\nu)c_t}{\nu(1-n_t)} = \frac{(1-\alpha)F(t)}{n_t} = (1-\alpha)A\left[\frac{k_t}{n_t}\right]^{\alpha}$$

$$k_{t+1}: \qquad (1+g)\frac{c_{t+1}}{c_t} = \beta\left[1-\delta+\alpha A\left[\frac{n_{t+1}}{k_{t+1}}\right]^{1-\alpha}\right]$$

$$Feas_t: \qquad c_t + (1+g)k_{t+1} = Ak_t^{\alpha} n_t^{1-\alpha} + (1-\delta)k_t.$$

Assuming that we are in a steady state, these become:

$$n: \qquad \frac{(1-\nu)c}{\nu(1-n)} = (1-\alpha)A\left[\frac{k}{n}\right]^{\alpha}$$

$$k: \qquad (1+g) = \beta\left[1-\delta+\alpha A\left[\frac{n}{k}\right]^{1-\alpha}\right]$$

$$Feas: \qquad c + (1+g)k = Ak^{\alpha}n^{1-\alpha} + (1-\delta)k.$$

Given the deep parameters, k gives n/k. Given this $Feas$ gives c/k. Given these, n gives $(1-n)/k$.

And we need $k_0 = k_{ss}$.

2.1 Parameters

Note tha the solution to the growth model above is homogeneous in (A, k_0), that is doubling A and k_0 doubles all the real variables in the solution, holding n and ℓ fixed. Thus, I will solve the model with $A = 1$ and then renormalize to match the initial conditions of the US time series:

Assume that $\beta = .96$, $\delta = .1$, $1+g = 1.02$, $\alpha = .33$, $A = 1$ (normalization).

Then k is:

$$k: \qquad 1.02 = .96\left[.9 + .33\left[\frac{n}{k}\right]^{1-.33}\right], \text{ Solution is : } \left\{\frac{n}{k} = .34738\right\},$$

$Feas: \quad \frac{c}{k} + (1+g) = A\left[\frac{n}{k}\right]^{1-\alpha} + (1-\delta).$

$Feas: \quad \frac{c}{k} + 1.02 = [.347\,38]^{1-.33} + 0.9.$, Solution is : $\left\{\frac{c}{k} = .372\,42\right\}$

$n: \quad \frac{(1-\nu)c}{\nu(1-n)} = (1-\alpha)A\left[\frac{k}{n}\right]^{\alpha}$

$n: \quad \frac{(1-\nu)\frac{c}{k}}{\nu\frac{(1-n)}{k}} = (1-\alpha)A\left[\frac{k}{n}\right]^{\alpha}$

$n: \quad \frac{(1-\nu)\frac{c}{k}}{\nu}\left[\frac{n}{k}\right]^{\alpha} = \frac{(1-n)}{k}(1-\alpha)A$

$n: \quad \frac{(1-\nu).372\,42}{\nu}[.347\,38]^{.33} = \frac{(1-n)}{k}(1-.33)$

Assume $\nu = .33$.

$n: \quad \frac{(1-.33)(.372\,42)}{.33}[.347\,38]^{.33} = \frac{(1-n)}{k}(1-.33)$, Solution is : $\left\{\frac{1-n}{k} = .796\,13\right\}$

Now, to solve for the levels:

I have $\frac{n}{k} = a$ and $\frac{(1-n)}{k} = b$, or,

$n = ak$ and $(1-n) = bk$.

So, $1 - ak = bk$, or

$1 = (a+b)k$

or,

$1 = (.347\,38 + .796\,13)k$, Solution is : $\{k = .874\,5\}$

$n = (.347\,38)(.8745)$, Solution is : $\{n = .303\,78\}$

$c = (.372\,42)(.8745)$, Solution is : $\{c = .325\,68\}$

$y = c + (1+g)k - (1-\delta)k$

So, $y = .32568 + 1.02(.8745) - .9(.8745)$, Solution is : $\{y = .430\,62\}$

$\frac{c}{y} = \frac{.32568}{.43062} = .7563.$

2.2 Solution is:

$\max_{\{\hat{c}_t,\hat{n}_t,\hat{\ell}_t,\hat{x}_t,\hat{k}_t\}} \quad \sum_{t=0}^{\infty} .96^t \left[33 \log(\hat{c}_t) + (1 - .33) \log(\hat{\ell}_t)\right]$

subject to:

$\hat{c}_t + \hat{x}_t \leq (55, 100)\hat{k}_t^{.33} \left[1.02^t \hat{n}_t\right]^{1-.67}$

$\hat{k}_{t+1} \leq (1 - .1)\hat{k}_t + \hat{x}_t$

$\hat{n}_t + \hat{\ell}_t \leq 1$

$\hat{k}_0$ given.

Is given by:

$\hat{k}_0 = (55.100)(.8745)$, Solution is : $\{k_0 = 48.185\}$

$\hat{c}_0 = (55.100)(.32568)$, Solution is : $\{\hat{c}_0 = 17.945\}$

$\hat{y}_0 = \hat{c}_0/.7563$ so, $\hat{y}_0 = 17.945/.7563$, Solution is : $\{\hat{y}_0 = 23.727\}$

$\hat{x}_0 = \hat{y}_0 - \hat{c}_0$ so $\hat{x}_0 = 23.727 - 17.945$, Solution is : $\{x_0 = 5.782\}$
$\hat{k}_t = (1.02)^t * 55,100 * .8745$

$\hat{c}_t = (1.02)^t * 55,100 * .32568$

$\hat{y}_t = (1.02)^t * 23.727$

$\hat{n}_t = .30378$

3 Approximating the Policy Function Near the Steady State

In these notes I show how to approximate the policy function for the one-sector growth model with exogenous labor supply near the steady state capital stock. The idea is to use the FOC and envelope conditions and expand them in a Taylor's series around the steady state value of the capital stock. This gives a quadratic equation in $g'(k_{ss})$ which can be solved numerically.

This approximation can then be used to describe, approximately, off steady state dynamics when the system is close to the steady state (or BGP). That is,

$$g(k) \approx g(k_{ss}) + (k - k_{ss})g'(k_{ss}) + ...$$

This is only one way to do approximations like this, there are many others. You should try some of them yourself!

3.1 Expansion around the Steady State

The FOC and Envelope conditions for the growth model are:

FOC $\quad u'(f(k) - g(k)) = \beta V'(g(k))$

ENV $\quad V'(k) = u'(f(k) - g(k))f'(k)$

Rewriting ENV at $g(k)$ gives:

ENV $\quad V'(g(k)) = u'(f(g(k)) - g(g(k)))f'(g(k))$

Thus, we have the following IDENTITY in k:

$$u'(f(k) - g(k)) \equiv \beta u'(f(g(k)) - g(g(k)))f'(g(k))$$

or,

$$H_1(k) \equiv H_2(k),$$

where,

$$H_1(k) = u'(f(k) - g(k)),$$

and,

$$H_2(k) = \beta u'(f(g(k)) - g(g(k)))f'(g(k))$$

Since this is an identity, it must also hold that $H_1'(k) \equiv H_2'(k)$ and, in particular, $H_1'(k_{ss}) = H_2'(k_{ss})$.

With a little work, we find:

$$H_1'(k_{ss}) = u''(f(k_{ss}) - g(k_{ss}))\left[f'(k_{ss}) - g'(k_{ss})\right]$$

and,

$$H_2'(k_{ss}) = \beta \times [A + B]$$

where

$$A = u''(f(g(k_{ss})) - g(g(k_{ss}))f'(g(k_{ss}))\,[f'(g(k_{ss}))g'(k_{ss}) - g'(g(k_{ss}))g'(k_{ss})]$$

and

$$B = u'(f(g(k_{ss})) - g(g(k_{ss})))f''(g(k_{ss}))g'(k_{ss}).$$

Since $g(k_{ss}) = k_{ss}$ these can be simplified to obtain:

$$A = u''(f(k_{ss}) - k_{ss})f'(k_{ss})\left[f'(k_{ss})g'(k_{ss}) - [g'(k_{ss})]^2\right]$$

and

$$B = u'(f(k_{ss}) - k_{ss})f''(k_{ss})g'(k_{ss}).$$

Thus, dividing through both H_1' and H_2', by u'', we obtain:

$$f' - g' = \beta\left[f'\left[f'g' - (g')^2\right] + \tfrac{u'(f-k_{ss})}{u''(f-k_{ss})}f''g'\right]$$

where all functions are evaluated at $k = k_{ss}$. At k_{ss}, $f' = \frac{1}{\beta}$. Making this substitution, we obtain:

$$\tfrac{1}{\beta} - g' = \beta\left[\tfrac{1}{\beta}\left[\tfrac{1}{\beta}g' - (g')^2\right] + \tfrac{u'(f-k_{ss})}{u''(f-k_{ss})}f''g'\right].$$

Rearranging terms, we obtain:

$$(g')^2 - \left(\tfrac{1}{\beta} + 1 + \beta Z\right)g' + \tfrac{1}{\beta} = 0,$$

where,

$$Z = \tfrac{u'(f(k_{ss})-k_{ss})}{u''(f(k_{ss})-k_{ss})}f''(k_{ss}).$$

Note that $Z > 0$.

This is of the form:

$$a\,(g')^2 - bg' + c = 0,$$

where, $a = 1$, $b = \left(\frac{1}{\beta} + 1 + \beta Z\right)$ and $c = \frac{1}{\beta}$.

Note that this is positive at $g' = 0$ and the slope at $g' = 0$ is negative. It follows that all roots, should any exist, are postive.

What do we know about the roots of this quadratic? For one thing, we know that the true value of $g'(k_{ss})$ is one of the roots. However there are two roots, how do we know which one is the correct one?

Recall from the theory we know that the system is globally stable, this implies that $g'(k_{ss}) < 1$. We will show that there is only one root satisfying this restriction. To see this, use the quadratic formula, to obtain the two roots:

$$g' = \frac{-b \pm \sqrt[2]{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt[2]{b^2 - 4/\beta}}{2}.$$

Call these two roots r_1 and r_2. Then, we have:

$$r_1 + r_2 = \frac{-b + \sqrt[2]{b^2 - 4/\beta}}{2} + \frac{-b - \sqrt[2]{b^2 - 4/\beta}}{2} = -\frac{2b}{2} = -b.$$

From above,

$$-b = \frac{1}{\beta} + 1 + \beta Z > \frac{1}{\beta} + 1 > 2$$

where the first inequality follows since $Z > 0$ and the second follows since $1/\beta > 1$.

Thus, since one of the roots we know is less than one, the other must be larger than one. And thus, it follows that we want the smaller of the two roots:

$$g'(k_{ss}) = \frac{-b - \sqrt[2]{b^2 - 4/\beta}}{2}.$$

(It can also be shown that $r_1 * r_2 = \frac{1}{\beta}$.)

3.2 Finding Z

To go any further than this, we need to know Z.

$$Z = \frac{u'(f(k_{ss}) - k_{ss})}{u''(f(k_{ss}) - k_{ss})} f''(k_{ss}) = \frac{u'(c_{ss})}{u''(c_{ss})} f''(k_{ss}).$$

Assuming that $u(c) = c^{1-\sigma}/(1-\sigma)$, we see that $u' = c^{-\sigma}$, and $u'' = -\sigma c^{-\sigma-1}$. Thus,

$$\frac{u'}{u''} = \frac{c^{-\sigma}}{-\sigma c^{-\sigma-1}} = -\frac{c}{\sigma}.$$

Assuming that $f(k) = Ak^\alpha + (1-\delta)k$, we have that $f' = \alpha A k^{\alpha-1} + (1-\delta)$ and $f'' = \alpha(\alpha-1)Ak^{\alpha-2} = -\alpha(1-\alpha)Ak^{\alpha-2}$.

To find k_{ss} use the fact that $\frac{1}{\beta} = f'(k_{ss})$ and so:

$$\frac{1}{\beta} = \alpha A k_{ss}^{\alpha-1} + (1-\delta).$$

Thus,

$$\alpha A k_{ss}^{\alpha-1} = \frac{1}{\beta} + \delta - 1, \text{ or,}$$

$$k_{ss} = \left[\frac{1}{\alpha A}\left(\frac{1}{\beta} + \delta - 1\right)\right]^{1/(\alpha-1)}.$$

From this it follows that

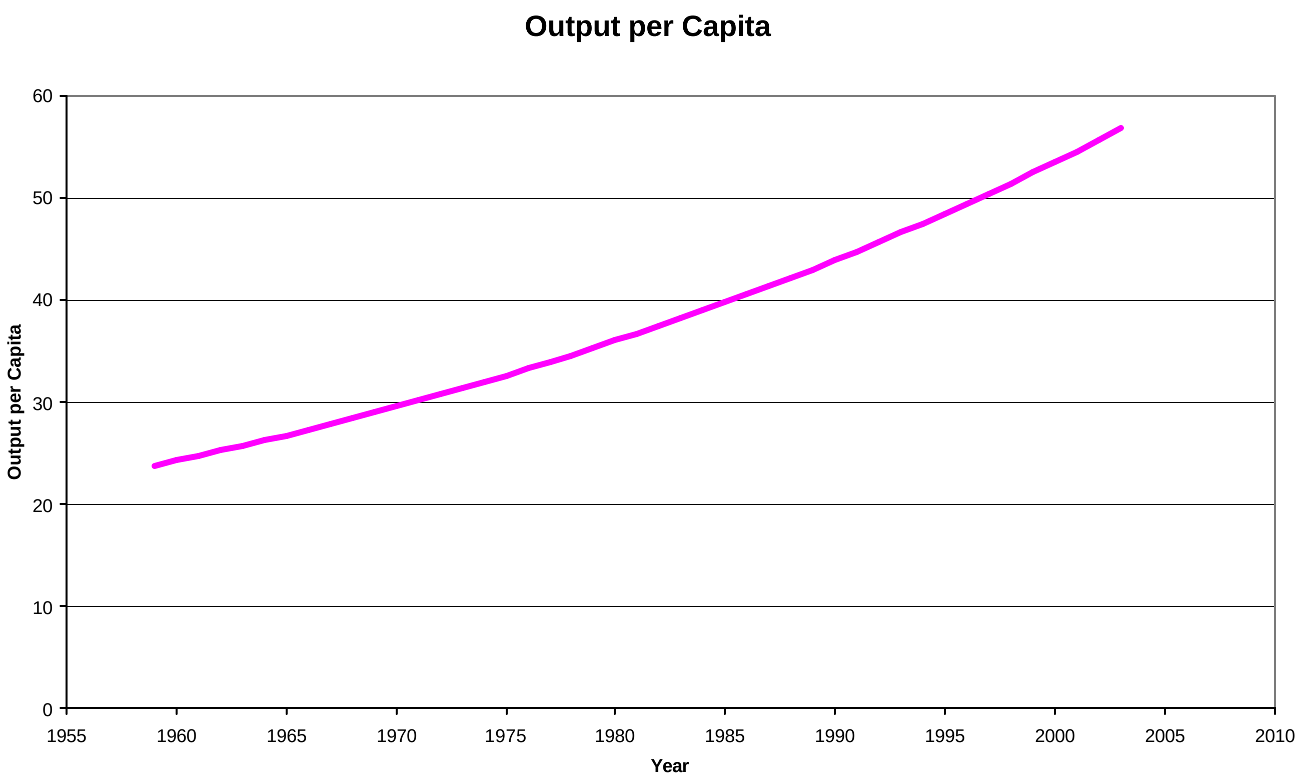
$$c_{ss} = y_{ss} - \delta k_{ss} = A k_{ss}^{\alpha} - \delta k_{ss}.$$

Thus,

$$Z = \frac{\alpha(1-\alpha) A k_{ss}^{\alpha-2} c_{ss}}{\sigma}.$$

From the expressions above, once α, σ, β, δ and A are chosen, Z can be calculated directly.

4 Figures for the Examples


Output per Capita
60
50
40
30
20
10
0
Output per Capita
1955
1960
1965
1970
1975
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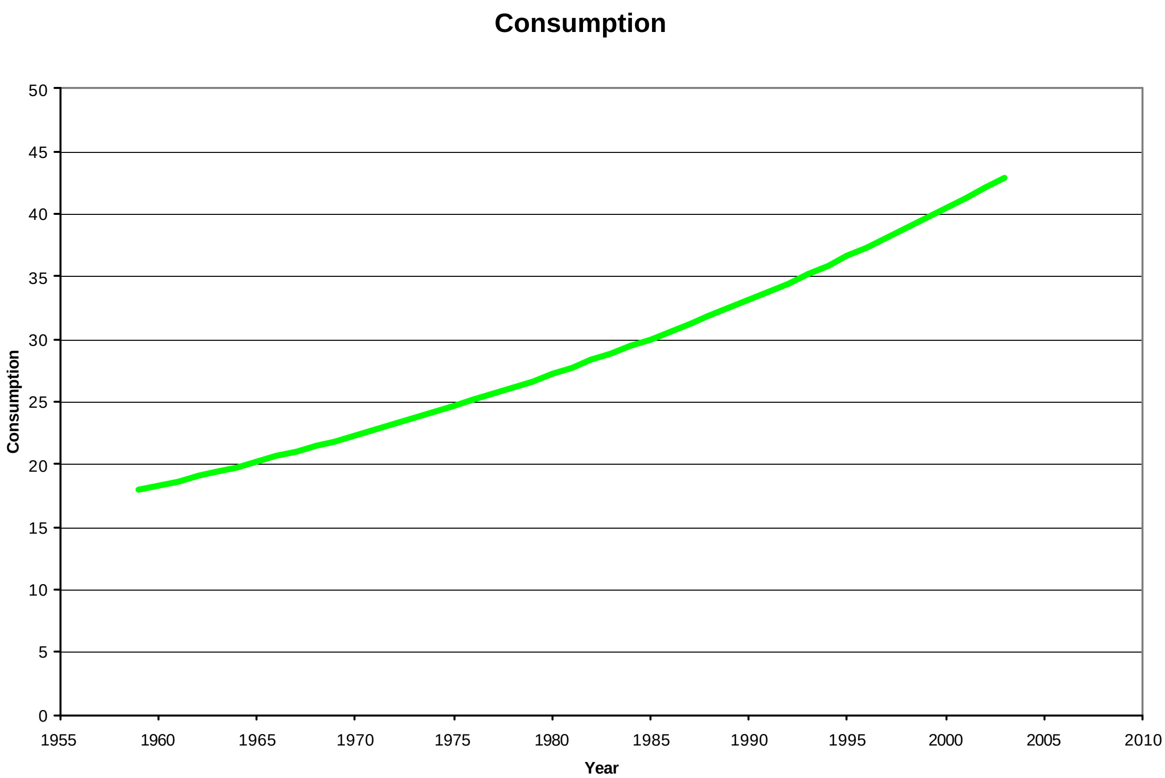
2000
2005
2010
Year

Figure 1:


Consumption
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Consumption
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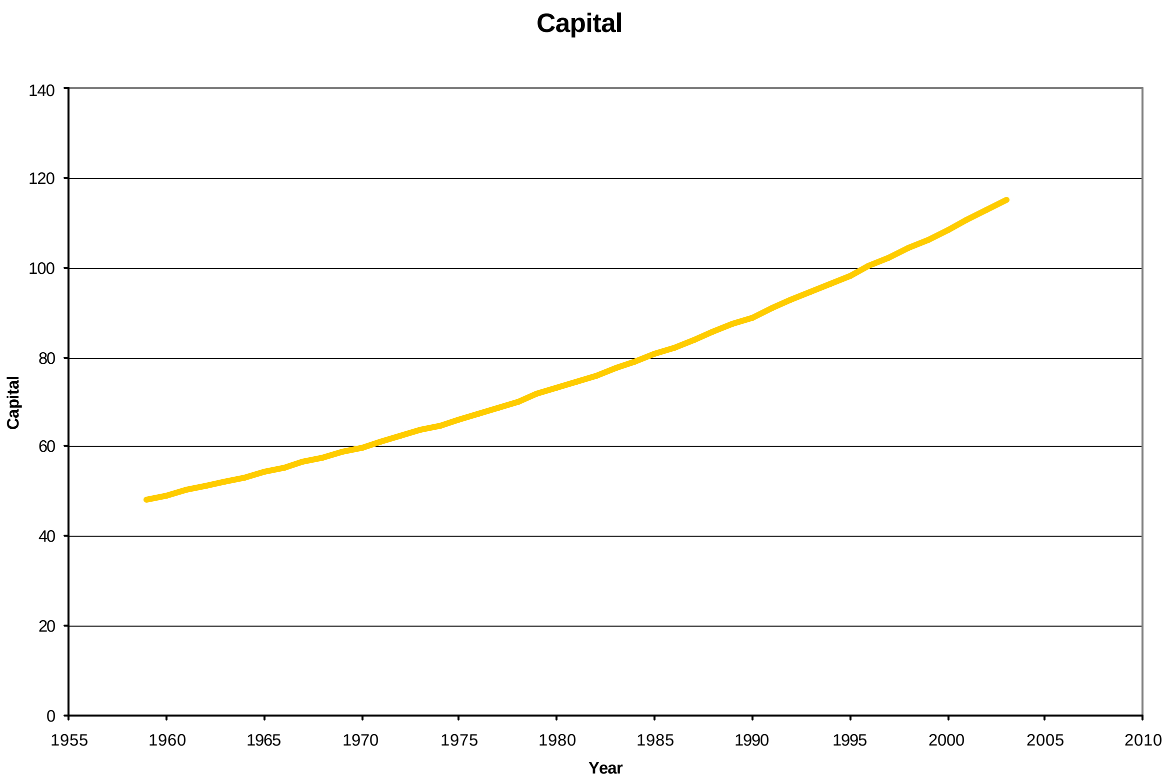
Year

Figure 2:



Capital
140
120
100
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Capital
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Year

Figure 3:

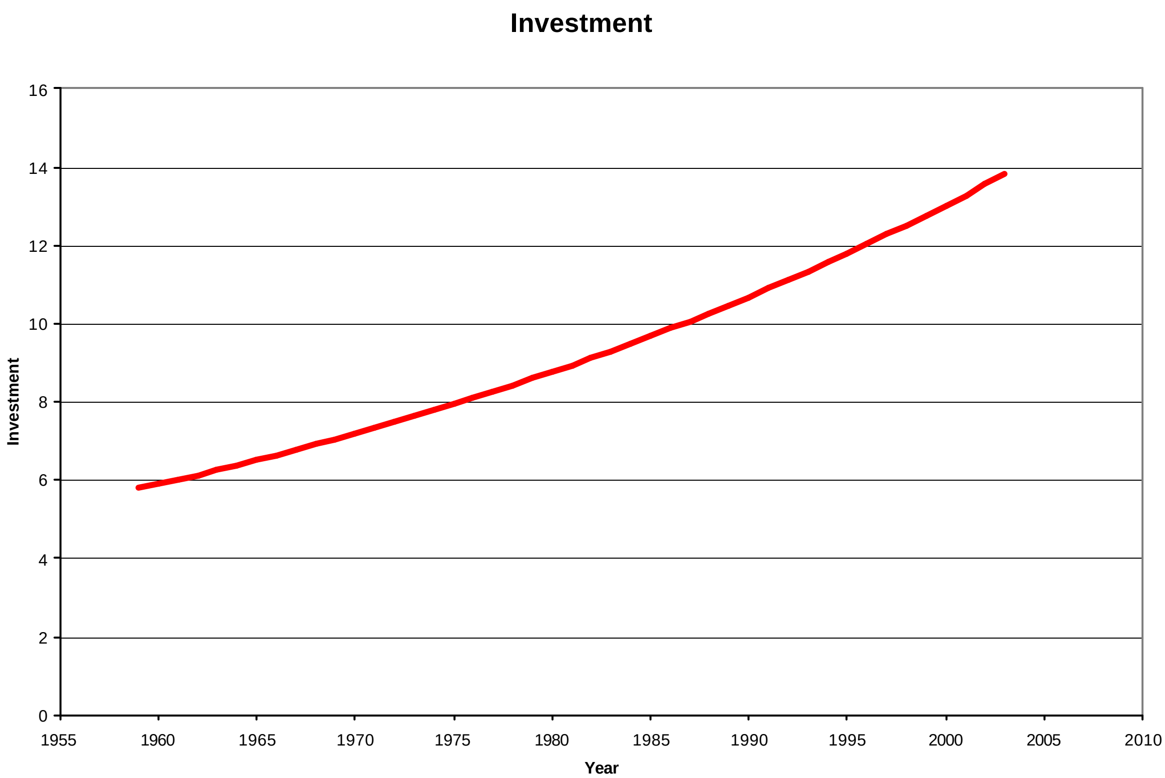
Investment
Investment
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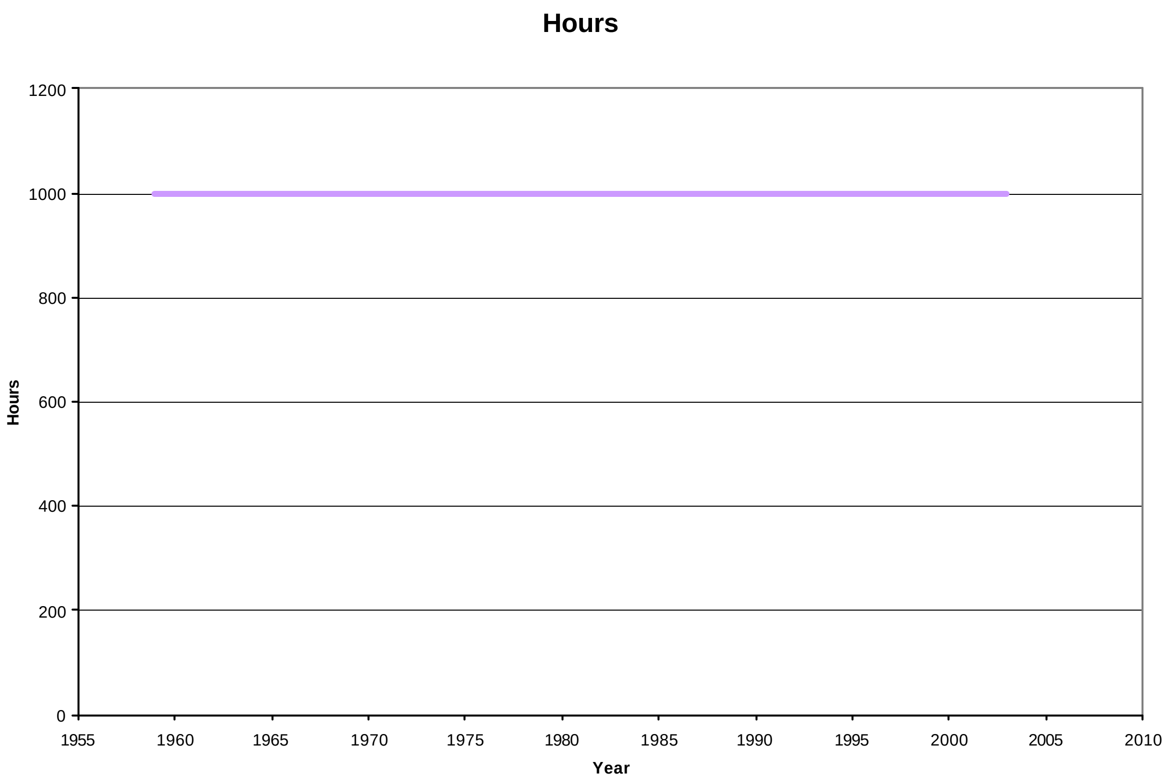
Figure 4:


Hours
1200
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400
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Hours
1955
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2010
Year

Figure 5:

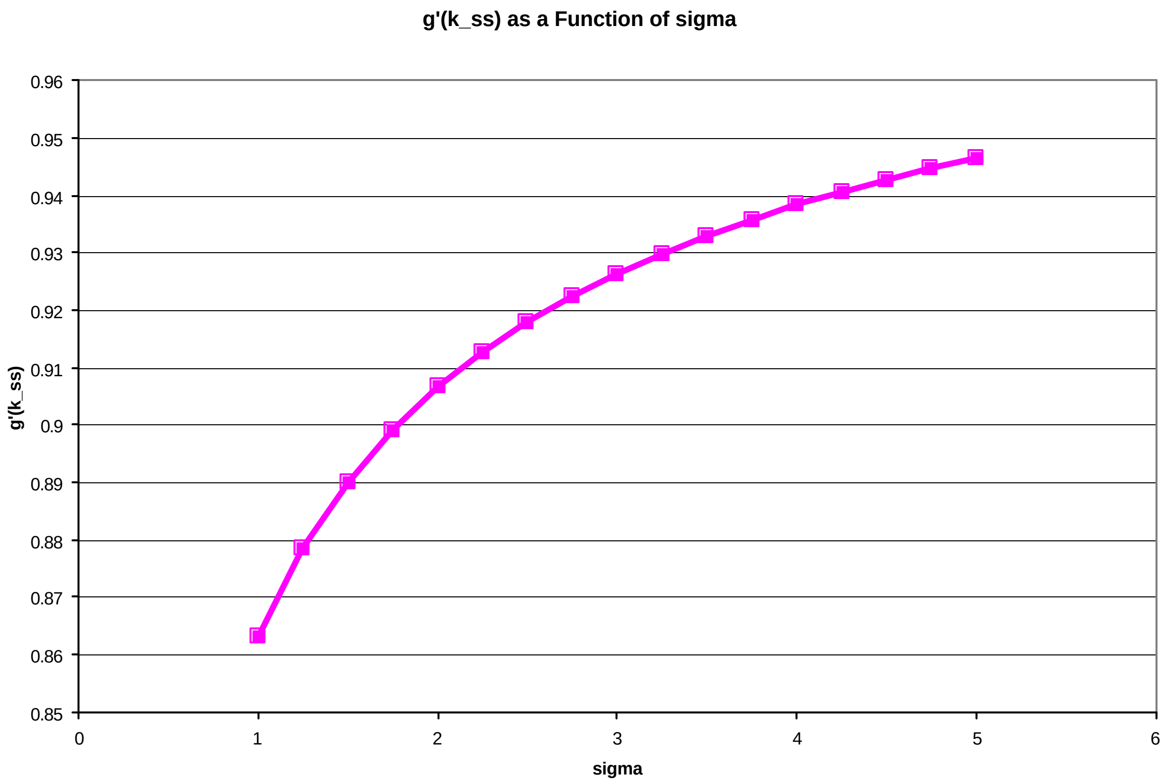
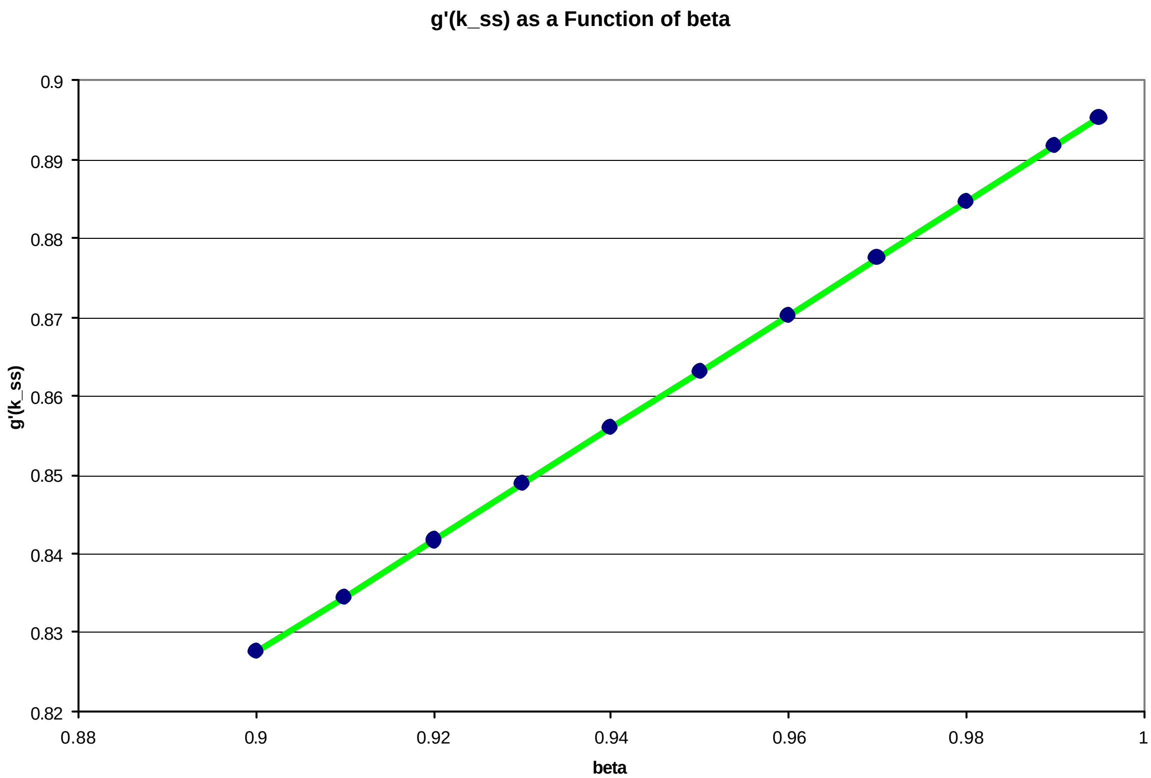

g'(k_ss) as a Function of sigma
g'(k_ss)
0.96
0.95
0.94
0.93
0.92
0.91
0.9
0.89
0.88
0.87
0.86
0.85
0
1
2
3
4
5
6
sigma

Figure 6:


g'(k_ss) as a Function of beta
g'(k_ss)
beta
0.9
0.89
0.88
0.87
0.86
0.85
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0.83
0.82
0.88
0.9
0.92
0.94
0.96
0.98
1

Figure 7: