

Selection, Growth, and the Size Distribution of Firms

—Technical Appendix—

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Abstract

This proves Lemma 2 and Proposition 3 in “Selection, Growth and the Size Distribution of Firms,” *Quarterly Journal of Economics*, forthcoming, referred to hereafter as “the paper.” In addition, the selection of the stationary distribution defined in Lemma 2 is discussed. Some omitted calculations for other parts of the paper are also reported.

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This contains detailed algebra for Section 6.

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1. DEFINITIONS

The notation is as in the paper, except that the size measure s is everywhere replaced by $x = s - b$. Also, λ and θ appear in this appendix without subscripts, and these are not the same parameters as the subscripted parameters in the paper.

1.1 The Density of Entrants

The stationary density constructed in Section V.A. of the paper is:

$$f(x) = \zeta^2 x e^{-\zeta x}$$

where $x \in [0, \infty)$ and:

$$\zeta = -\left(\frac{\mu}{\sigma^2} + \frac{1}{\delta}\right) + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{1}{\delta^2} + \frac{\eta}{\sigma^2/2}} \quad (1)$$

and recall that equilibrium conditions imply $\zeta > 1$. Note that:

$$\mu + \zeta\sigma^2 = -\frac{\sigma^2}{\delta} + \sqrt{\mu^2 + \left(\frac{\sigma^2}{\delta}\right)^2 + 2\sigma^2\eta} > 0$$

since η is assumed to be non-negative. The density of entrants can be written as:

$$\frac{f(x+\delta)}{\int_0^\infty f(x+\delta)dx} = \left(\frac{\zeta\delta}{1+\zeta\delta}\right) f_e(x) + \left(\frac{1}{1+\zeta\delta}\right) f_g(x) \quad (2)$$

where:

$$f_e(x) = \zeta e^{-\zeta x}, \quad f_g(x) = \zeta^2 x e^{-\zeta x}$$

are exponential and gamma densities.

1.2 The Conditional Survivor Function

Section III.A. of the paper shows that the survivor function conditional on the initial state x is given by:

$$\Lambda(a|x) = \Phi\left(\frac{x+\mu a}{\sigma\sqrt{a}}\right) - \exp\left(-\frac{\mu x}{\sigma^2/2}\right) \Phi\left(\frac{-x+\mu a}{\sigma\sqrt{a}}\right) \quad (3)$$

The conditional hazard rate $h(a|x) = -D\Lambda(a|x)/\Lambda(a|x)$ is:

$$h(a|x) = \frac{\frac{x-1}{a\sigma\sqrt{a}}\phi\left(\frac{x+\mu a}{\sigma\sqrt{a}}\right)}{\Phi\left(\frac{x+\mu a}{\sigma\sqrt{a}}\right) - \exp\left(-\frac{\mu x}{\sigma^2/2}\right) \Phi\left(\frac{-x+\mu a}{\sigma\sqrt{a}}\right)}$$

This conditional hazard rate is non-monotone in age. One can verify that:

$$\lim_{a \rightarrow \infty} h(a|x) = \frac{1}{2} \left(\left[\frac{-\mu}{\sigma} \right]^+ \right)^2 \quad (4)$$

This is the underlying reason for (18) below.

1.3 Auxiliary Parameters and Variables

Define the variables:

$$u = -\frac{\mu a}{\sigma \sqrt{a}}, \quad v = \frac{[\mu + \zeta \sigma^2] a}{\sigma \sqrt{a}}, \quad w = \frac{\mu a}{\sigma \sqrt{a}} \quad (5)$$

Thus $v > 0$, $u > 0$ if $\mu < 0$, and $w > 0$ if $\mu > 0$. Note that $(a/u)\partial u/\partial a$, $(a/v)\partial v/\partial a$ and $(a/w)\partial w/\partial a$ are all equal to $1/2$.

Define the parameters:

$$\chi = \frac{1}{\frac{1}{2}\sigma^2} \left[\mu + \frac{1}{2}\sigma^2\zeta \right] \quad (6)$$

and:

$$\lambda = \frac{\zeta}{\chi} \quad (7)$$

and:

$$\omega = \frac{1}{2} \frac{u^2 - v^2}{v^2} \quad (8)$$

Note that ω does not depend on a .

The following relations will be used below:

$$-\mu\chi + \frac{1}{2}\sigma^2\chi^2 = \left(\mu + \frac{1}{2}\sigma^2\zeta \right) \zeta \quad (9)$$

$$-\mu + \sigma^2\chi = \mu + \sigma^2\zeta \quad (10)$$

and:

$$\frac{u^2 - v^2}{2a} = - \left(\mu + \frac{1}{2}\zeta\sigma^2 \right) \zeta \quad (11)$$

and:

$$\frac{u}{v} = \frac{\lambda^2 - 1}{(\lambda + 1)^2} = \frac{1}{2} \frac{\omega}{\lambda} (1 - \lambda^2) \quad (12)$$

and:

$$2uv = -2wv = -(\zeta + \chi)\mu a \quad (13)$$

The parameter λ is only well defined if $\chi \neq 0$. Note that $u^2 = v^2$ precisely when $\chi = 0$. The conclusions about survivor functions and hazard rates derived below continue to hold in this limit case.

2. MISCELLANEOUS

This section computes the normalizing constant for the stationary density derived in Section III of the paper. It also computes the exit rate used in Section V.

2.1 Normalizing Constant for the Age-Size Density

Take $p > 0$ and $q > 0$. For any $T > 0$:

$$\int_0^T \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{p+qt^2}{2t}\right) dt = \frac{1}{\sqrt{q}} \left[e^{-\sqrt{pq}} \Phi\left(\frac{-\sqrt{p}+T\sqrt{q}}{\sqrt{T}}\right) - e^{\sqrt{pq}} \Phi\left(\frac{-\sqrt{p}-T\sqrt{q}}{\sqrt{T}}\right) \right]$$

This yields:

$$\int_0^\infty \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{p+qt^2}{2t}\right) dt = \frac{1}{\sqrt{q}} e^{-\sqrt{pq}}$$

From Section III of the paper, recall:

$$\psi(a, y|x) = \frac{1}{\sigma\sqrt{a}} \left[\phi\left(\frac{y-x-\mu a}{\sigma\sqrt{a}}\right) - \exp\left(-\frac{\mu x}{\sigma^2/2}\right) \phi\left(\frac{y+x-\mu a}{\sigma\sqrt{a}}\right) \right] \quad (14)$$

The normalizing constant in (18) of the paper follows from:

$$\int_0^\infty e^{-\eta a} \psi(a, y|x) da = I_1 - e^{-\frac{\mu x}{\sigma^2/2}} I_2$$

where:

$$I_1 = \int_0^\infty \frac{e^{-\eta a}}{\sigma\sqrt{a}} \phi\left(\frac{y-x-\mu a}{\sigma\sqrt{a}}\right) da, \quad I_2 = \int_0^\infty \frac{e^{-\eta a}}{\sigma\sqrt{a}} \phi\left(\frac{y+x-\mu a}{\sigma\sqrt{a}}\right) da$$

and note that:

$$\frac{\alpha + \alpha_*}{2} = \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}, \quad \alpha - \alpha_* = -\frac{\mu}{\sigma^2/2}, \quad \alpha\alpha_* = \frac{\eta}{\sigma^2/2}$$

Then:

$$\begin{aligned} I_1 &= \int_0^\infty \frac{e^{-\eta a}}{\sigma\sqrt{2\pi a}} \exp\left(-\frac{1}{2}\left(\frac{y-x-\mu a}{\sigma\sqrt{a}}\right)^2\right) da \\ &= \frac{1}{\sigma} \exp\left(\frac{(y-x)\mu}{\sigma^2}\right) \int_0^\infty \frac{1}{\sqrt{2\pi a}} \exp\left(-\frac{(y-x)^2 + (\mu^2 + 2\sigma^2\eta)a^2}{2\sigma^2 a}\right) da \\ &= \frac{1}{\sigma} \exp\left(\frac{(y-x)\mu}{\sigma^2}\right) \frac{1}{\sqrt{\frac{\mu^2 + 2\sigma^2\eta}{\sigma^2}}} \exp\left(-\frac{|y-x|}{\sigma} \sqrt{\frac{\mu^2 + 2\sigma^2\eta}{\sigma^2}}\right) \\ &= \frac{\min\{e^{\alpha_*(y-x)}, e^{-\alpha(y-x)}\}}{\sqrt{\mu^2 + 2\sigma^2\eta}} \end{aligned}$$

and:

$$\begin{aligned}
I_2 &= \int_0^\infty \frac{e^{-\eta a}}{\sigma\sqrt{2\pi a}} \exp\left(-\frac{1}{2}\left(\frac{y+x-\mu a}{\sigma\sqrt{a}}\right)^2\right) da \\
&= \frac{1}{\sigma} \exp\left(\frac{(y+x)\mu}{\sigma^2}\right) \int_0^\infty \frac{1}{\sqrt{2\pi a}} \exp\left(-\frac{(y+x)^2 + (\mu^2 + 2\sigma^2\eta)a^2}{2\sigma^2 a}\right) da \\
&= \frac{1}{\sigma} \exp\left(\frac{(y+x)\mu}{\sigma^2}\right) \frac{1}{\sqrt{\frac{\mu^2 + 2\sigma^2\eta}{\sigma^2}}} \exp\left(-\frac{|y+x|}{\sigma} \sqrt{\frac{\mu^2 + 2\sigma^2\eta}{\sigma^2}}\right) \\
&= \frac{\exp(-\alpha[y+x])}{\sqrt{\mu^2 + 2\sigma^2\eta}}
\end{aligned}$$

The last line relies on $y+x > 0$. Combining these two integrals, gives:

$$\int_0^\infty e^{-\eta a} \psi(a, y|x) da = \frac{1}{-\mu} \frac{\alpha - \alpha_*}{\alpha + \alpha_*} \frac{\min\{e^{[\alpha+\alpha_*]y}, e^{[\alpha+\alpha_*]x}\} - 1}{e^{\alpha_* x} e^{\alpha y}}$$

This determines the shape of the marginal density of size given initial size. Integrating over y gives:

$$\int_0^\infty \left(\frac{\min\{e^{[\alpha+\alpha_*]y}, e^{[\alpha+\alpha_*]x}\} - 1}{e^{\alpha y}}\right) dy = \frac{\alpha + \alpha_*}{\alpha\alpha_*} [e^{\alpha_* x} - 1]$$

Therefore:

$$\int_0^\infty \int_0^\infty e^{-\eta a} \psi(a, y|x) da dy = \frac{1 - e^{-\alpha_* x}}{\eta}$$

This determines the normalizing constant for the age-size density.

2.2 Cohort and Aggregate Exit Rates

Take $x > 0$ and suppose:

$$x_t = x + \mu t + \sigma W_t$$

Let $y_t = x_t$ as long as x_t has never reached zero, and let $y_t = 0$ otherwise. Define $p(t, y)$ to be the density of y_t . For all positive t and y , this density satisfies the Kolmogorov Forward Equation:

$$D_t p(t, y) = -\mu D_y p(t, y) + \frac{1}{2} \sigma^2 D_{yy} p(t, y)$$

Also, $p(t, 0) = 0$ and $\lim_{y \rightarrow \infty} p(t, y) = 0$. The fraction sample paths that has not reached zero by time t is:

$$S(t) = \int_0^\infty p(t, y) dy$$

Then:

$$DS(t) = \int_0^\infty D_t p(t, y) dy = \int_0^\infty \left[-\mu D_y p(t, y) + \frac{1}{2} \sigma^2 D_{yy} p(t, y) \right] dy = -\frac{1}{2} \sigma^2 D_y p(t, 0) \quad (15)$$

Thus the probability of a sample path not having reached zero declines at a rate determined by the slope of $p(t, y)$ at the zero exit barrier.

In Section V of the paper, the size density $f(s)$ is a weighted average of cohort size densities. The aggregate exit rate is an average of the cohort exit rates, constructed using the same weights. For each of the cohorts, the rate of exit is determined by the slope of the cohort size density at the exit barrier b , as in (15). This implies that the aggregate exit rate is given by $\frac{1}{2} \sigma^2 Df(b)$.

3. THE CONSTANT HAZARD RATE RESULT

The following proves the first part of Proposition 3 in the paper.

Consider a process $x_t = x + \mu t + \sigma W_t$ where $x > 0$. Let $\{\mathcal{F}_t\}_{t \geq 0}$ be the filtration generated by $\{x_t\}_{t \geq 0}$ and define T to be the first hitting time for 0. The survivor function is defined as:

$$S(a|x) = \Pr [T \geq a | x_0 = x]$$

Let y_t be the process x_t killed at 0:

$$y_t = x_t \mathbf{1}_{[T > t]}$$

For any $a > t$, this gives:

$$S(a - t | y_t) = \Pr [T \geq a | \mathcal{F}_t]$$

The right-hand side is a martingale. Its drift must be zero. Ito's lemma implies:

$$D_a S(a|x) = \mu D_x S(a|x) + \frac{1}{2} \sigma^2 D_{xx} S(a|x)$$

Thus $S(a|x)$ satisfies the Kolmogorov Backward Equation. One boundary condition is $S(a|0) = 0$ and another is $S(0|x) = 1$ for $x > 0$. These results can be verified directly using (3).

Suppose the initial conditions are given by some smooth density $g(x)$ that satisfies $g(0) = 0$. We are interested in the hazard rate of:

$$S(a) = \int_0^\infty S(a|x) g(x) dx$$

Integrate the backward equation for $S(a|x)$ against the initial conditions to obtain:

$$\begin{aligned}
\int_0^\infty D_a S(a|x)g(x)dx &= \mu \int_0^\infty D_x S(a|x)g(x)dx + \frac{1}{2}\sigma^2 \int_0^\infty D_{xx} S(a|x)g(x)dx \\
&= -\mu \int_0^\infty S(a|x)Dg(x)dx - \frac{1}{2}\sigma^2 \int_0^\infty D_x S(a|x)Dg(x)dx \\
&= \int_0^\infty S(a|x) \left[-\mu Dg(x) + \frac{1}{2}\sigma^2 D^2g(x) \right] dx \tag{16}
\end{aligned}$$

The boundary terms disappear because $g(0) = 0$ and $S(a|0) = 0$.

Suppose the density g is smooth and satisfies $g(0) = 0$ and:

$$-\mu Dg(x) + \frac{1}{2}\sigma^2 D^2g(x) = -\vartheta g(x) \tag{17}$$

for some ϑ . That is, g is an eigenvector of the forward differential operator. Then (16) and (17) imply:

$$DS(a) = \int_0^\infty D_a S(a|x)g(x)dx = -\vartheta \int_0^\infty S(a|x)g(x)dx = -\vartheta S(a)$$

so that $\vartheta = -DS(a)/S(a)$ is the constant hazard rate of $S(a)$.

The stationary size distribution must satisfy equation (23) in Section V.A. of the paper. When $\delta = 0$, this equation is just (17) with $\vartheta = \varepsilon_A - \eta$. The resulting hazard rate is therefore $\varepsilon_A - \eta$. This equals $\varepsilon_S - \eta$ because all entry attempts are successful in this case. The limit of (1) as δ goes to zero is $\zeta = -\mu/\sigma^2$. Lemma 2 in the paper then gives $\varepsilon_A - \eta = \frac{1}{2}(\mu/\sigma)^2$. Since $\mu < 0$ in this case, this matches the limiting hazard rate shown in (4).

4. MILL'S RATIO AND RELATED EXPRESSIONS

Define:

$$\phi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}, \quad \Phi(x) = \int_{-\infty}^x \phi(z)dz$$

and:

$$\Psi(x) = \frac{x\Phi(-x)}{\phi(x)}$$

Observe:

$$\begin{aligned}
\int_0^\infty \frac{x}{s} \phi\left(\frac{x-m}{s}\right) dx &= s \int_{-m/s}^\infty z \phi(z) dz + m \int_{-m/s}^\infty \phi(z) dz \\
&= s \phi\left(\frac{m}{s}\right) + m \Phi\left(\frac{m}{s}\right) \\
&= m \left[1 - \Psi\left(\frac{m}{s}\right) \right] \frac{\phi\left(\frac{m}{s}\right)}{\frac{m}{s}}
\end{aligned}$$

Mill's ratio is defined as:

$$R(x) = \frac{\Phi(-x)}{\phi(x)} = e^{x^2/2} \int_x^\infty e^{-z^2/2} dz$$

Define also:

$$P(x) = 1 - \Psi(x), \quad Q(x) = [1 - \Psi(x)]^{-1}$$

Lemma 1: *The functions $R(x)$ and $\Psi(x)$ satisfy:*

$$DR(x) < 0, \quad D\Psi(x) > 0$$

and:

$$\lim_{x \rightarrow \infty} R(x) = 0, \quad \lim_{x \rightarrow \infty} \Psi(x) = 1$$

Proof: Change of variables $y = z^2 - x^2$ and note that $dy/dz = 2z = 2\sqrt{y + x^2}$.
Therefore:

$$\begin{aligned} R(x) &= e^{x^2/2} \int_x^\infty e^{-z^2/2} dz = \frac{1}{2} \int_0^\infty \frac{e^{-z/2}}{\sqrt{x^2 + z}} dz \\ \Psi(x) &= xe^{x^2/2} \int_x^\infty e^{-z^2/2} dz = \frac{1}{2} \int_0^\infty \frac{e^{-z/2}}{\sqrt{1 + \frac{z}{x^2}}} dz \end{aligned}$$

The result then follows from the dominated convergence theorem. ■

Lemma 2: *The function $P(x)$ satisfies $P(0) = 1$,*

$$DP(x) < 0$$

and:

$$\lim_{x \rightarrow \infty} x^2 P(x) = 1$$

Proof: The first two results follow immediately from the properties of $\Psi(x)$. Observe that:

$$x^2 P(x) = x^2 \left(1 - \int_0^\infty \frac{e^{-z/2}}{2\sqrt{1 + \frac{z}{x^2}}} dz \right) = \frac{1}{2} \int_0^\infty \left(\frac{\sqrt{x^4 + zx^2 - x^2}}{\sqrt{1 + \frac{z}{x^2}}} \right) e^{-z/2} dz$$

and:

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^4 + zx^2} - x^2}{\sqrt{1 + \frac{z}{x^2}}} = \frac{z}{2}$$

$$\frac{1}{4} \int_0^\infty ze^{-z/2} dz = 1$$

The desired result again follows from the dominated convergence theorem. ■

Lemma 3: *The function $P(x)$ satisfies:*

$$\frac{\partial}{\partial x} \left(\frac{1}{P(x)} - x^2 \right) > 0$$

and this implies:

$$\frac{\partial}{\partial x} \left[\frac{P(\theta x)}{P(x)} \right] > 0$$

for all $\theta \in (0, 1)$.

Proof: To see the second part, note that:

$$DP(x) = -(x + x^{-1}) [1 - P(x)] + x = -\frac{1}{x} [1 - (1 + x^2) P(x)]$$

and therefore:

$$\frac{xDP(x)}{P(x)} = - \left[\frac{1}{P(x)} - (1 + x^2) \right]$$

The result then follows from:

$$\frac{\partial}{\partial x} \left[\frac{P(\theta x)}{P(x)} \right] = \left[\frac{\theta DP(\theta x)}{P(\theta x)} - \frac{DP(x)}{P(x)} \right] \frac{P(\theta x)}{P(x)}$$

To see the first part, write:

$$\frac{1}{P(x)} - x^2 = \frac{1}{1 - xR(x)} - x^2$$

and observe that:

$$DR(x) = -[1 - xR(x)]$$

Thus we want to show that:

$$(1 + x^2) R(x) - x - 2x [1 - xR(x)]^2 \geq 0$$

or:

$$\frac{8x^2 + (1 + x^2)^2}{(4x^3)^2} = \left(\frac{1 + 5x^2}{4x^3} \right)^2 - \frac{3}{2x^2} \geq \left(R(x) - \left(\frac{1 + 5x^2}{4x^3} \right) \right)^2$$

This requires that:

$$\frac{1}{4x^3} \left(1 + 5x^2 - \sqrt{1 + 10x^2 + x^4} \right) \leq R(x) \leq \frac{1}{4x^3} \left(1 + 5x^2 + \sqrt{1 + 10x^2 + x^4} \right)$$

The upper bound is easy to show using $R(x) \leq 1/x$. The lower bound is shown in (44). ■

Lemma 4: As a function of y ,

$$\left(1 + \frac{1}{2} \left(\frac{x^2}{y^2} - 1 \right) [Q(y) - (1 + y^2)] \right) Q(y)$$

is decreasing in $y < x$ and increasing in $y > x$.

Proof: The slope is:

$$\begin{aligned} & \left(1 + \frac{1}{2} \left(\frac{x^2}{y^2} - 1 \right) [Q(y) - (1 + y^2)] \right) DQ(y) \\ & + \frac{1}{2} \left(\frac{x^2}{y^2} - 1 \right) (DQ(y) - 2y) Q(y) - \left(\frac{x^2}{y^3} \right) [Q(y) - (1 + y^2)] Q(y) \end{aligned}$$

Note that:

$$DQ(y) = Q(y) [y^{-1}Q(y) - (y + y^{-1})]$$

Thus for $y < x$ we need to check that:

$$\begin{aligned} & \left(1 + \frac{1}{2} \left(\frac{x^2}{y^2} - 1 \right) [Q(y) - (1 + y^2)] \right) Q(y) [y^{-1}Q(y) - (y + y^{-1})] \\ & + \frac{1}{2} \left(\frac{x^2}{y^2} - 1 \right) (Q(y) [y^{-1}Q(y) - (y + y^{-1})] - 2y) Q(y) - \left(\frac{x^2}{y^3} \right) [Q(y) - (1 + y^2)] Q(y) \leq 0 \end{aligned}$$

or:

$$\begin{aligned} & \frac{1}{2} \left(\frac{x^2}{y^2} - 1 \right) [Q(y) - (1 + y^2)]^2 \\ & + \frac{1}{2} \left(\frac{x^2}{y^2} - 1 \right) (Q(y) [Q(y) - (1 + y^2)] - 2y^2) - \left(\frac{x^2}{y^2} - 1 \right) [Q(y) - (1 + y^2)] \leq 0 \end{aligned}$$

Canceling a factor $(x^2 - y^2)/2 > 0$ gives:

$$3 + 2y^2 + y^4 - (5 + 3y^2) Q(y) + 2[Q(y)]^2 \leq 0$$

This is equivalent to:

$$\frac{1}{4} \left(5 + 3y^2 - \sqrt{1 + 14y^2 + y^4} \right) \leq Q(y) \leq \frac{1}{4} \left(5 + 3y^2 + \sqrt{1 + 14y^2 + y^4} \right)$$

This is shown in (45). This condition also implies the result for $y > x$. ■

Lemma 5: *The function P satisfies:*

$$\lim_{x \rightarrow \infty} \left(\frac{P(x)}{1 - (1 + x^2)P(x)} \right) = \frac{1}{2}$$

Proof: Write this as:

$$\frac{P(x)}{1 - (1 + x^2)P(x)} = \frac{x^2 P(x)}{x^2 (1 - x^2 [1 - xR(x)]) - x^2 P(x)}$$

From Lemma 2 we know that $x^2 P(x)$ converges to 1. The following continued fraction bounds for $\Psi(x) = xR(x)$ hold:

$$L(x) = \frac{x^2(x^2 + 5)}{x^4 + 6x^2 + 3} = \frac{x}{x + \frac{1}{x + \frac{2}{3}}} \leq \Psi(x) \leq \frac{x}{x + \frac{1}{x + \frac{2}{x}}} = \frac{x^2 + 2}{x^2 + 3} = U(x)$$

Furthermore:

$$L(x) \in (0, 1) \text{ and } U(x) \in (0, 1)$$

This implies:

$$1 - x^2 [1 - L(x)] \leq 1 - x^2 [1 - \Psi(x)] \leq 1 - x^2 [1 - U(x)]$$

In addition:

$$1 - x^2 [1 - L(x)] \geq 0$$

and therefore:

$$x^2 (1 - x^2 [1 - L(x)]) \leq x^2 (1 - x^2 [1 - \Psi(x)]) \leq x^2 (1 - x^2 [1 - U(x)])$$

Now:

$$\lim_{x \rightarrow \infty} x^2 (1 - x^2 [1 - L(x)]) = \lim_{x \rightarrow \infty} x^2 (1 - x^2 [1 - U(x)]) = 3$$

From this the result follows. ■

5. WEIGHTED SURVIVOR FUNCTIONS

The hazard rate of a survivor function that is a weighted average of survivor functions with downward sloping hazards is downward sloping. To see this, let $S_0(a)$ and $S_1(a)$ be two survivor functions, and consider the survivor function:

$$S(a) = \omega S_0(a) + (1 - \omega)S_1(a)$$

where $\omega \in [0, 1]$. The resulting hazard rate is:

$$-\frac{DS(a)}{S(a)} = \frac{\omega S_0(a)}{S(a)} \left[-\frac{DS_0(a)}{S_0(a)} \right] + \frac{(1 - \omega)S_1(a)}{S(a)} \left[-\frac{DS_1(a)}{S_1(a)} \right]$$

The slope of this hazard rate is:

$$\begin{aligned} D \left[-\frac{DS(a)}{S(a)} \right] &= -\frac{\omega S_0(a)}{S(a)} \frac{(1 - \omega)S_1(a)}{S(a)} \left(\left[-\frac{DS_0(a)}{S_0(a)} \right] - \left[-\frac{DS_1(a)}{S_1(a)} \right] \right)^2 \\ &\quad + \frac{\omega S_0(a)}{S(a)} D \left[-\frac{DS_0(a)}{S_0(a)} \right] + \frac{(1 - \omega)S_1(a)}{S(a)} D \left[-\frac{DS_1(a)}{S_1(a)} \right] \end{aligned}$$

The first term is negative, and hence the result.

6. THE UNCONDITIONAL SURVIVOR FUNCTION

The survivor function implied by the entry density (2) and the conditional survivor function (3) is:

$$\Lambda(a) = \frac{\int_0^\infty \Lambda(a|x)f(x + \delta)dx}{\int_0^\infty f(x + \delta)dx}$$

This can be written as:

$$\Lambda(a) = \frac{\zeta\delta}{1 + \zeta\delta} \int_0^\infty \Lambda(a|x)f_e(x)dx + \frac{1}{1 + \zeta\delta} \int_0^\infty \Lambda(a|x)f_g(x)dx$$

The unconditional hazard rate is therefore:

$$-\frac{D\Lambda(a)}{\Lambda(a)} = \frac{h_e(a) \left(\frac{\zeta\delta}{1 + \zeta\delta} \right) \Lambda_e(a) + h_g(a) \left(\frac{1}{1 + \zeta\delta} \right) \Lambda_g(a)}{\left(\frac{\zeta\delta}{1 + \zeta\delta} \right) \Lambda_e(a) + \left(\frac{1}{1 + \zeta\delta} \right) \Lambda_g(a)}$$

where:

$$\begin{aligned} \Lambda_e(a) &= \int_0^\infty \Lambda(a|x)f_e(x)dx, & h_e(a) &= \frac{\int_0^\infty h(a|x)\Lambda(a|x)f_e(x)dx}{\int_0^\infty \Lambda(a|x)f_e(x)dx} \\ \Lambda_g(a) &= \int_0^\infty \Lambda(a|x)f_g(x)dx, & h_g(a) &= \frac{\int_0^\infty h(a|x)\Lambda(a|x)f_g(x)dx}{\int_0^\infty \Lambda(a|x)f_g(x)dx} \end{aligned}$$

The following two sections will show, respectively, that $h_e(a)$ and $h_g(a)$ are downward sloping. The same must then be true for the unconditional hazard rate. In addition, it will be shown that:

$$\lim_{a \rightarrow \infty} h_e(a) = \lim_{a \rightarrow \infty} h_g(a) = \frac{1}{2} \left(\left[\frac{-\mu}{\sigma} \right]^+ \right)^2 \quad (18)$$

which then proves the limit shown in Proposition 3 of the paper.

Detailed calculations are reported in Appendix A for the survivor functions and in Appendix B for the hazard rates.

6.1 Exponential Initial Conditions

6.1.1 The Survivor Function

Integration-by-parts gives:

$$\Lambda_e(a) = \frac{\mu}{\mu + \frac{1}{2}\sigma^2\zeta} \left[\frac{\left(\frac{\mu a}{\sigma\sqrt{a}} \right) \Phi \left(\frac{\mu a}{\sigma\sqrt{a}} \right)}{\phi \left(\frac{\mu a}{\sigma\sqrt{a}} \right)} - \frac{\left(-\frac{[\mu + \sigma^2\zeta]a}{\sigma\sqrt{a}} \right) \Phi \left(-\frac{[\mu + \sigma^2\zeta]a}{\sigma\sqrt{a}} \right)}{\phi \left(-\frac{[\mu + \sigma^2\zeta]a}{\sigma\sqrt{a}} \right)} \right] \frac{1}{\frac{\mu a}{\sigma\sqrt{a}}} \phi \left(\frac{\mu a}{\sigma\sqrt{a}} \right)$$

This can be written more compactly using u and v :

$$\Lambda_e(a) = \frac{-\mu}{\mu + \frac{1}{2}\sigma^2\zeta} [\Psi(v) - \Psi(u)] \frac{1}{u} \phi(u)$$

or, using w and v :

$$\Lambda_e(a) = \frac{\mu}{\mu + \frac{1}{2}\sigma^2\zeta} \left\{ 1 + [\Psi(v) - \Psi(w)] \frac{\phi(w)}{w} \right\}$$

If $\mu < 0$ then $u > 0$ and Lemma 1 implies $\lim_{a \rightarrow \infty} \Lambda_e(a) = 0$. If $\mu > 0$ then $w > 0$ and Lemma 1 implies $\lim_{a \rightarrow \infty} \Lambda_e(a) = \mu / (\mu + \frac{1}{2}\sigma^2\zeta)$. A non-zero fraction of entrants will survive if $\mu > 0$.

6.1.2 The Hazard Rate

A differentiation yields:

$$h_e(a) = \omega \left(\frac{\mu + \zeta\sigma^2}{\sigma} \right)^2 \frac{1 - \Psi(v)}{\Psi(u) - \Psi(v)} \quad (19)$$

or:

$$h_e(a) = -\omega \left(\frac{\mu + \zeta\sigma^2}{\sigma} \right)^2 \frac{1 - \Psi(v)}{\Psi(v) - \Psi(w) + \frac{w}{\phi(w)}} \quad (20)$$

Recall from (8) that $\omega > 0$ if and only if $u^2 > v^2$, and that $\mu > 0$ implies $w \in (0, v)$ and $\omega < 0$. Together with Lemma 1, this confirms that the expressions in (19) and (20) are positive. For $\mu < 0$:

$$\lim_{a \rightarrow \infty} h_e(a) = \lim_{a \rightarrow \infty} \frac{\omega \left(\frac{\mu + \zeta \sigma^2}{\sigma} \right)^2}{1 - \frac{1 - \Psi(u)}{1 - \Psi(v)}} = \lim_{a \rightarrow \infty} \frac{\omega \left(\frac{\mu + \zeta \sigma^2}{\sigma} \right)^2}{1 - \frac{v^2 u^2 N(u)}{u^2 v^2 N(v)}} = \frac{\omega \left(\frac{\mu + \zeta \sigma^2}{\sigma} \right)^2}{1 - \left(\frac{v}{u} \right)^2} = \frac{1}{2} \left(\frac{\mu}{\sigma} \right)^2$$

by Lemma 2. From (20), the hazard rate converges to zero when $\mu > 0$ and age grows without bound. This shows (18) for exponential initial conditions.

Since $\Psi(0) = 0$, it is immediate that $h_e(a) \rightarrow \infty$ as a goes to zero. This implies that the same must be true for the unconditional hazard rate $-\Delta\Lambda(a)/\Lambda(a)$.

The Case $\mu < 0$ Define $\theta = u/v$. Then (19) yields:

$$h_e(a) = \frac{1}{2} \left(\frac{\mu + \zeta \sigma^2}{\sigma} \right)^2 \frac{\theta^2 - 1}{1 - \frac{N(\theta v)}{N(v)}}$$

Suppose $\theta > 1$. Lemma 2 implies $1 - N(\theta v)/N(v) > 0$ and Lemma 3 implies that $N(\theta v)/N(v)$ is decreasing in v . Hence $h_e(a)$ is decreasing. Alternatively, suppose that $\theta \in (0, 1)$. Then Lemma 2 implies that $1 - N(\theta v)/N(v) < 0$ and Lemma 3 implies that $N(\theta v)/N(v)$ is increasing. Hence $h_e(a)$ is decreasing.

The Case $\mu > 0$ Define $\theta = w/v = \mu/(\mu + \zeta \sigma^2) \in (0, 1)$. Then (20) yields:

$$h_e(a) = \frac{\frac{1}{2} \left(\frac{\mu + \zeta \sigma^2}{\sigma} \right)^2 (1 - \theta^2)}{\left[\frac{\phi(\theta v)}{\theta v} N(v) \right]^{-1} + \frac{N(\theta v)}{N(v)} - 1}$$

Lemma 2 states that $N(v)$ is decreasing, and so is $\phi(\theta v)/v$. Lemma 3 implies that $N(\theta v)/N(v)$ is increasing. Hence $h_e(a)$ is decreasing.

6.2 Gamma Initial Conditions

6.2.1 The Survivor Function

Integration-by-parts gives:

$$\Lambda_g(a) = \frac{1}{\frac{\mu a}{\sigma \sqrt{a}}} \phi \left(\frac{\mu a}{\sigma \sqrt{a}} \right) \times$$

$$\left((\lambda^2 - 1) \left[\frac{-\frac{\mu a}{\sigma\sqrt{a}} \Phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right)}{\phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right)} \right] + (\lambda^2 + 1) \left(\frac{\mu}{\mu + \zeta\sigma^2} \right) \left[\frac{\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}} \Phi\left(-\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right)}{\phi\left(\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right)} \right] \right. \\ \left. + \lambda(\zeta + \chi) \mu a \left[1 - \frac{\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}} \Phi\left(-\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right)}{\phi\left(\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right)} \right] \right)$$

This can be written more compactly using u and v :

$$\Lambda_g(a) = (1 - \lambda^2) [\Psi(u) - \Psi(v) - \omega (1 - (1 + v^2) [1 - \Psi(v)])] \frac{\phi(u)}{u}$$

or, using w and v :

$$\Lambda_g(a) = (1 - \lambda^2) \left\{ 1 - [\Psi(w) - \Psi(v) - \omega (1 - (1 + v^2) [1 - \Psi(v)])] \frac{\phi(w)}{w} \right\}$$

If $\mu < 0$ then $u > 0$ and Lemma 2 and Lemma 5 imply $\lim_{a \rightarrow \infty} \Lambda_e(a) = 0$. If $\mu > 0$ then $w > 0$ and Lemma 1 and Lemma 6 imply $\lim_{a \rightarrow \infty} \Lambda_e(a) = 1 - \lambda^2 = \mu(\mu + \sigma^2\zeta) / (\mu + \frac{1}{2}\sigma^2\zeta)^2$. A non-zero fraction of entrants will survive if $\mu > 0$.

6.2.2 The Hazard Rate

A differentiation yields:

$$h_g(a) = \omega \left(\frac{\mu + \zeta\sigma^2}{\sigma} \right)^2 \left[1 + \frac{1}{\omega} \frac{\Psi(v) - \Psi(u)}{1 - (1 + v^2) [1 - \Psi(v)]} \right]^{-1} \quad (21)$$

or:

$$h_g(a) = \omega \left(\frac{\mu + \zeta\sigma^2}{\sigma} \right)^2 \left[1 + \frac{1}{\omega} \frac{\frac{w}{\phi(w)} + \Psi(v) - \Psi(w)}{1 - (1 + v^2) [1 - \Psi(v)]} \right]^{-1} \quad (22)$$

For any $v > 0$ and $\theta > 0$, define:

$$\Xi(v) = \frac{\Psi(v) - \Psi(\theta v)}{1 - (1 + v^2) [1 - \Psi(v)]} = \frac{P(\theta v) - P(v)}{1 - (1 + v^2)P(v)}$$

Lemma 2 and Lemma 5 imply that for any $\theta > 0$:

$$\lim_{v \rightarrow \infty} \Xi(v) = \lim_{v \rightarrow \infty} \frac{\frac{1}{\theta^2} \frac{(\theta v)^2 P(\theta v)}{v^2 P(v)} - 1}{\frac{1}{P(v)} - (1 + v^2)} = \frac{1}{2} \left(\frac{1}{\theta^2} - 1 \right)$$

Define $\theta = u/v$. If $\mu < 0$ then $u > 0$ and so $\theta > 0$. Then (21) yields:

$$\lim_{a \rightarrow \infty} h_g(a) = \frac{\omega \left(\frac{\mu + \zeta\sigma^2}{\sigma} \right)^2}{1 + \frac{1}{2\omega} \left(\frac{1}{\theta^2} - 1 \right)} = \frac{1}{2} \left(\frac{\mu}{\sigma} \right)^2$$

Alternatively, define $\theta = w/v$. If $\mu > 0$ then $w > 0$ and so $\theta > 0$. Thus $\Xi(v)$ still converges if $\mu > 0$, but $w/\phi(w)$ grows without bound and (22) then implies that $h_g(a)$ converges to zero. Thus (18) holds for gamma initial conditions.

The Slope of $\Xi(v)$ The derivative of $\Xi(v)$ is proportional to:

$$D [P(\theta v) - P(v)] [1 - (1 + v^2)P(v)] - [P(\theta v) - P(v)] D [1 - (1 + v^2)P(v)]$$

and this equals:

$$[\theta DP(\theta v) - DP(v)] [1 - (1 + v^2)P(v)] + [P(\theta v) - P(v)] [2vP(v) + (1 + v^2)DP(v)]$$

Note that:

$$DP(v) = -\frac{1}{v} [1 - (1 + v^2)P(v)]$$

Therefore:

$$\begin{aligned} D\Xi(v) \propto & -\left(\frac{1}{v} [1 - (1 + \theta^2 v^2)P(\theta v)] - \frac{1}{v} [1 - (1 + v^2)P(v)]\right) [1 - (1 + v^2)P(v)] \\ & + [P(\theta v) - P(v)] \left(2vP(v) - (1 + v^2)\frac{1}{v} [1 - (1 + v^2)P(v)]\right) \end{aligned}$$

This can also be written as:

$$D\Xi(v) \propto [(\theta^2 - 1) [1 - (1 + v^2)P(v)] + 2P(v)] P(\theta v) - 2[P(v)]^2$$

or:

$$D\Xi(v) \propto \left(1 + \frac{1}{2}(\theta^2 - 1) \left[\frac{1}{P(v)} - (1 + v^2)\right]\right) \frac{1}{P(v)} - \frac{1}{P(\theta v)}$$

or:

$$D\Xi(v) \propto \left(1 + \frac{1}{2} \left(\left[\frac{x}{y}\right]^2 - 1\right) [K(y) - (1 + y^2)]\right) K(y) - K(x)$$

where $x = \theta v$ and $y = v$. Lemma 4 implies that the right-hand side attains a minimum of 0 at $x = y$, and hence $D\Xi(v) > 0$.

The Case $\mu < 0$ Define $\theta = u/v$. The hazard rate (21) can be written as:

$$h_g(a) = \omega \left(\frac{\mu + \zeta\sigma^2}{\sigma}\right)^2 \left[1 + \frac{1}{\omega}\Xi(v)\right]^{-1}$$

The slope of this hazard rate is:

$$Dh_g(a) = -\left(\frac{\mu + \zeta\sigma^2}{\sigma}\right)^2 \left[1 + \frac{1}{\omega}\Xi(v)\right]^{-2} D\Xi(v) \frac{\partial v}{\partial a}$$

which has the same sign as $-D\Xi(v)$. Thus $Dh_g(a) < 0$.

The Case $\mu > 0$ Define $\theta = w/v$ and note that $-\omega = (1-\theta^2)/2$ and $\theta = \mu/(\mu+\zeta\sigma^2) \in (0, 1)$. So $-\omega > 0$. The hazard rate (22) can be written as:

$$\begin{aligned} h_g(a) &= \frac{1}{2} \left(\frac{\mu}{\sigma}\right)^2 \left(\frac{1}{\theta^2} - 1\right) \left[-1 + \frac{1}{\frac{1}{2}(1-\theta^2)} \frac{\frac{w}{\phi(w)} + \Psi(v) - \Psi(w)}{1 - (1+v^2)[1 - \Psi(v)]} \right]^{-1} \\ &= \frac{1}{2} \left(\frac{\mu}{\sigma}\right)^2 \left(\frac{1}{\theta^2} - 1\right) \left[\frac{1}{\frac{1}{2}(1-\theta^2)} \frac{\frac{\theta v}{\phi(\theta v)} + P(\theta v) - P(v)}{1 - (1+v^2)P(v)} - 1 \right]^{-1} \end{aligned}$$

Thus we need to examine:

$$\frac{\frac{\theta v}{\phi(\theta v)} + P(\theta v) - P(v)}{1 - (1+v^2)P(v)} = \left(\frac{1}{\theta} \phi(\theta v) \times \frac{1}{v} [1 - (1+v^2)P(v)] \right)^{-1} + \Xi(v)$$

We have already shown that $\Xi(v)$ is increasing. Note that $\phi(\theta v)/\theta$ is decreasing for $v > 0$. Observe:

$$\begin{aligned} \text{D} \left[\frac{1}{v} [1 - (1+v^2)P(v)] \right] &= -\frac{1}{v^2} [1 - (1+v^2)P(v)] - \frac{1}{v} \text{D} [(1+v^2)P(v)] \\ &= -\frac{1}{v^2} [1 - (1+v^2)P(v)] - \frac{1}{v} [2vP(v) + (1+v^2)\text{D}P(v)] \\ &= -\frac{1}{v^2} [1 - (1+v^2)P(v)] - \frac{1}{v} \left[2vP(v) - \left(\frac{1+v^2}{v} \right) [1 - (1+v^2)P(v)] \right] \end{aligned}$$

Thus:

$$\begin{aligned} -v^2 \text{D} \left[\frac{1}{v} [1 - (1+v^2)P(v)] \right] &= [1 - (1+v^2)P(v)] + 2v^2P(v) - (1+v^2) [1 - (1+v^2)P(v)] \\ &= v^2 (2P(v) - [1 - (1+v^2)P(v)]) \end{aligned}$$

or:

$$\text{D} \left[\frac{1}{v} [1 - (1+v^2)P(v)] \right] = 1 - (3+v^2)P(v)$$

The continued fraction bounds discussed in Appendix C imply:

$$\Psi(v) = \frac{v\Phi(-v)}{\phi(v)} < \frac{v}{v + \frac{1}{v+\frac{2}{v}}} = \frac{2+v^2}{3+v^2}$$

Hence $1 - (3+v^2)P(v) < 0$ and therefore $\text{D}h_g(a) < 0$.

7. PROOF OF LEMMA 2 IN THE PAPER

Define $q : \mathbb{C} \rightarrow \mathbb{C}$:

$$q(z) = \eta - \left[\mu z + \frac{1}{2} \sigma^2 z^2 + \varepsilon_A e^{-\delta z} \right]$$

Consider the characteristic equation $q(z) = 0$. Suppose $\delta = 0$. Then the solutions are:

$$z = -\frac{\mu}{\sigma^2} \pm \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta - \varepsilon_A}{\sigma^2/2}}$$

The only value of ε_A for which the two roots coincide is $\varepsilon_A = \eta + \frac{1}{2} \left(\frac{\mu}{\sigma}\right)^2$, and the resulting root is $z = -\mu/\sigma^2 = \zeta$.

Now suppose $\delta > 0$. Recall that $\mu < \delta\eta$ and $\varepsilon_A > \eta$. We want to show that there is a unique $\varepsilon_A > \eta$ for which the characteristic equation $q(z) = 0$ has only one solution in \mathbb{C} . Although the characteristic equation is not a polynomial, if it has a solution z that is complex, then the complex conjugate \bar{z} will also solve the characteristic equation. Thus we can restrict attention to values of ε_A for which $q(z) = 0$ has only real solutions.

When restricted to \mathbb{R} , the function q is strictly concave. Define:

$$v[\varepsilon_A] = \max_{z \in \mathbb{R}} q(z)$$

This maximum is attained by the unique solution $z \in \mathbb{R}$ to the first-order condition:

$$\delta\varepsilon_A e^{-\delta z} = \mu + \sigma^2 z$$

Let $z[\varepsilon_A]$ denote the solution. Note that $\mu < \delta\eta$ and $\varepsilon_A > \eta$ imply:

$$\delta\eta e^{-\delta z} < \delta\varepsilon_A e^{-\delta z} = \mu + \sigma^2 z < \delta\eta + \sigma^2 z$$

Thus $z[\varepsilon_A] > 0$. The envelope condition implies that $v[\varepsilon_A]$ is strictly decreasing. Thus $v[\varepsilon_A] = 0$ can hold for at most one value of ε_A .

To construct this value, note that the first-order condition implies $\varepsilon_A e^{-\delta z[\varepsilon_A]} = (\mu + \sigma^2 z[\varepsilon_A])/\delta$ substitute this into the expression for $q(z)$ to conclude that ε_A must satisfy:

$$\eta = \mu z[\varepsilon_A] + \frac{1}{2}\sigma^2 z^2[\varepsilon_A] + \frac{1}{\delta}(\mu + \sigma^2 z[\varepsilon_A])$$

This condition only depends on ε_A via $z[\varepsilon_A]$. Solving this quadratic equation for $z[\varepsilon_A]$ gives two possible values for $z[\varepsilon_A]$. The fact that $\mu < \delta\eta$ implies that the only positive solution is:

$$z[\varepsilon_A] = -\left(\frac{\mu}{\sigma^2} + \frac{1}{\delta}\right) + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{1}{\delta^2} + \frac{\eta}{\sigma^2/2}}$$

Thus $z[\varepsilon_A] = \zeta$. The associated value of ε_A follows from $q(z[\varepsilon_A]) = 0$:

$$\varepsilon_A = \left(\eta - \left[\mu\zeta + \frac{1}{2}\sigma^2\zeta^2\right]\right) e^{\delta\zeta} \quad (23)$$

This shows that the entry rate described in Lemma 2 of the paper is well defined.

8. OUT-OF-STEADY STATE CONSIDERATIONS

Throughout this section, size is assumed to evolve according to a Brownian motion with drift μ and local variance σ^2 , as long as the zero exit barrier is not reached. Entry and population dynamics are described when entry rates are not necessarily constant and initial conditions are not necessarily consistent with a steady state. This is used to examine stability in the special case of $\delta = 0$. Along the way, more detailed derivations are given of (14) and (23) in the paper.

8.1 Non-Stationary Entry and Population Dynamics

Let $n(t, a, y)$ be the date- t density of firms of age a and size y . Let $n(t, y)$ be the date- t density of firms of size y :

$$n(t, y) = \int_0^\infty n(t, a, y) da$$

The size of the population at time t is denoted by:

$$N_t = \int_0^\infty n(t, z) dz$$

Note that $n(t, \cdot, \cdot)$ is not assumed to be a probability density. The date- t probability density of firms of size y is $f(t, y) = n(t, y)/N_t$.

The density $n(t, a, y)$ satisfies the Kolmogorov Forward Equation:

$$D_t n(t, a, y) = -D_a n(t, a, y) - \mu D_y n(t, a, y) + \frac{1}{2} \sigma^2 D_{yy} n(t, a, y) \quad (24)$$

Exit at zero implies $n(t, a, 0) = 0$. Suppose there are I_t entry attempts per unit of time and let $G(t, y)$ describe the size distribution among potential entrants. Size draws below zero result in failure to enter. The measure of age-0 firms should correspond to the measure $I_t[G(t, y) - G(t, 0)]$:

$$\lim_{a \downarrow 0} \int_0^y n(t, a, z) dz = I_t [G(t, y) - G(t, 0)] \quad (25)$$

The number of successful entries per unit of time is $I_t[1 - G(t, 0)]$.

Suppose that $G(t, y)$ has a density $g(t, y)$. Integrating (24) over all ages using (25) gives:

$$D_t n(t, y) = -\mu D_y n(t, y) + \frac{1}{2} \sigma^2 D_{yy} n(t, y) + I_t g(t, y)$$

Take some $\delta \geq 0$ and suppose now that for all $y + \delta \geq 0$:

$$g(t, y) = f(t, y + \delta)$$

Potential entrants draw a random incumbent and are then set back by δ , as in Section V of the paper. This process of imitation gives:

$$D_t n(t, y) = -\mu D_y n(t, y) + \frac{1}{2} \sigma^2 D_{yy} n(t, y) + I_t f(t, y + \delta) \quad (26)$$

Recall that I_t is the number of entry attempts per unit of time. The number of successful entry attempts is then:

$$E_t = I_t \int_0^\infty f(t, z + \delta) dz$$

per unit of time.

8.1.1 Steady State

In the paper, the number of entry attempts grows exponentially according to $I_t = Ie^{\eta t}$. In a steady state, this yields $n(t, a, y) = m(a, y)Ie^{\eta t}$. Hence $D_t n(t, a, y) = \eta n(t, a, y)$. Equation (14) in the paper follows from this and (24) above. Write:

$$m(y) = \int_0^\infty m(a, y) da$$

for the size marginal scaled by the entry rate. Then:

$$N_t = Ie^{\eta t} \int_0^\infty m(z) dz$$

is the steady state number of firms. In a steady state, $f(t, y)$ becomes:

$$f(y) = \left[\int_0^\infty m(z) dz \right]^{-1} m(y)$$

The imitation dynamics (26) can now be written as:

$$\eta m(y) = -\mu D m(y) + \frac{1}{2} \sigma^2 D^2 m(y) + f(y + \delta)$$

Normalizing to replace the density $m(y)$ by the probability density $f(y)$ yields:

$$\eta f(y) = -\mu D f(y) + \frac{1}{2} \sigma^2 D^2 f(y) + \left[\int_0^\infty m(z) dz \right]^{-1} f(y + \delta)$$

As a fraction of the number of incumbent firms, the number of entry attempts per unit of time equals:

$$\varepsilon_A = \frac{I_t}{N_t} = \left[\int_0^\infty m(z) dz \right]^{-1}$$

Hence:

$$\eta f(y) = -\mu Df(y) + \frac{1}{2}\sigma^2 D^2 f(y) + \varepsilon_A f(y + \delta)$$

which is equation (23) in the paper. As a fraction of the number of incumbent firms, the number of successful entry attempts per unit of time is now:

$$\varepsilon_S = \frac{E_t}{N_t} = \frac{I_t \int_0^\infty f(z + \delta) dz}{N_t} = \varepsilon_A \int_0^\infty f(z + \delta) dz$$

as expected.

8.2 Further Interpretation of Lemma 2 in the Paper

Consider an infinitesimal “industry” that starts at some initial date 0 with a certain initial size density $n(0, y)$. Suppose that entry into this industry takes place at a constant rate $\varepsilon_A > 0$ and suppose that new entrants can perfectly imitate. That is, $\delta = 0$.

The size density of this industry then evolves with age according to:

$$D_a n(a, y) = -\mu D_s n(a, y) + \frac{1}{2}\sigma^2 D_{ss} n(a, y) + \varepsilon_A n(a, y) \quad (27)$$

and exit at the boundary 0 implies the boundary condition $n(a, 0) = 0$. If $n(0, y)$ is a point mass at x , then the solution is $n(a, y) = e^{\varepsilon_A a} \psi(a, y|x)$. This has a spectral representation, mentioned in footnote 15 of the paper, given by:

$$n(a, y) = \frac{2}{\pi} \int_0^\infty e^{[-\frac{1}{2}\sigma^2 \omega^2 - \frac{1}{2}(\frac{\mu}{\sigma})^2 + \varepsilon_A]a} e^{(y-x)\mu/\sigma^2} \sin(\omega y) \sin(\omega x) d\omega \quad (28)$$

The coefficients $-\frac{1}{2}\sigma^2 \omega^2 - \frac{1}{2}(\mu/\sigma)^2 + \varepsilon_A$ are eigenvalues of the operator acting on $n(a, \cdot)$ as defined by the right-hand side of (27). The corresponding eigenfunctions are $e^{y\mu/\sigma^2} \sin(\omega y)$. The largest eigenvalue mentioned in footnote 15 is $-\frac{1}{2}(\mu/\sigma)^2 + \varepsilon_A$, which corresponds to the $\omega = 0$ term in the integral (28). As a becomes large, small- ω terms in (28) decay more slowly than large- ω terms, and the small- a terms will come to dominate the shape of $n(a, y)$.

For any $x > 0$, the density $e^{\varepsilon_A a} \psi(a, y|x)$ converges in distribution to the gamma distribution of Lemma 2. This is not difficult to show directly, without reference to (28), using the expression for $\psi(a, y|x)$ given in Lemma 1 of the paper. This convergence result generalizes to compactly supported initial distributions. Both results are proven in the next subsection. Convergence to the gamma density cannot hold without some restrictions: if the initial size distribution is one of the alternative stationary distributions, then convergence to the gamma density must necessarily fail.

To interpret this further and motivate the construction of the equilibrium entry rate in Lemma 2 of the paper, consider solutions to (27) of the form $n(a, y) = \exp(ga - hy)$. For a given candidate eigenvalue g , the coefficient h must solve the characteristic polynomial:

$$g = \mu h + \frac{1}{2}\sigma^2 h^2 + \varepsilon_A \quad (29)$$

The roots in \mathbb{C} are:

$$h_{\pm} = -\frac{\mu}{\sigma^2} \pm \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{g - \varepsilon_A}{\sigma^2/2}}$$

The difference between two solutions with the same eigenvalue g satisfies the boundary condition $n(a, 0) = 0$. The solutions to (27) that satisfy this boundary condition are therefore:

$$n(a, y) = e^{ga} \left[\frac{e^{-h_+ y} - e^{-h_- y}}{h_- - h_+} \right]$$

where normalization is chosen so that the $h_+ = h_-$ case can be interpreted as a limit.

Define γ to be the unique value of g for which the characteristic polynomial (29) only has a single real root:

$$\gamma = -\frac{\sigma^2}{2} \left(\frac{\mu}{\sigma}\right)^2 + \varepsilon_A$$

This gives rise to the solution $n(a, y) = e^{\gamma a} y e^{-\zeta y}$, where $\zeta = -\mu/\sigma^2$. This is proportional to the solution given in Lemma 2 of the paper. The population of firms is supposed to grow at a rate η , and so the entry rate ε_A must be such that $\gamma = \eta$, as in Lemma 2 of the paper.

For $g > \gamma$, one obtains the alternative stationary distributions described in the second paragraph following Lemma 2 of the paper. For $g < \gamma$, the solutions to the characteristic polynomial are of the form $h_{\pm} = \zeta \pm i\omega$ and this yields $n(a, y) = e^{ga - \zeta y} \sin(\omega y)$. Note that ω is given by:

$$\omega = i \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{g - \varepsilon_A}{\sigma^2/2}}$$

and thus:

$$g = -\frac{1}{2} \left(\frac{\mu}{\sigma}\right)^2 + \varepsilon_A - \frac{1}{2}\sigma^2\omega^2$$

Letting g range through all $g < \gamma$ corresponds to letting ω range from 0 to ∞ . The eigenfunctions $e^{-\zeta y} \sin(\omega y)$, $\omega > 0$, are the ones that appear in the spectral representation (28). The supremum of the eigenvalues g associated with these eigenfunctions is obtained by letting ω go to zero. In sum, we obtain convergence to the eigenfunction

corresponding to the supremum of the eigenvalues associated with the eigenfunctions that appear in the spectral representation (28).

The characteristic equation for $\delta > 0$ is:

$$g = \mu h + \frac{1}{2}\sigma^2 h^2 + \varepsilon_A e^{-\delta h}$$

Although this characteristic equation is no longer a polynomial in h , the solutions to this equation are similar to the ones obtained for $\delta = 0$. As shown in the proof of Lemma 2 of the paper, there is again an eigenvalue $g = \gamma$ that has the gamma density as an eigenfunction. For $g > \gamma$ one again obtains the alternative stationary distributions. And for $g < \gamma$, the roots of the characteristic equation again come in complex pairs. These now give rise to solutions of the form $e^{-a[\omega]y} \sin(\omega y)$, where $a[\omega]$ is an oscillating function that converges to ζ as ω grows without bound. If these eigenfunctions can again be used to construct a spectral representation like (28), then one would expect convergence to the gamma density shown in Lemma 2 of the paper. The question remains whether or not this is indeed the case.

8.3 The Gamma Limit

Recall the definition of $\psi(a, y|x)$ in (14) and note that:

$$\begin{aligned} \int_0^y \psi(a, s|x) ds &= \Phi\left(\frac{y-x-\mu a}{\sigma\sqrt{a}}\right) - \exp\left(-\frac{\mu x}{\sigma^2/2}\right) \Phi\left(\frac{y+x-\mu a}{\sigma\sqrt{a}}\right) \\ &\quad - \left[\Phi\left(\frac{-x-\mu a}{\sigma\sqrt{a}}\right) - \exp\left(-\frac{\mu x}{\sigma^2/2}\right) \Phi\left(\frac{x-\mu a}{\sigma\sqrt{a}}\right) \right] \end{aligned}$$

In particular:

$$\Lambda(a|x) = \int_0^\infty \psi(a, s|x) ds = \Phi\left(\frac{x+\mu a}{\sigma\sqrt{a}}\right) - \exp\left(-\frac{\mu x}{\sigma^2/2}\right) \Phi\left(\frac{-x+\mu a}{\sigma\sqrt{a}}\right)$$

For any $x > 0$, define:

$$P(a, y|x) = \frac{1}{\Lambda(a|x)} \int_0^y \psi(a, s|x) ds$$

This is the size distribution among survivors at age a of a cohort with identical initial conditions.

The definition (1) of ζ reduces to $\zeta = -\mu/\sigma^2$ when $\delta = 0$. Since $\zeta > 1$ in equilibrium, we can restrict attention to the case $\mu < 0$. This implies that $\Lambda(a|x) \rightarrow 0$ as $a \rightarrow \infty$. Suppose $x > 0$ so that $\Lambda(a|x) > 0$. Observe:

$$P(a, y|x) = \frac{\Phi\left(\frac{y-x-\mu a}{\sigma\sqrt{a}}\right) - \Phi\left(\frac{-x-\mu a}{\sigma\sqrt{a}}\right) - e^{-\frac{\mu x}{\sigma^2/2}} \left[\Phi\left(\frac{y+x-\mu a}{\sigma\sqrt{a}}\right) - \Phi\left(\frac{x-\mu a}{\sigma\sqrt{a}}\right) \right]}{\Phi\left(\frac{x+\mu a}{\sigma\sqrt{a}}\right) - e^{-\frac{\mu x}{\sigma^2/2}} \Phi\left(\frac{-x+\mu a}{\sigma\sqrt{a}}\right)}$$

or:

$$P(a, y|x) = \frac{\Phi\left(\frac{x+\mu a}{\sigma\sqrt{a}}\right) - \Phi\left(\frac{-y+x+\mu a}{\sigma\sqrt{a}}\right) - e^{-\frac{\mu x}{\sigma^2/2}} \left[\Phi\left(\frac{-x+\mu a}{\sigma\sqrt{a}}\right) - \Phi\left(\frac{-y-x+\mu a}{\sigma\sqrt{a}}\right) \right]}{\Phi\left(\frac{x+\mu a}{\sigma\sqrt{a}}\right) - e^{-\frac{\mu x}{\sigma^2/2}} \Phi\left(\frac{-x+\mu a}{\sigma\sqrt{a}}\right)}$$

or:

$$P(a, y|x) = 1 - \frac{\Phi\left(\frac{-y+x+\mu a}{\sigma\sqrt{a}}\right) - e^{-\frac{\mu x}{\sigma^2/2}} \Phi\left(\frac{-y-x+\mu a}{\sigma\sqrt{a}}\right)}{\Phi\left(\frac{x+\mu a}{\sigma\sqrt{a}}\right) - e^{-\frac{\mu x}{\sigma^2/2}} \Phi\left(\frac{-x+\mu a}{\sigma\sqrt{a}}\right)} \quad (30)$$

Both the numerator and the denominator on the right-hand side of this expression go to zero as a goes to infinity.

We want to examine the limit of:

$$\frac{\Phi\left(\frac{-y+x+\mu a}{\sigma\sqrt{a}}\right) - e^{-\frac{\mu x}{\sigma^2/2}} \Phi\left(\frac{-y-x+\mu a}{\sigma\sqrt{a}}\right)}{\Phi\left(\frac{x+\mu a}{\sigma\sqrt{a}}\right) - e^{-\frac{\mu x}{\sigma^2/2}} \Phi\left(\frac{-x+\mu a}{\sigma\sqrt{a}}\right)}$$

as a becomes large. Write $h = 1/\sigma\sqrt{a}$. Then we need to study $f(h, x)/g(h, x)$, where:

$$f(h, x) = \Phi\left(\left[-y + x - \frac{\zeta}{h^2}\right] h\right) - e^{2\zeta x} \Phi\left(\left[-y - x - \frac{\zeta}{h^2}\right] h\right)$$

and:

$$g(h, x) = \Phi\left(\left[x - \frac{\zeta}{h^2}\right] h\right) - e^{2\zeta x} \Phi\left(\left[-x - \frac{\zeta}{h^2}\right] h\right)$$

Since $\zeta > 0$, it follows that $f(h, x) \rightarrow 0$ and $g(h, x) \rightarrow 0$. The limit of $f(h, x)/g(h, x)$ can be computed using l'Hôpital's rule.

Observe:

$$D_1 f(h, x) = \left[-y + x + \frac{\zeta}{h^2}\right] \phi\left(\left[-y + x - \frac{\zeta}{h^2}\right] h\right) - \left[-y - x + \frac{\zeta}{h^2}\right] e^{2\zeta x} \phi\left(\left[-y - x - \frac{\zeta}{h^2}\right] h\right)$$

Note that:

$$\begin{aligned} -\frac{1}{2} \left(\left(x - \frac{\zeta}{h^2}\right) h\right)^2 - \zeta y + y(x - y/2)h^2 &= -\frac{1}{2} \left(\left(-y + x - \frac{\zeta}{h^2}\right) h\right)^2 \\ -\frac{1}{2} \left(\left(x - \frac{\zeta}{h^2}\right) h\right)^2 - \zeta y - y(x + y/2)h^2 &= 2\zeta x - \frac{1}{2} \left(\left(-y - x - \frac{\zeta}{h^2}\right) h\right)^2 \end{aligned}$$

and so:

$$\begin{aligned} \phi\left(\left[-y + x - \frac{\zeta}{h^2}\right] h\right) &= \phi\left(\left[x - \frac{\zeta}{h^2}\right] h\right) \exp(-\zeta y) \exp(y(x - y/2)h^2) \\ e^{2\zeta x} \phi\left(\left[-y - x - \frac{\zeta}{h^2}\right] h\right) &= \phi\left(\left[x - \frac{\zeta}{h^2}\right] h\right) \exp(-\zeta y) \exp(-y(x + y/2)h^2) \end{aligned}$$

Hence:

$$D_1f(h, x) = \phi\left(xh - \frac{\zeta}{h}\right) \exp(-\zeta y) \times \left\{ \left[-y + x + \frac{\zeta}{h^2} \right] \exp(y(x - y/2)h^2) - \left[-y - x + \frac{\zeta}{h^2} \right] \exp(-y(x + y/2)h^2) \right\}$$

Observe:

$$Dg(h, x) = \left(\left[x + \frac{\zeta}{h^2} \right] \right) \phi\left(\left[x - \frac{\zeta}{h^2} \right] h\right) - e^{2\zeta x} \left[-x + \frac{\zeta}{h^2} \right] \phi\left(\left[-x - \frac{\zeta}{h^2} \right] h\right)$$

Note that:

$$-\frac{1}{2} \left(\left(x - \frac{\zeta}{h^2} \right) h \right)^2 = 2\zeta x - \frac{1}{2} \left(\left(-x - \frac{\zeta}{h^2} \right) h \right)^2$$

Thus:

$$Dg(h, x) = 2x\phi\left(xh - \frac{\zeta}{h}\right)$$

The ratio of derivatives is therefore:

$$\frac{Df(h, x)}{Dg(h, x)} = \frac{e^{-\zeta y}}{2x} \left[\left(y + x - \frac{\zeta}{h^2} \right) \exp(-y(x + y/2)h^2) - \left(y - x - \frac{\zeta}{h^2} \right) \exp(y(x - y/2)h^2) \right]$$

Now:

$$\lim_{h \rightarrow 0} \frac{1}{2x} \left[(y + x) \exp(-y(x + y/2)h^2) - (y - x) \exp(y(x - y/2)h^2) \right] = 1 \quad (31)$$

and:

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{1}{2x} \left[\left(-\frac{\zeta}{h^2} \right) \exp(-y(x + y/2)h^2) - \left(-\frac{\zeta}{h^2} \right) \exp(y(x - y/2)h^2) \right] \\ &= \frac{\zeta}{2x} \lim_{h \rightarrow 0} \left[\frac{1 - \exp(-y(x + y/2)h^2)}{h^2} + \frac{\exp(y(x - y/2)h^2) - 1}{h^2} \right] \\ &= \frac{\zeta}{2x} [y(x + y/2) + y(x - y/2)] = \zeta y \end{aligned} \quad (32)$$

Hence:

$$\lim_{h \rightarrow 0} \frac{Df(h, x)}{Dg(h, x)} = (1 + \zeta y)e^{-\zeta y} \quad (33)$$

By l'Hôpital's rule, $f(h, x)/g(h, x)$ converges to $(1 + \zeta y)e^{-\zeta y}$. The limits (31)-(32) and thus (33) are uniform on any compact $X \subset (0, \infty)$. In turn, this implies that the convergence of $f(h, x)/g(h, x)$ to $(1 + \zeta y)e^{-\zeta y}$ is uniform on any compact $X \subset (0, \infty)$.

Therefore:

$$\lim_{a \rightarrow \infty} P(a, y|x) = 1 - (1 + \zeta y)e^{-\zeta y} \quad (34)$$

and the convergence is uniform in $x \in X$, for any compact $X \subset (0, \infty)$. This is the distribution function of the gamma distribution with density $\zeta^2 y e^{-\zeta y}$. Write $P(y)$ for this distribution.

Now suppose the initial conditions are distributed according to a distribution G with compact support $X \subset (0, \infty)$. Let $P_G(a, y)$ denote the distribution at age a . Then:

$$P_G(a, y) = \frac{\int [\int_0^y \psi(a, s|x) ds] dG(x)}{\int [\int_0^\infty \psi(a, s|x) ds] dG(x)} = \int w(a|x) P(a, y|x) dG(x)$$

where:

$$w(a, x) = \frac{\Lambda(a|x)}{\int \Lambda(a|x) dG(x)}$$

Clearly, these weights integrate to 1 against the distribution G . Fix some $y > 0$. Take any $\varepsilon > 0$. The uniform convergence in (34) implies that there is a a_ε such that:

$$|P(a, y|x) - P(y)| \leq \varepsilon$$

for all $x \in X$ and all $a \geq a_\varepsilon$. Hence:

$$|P_G(a, y) - P(y)| \leq \int_X w(a, x) |P(a, y|x) - P(y)| dG(x) \leq \varepsilon$$

for all $a \geq a_\varepsilon$. Thus:

$$\lim_{a \rightarrow \infty} P_G(a, y) = P(y) \tag{35}$$

Thus $P_G(a, \cdot)$ converges in distribution to the gamma distribution with density $\zeta^2 y e^{-\zeta y}$, which is the stationary solution to (27) selected in Lemma 2 of the paper. None of the stationary distributions that solve (27) have compact support, and so this convergence result does not apply to these distributions.

APPENDIX

This appendix shows detailed calculations.

A CALCULATING UNCONDITIONAL SURVIVOR FUNCTIONS

In the following, note that:

$$\frac{\phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right)}{\phi\left(\frac{[\mu+\zeta\sigma^2]a}{\sigma\sqrt{a}}\right)} = \exp\left(-\frac{1}{2}\left(\frac{\mu a}{\sigma\sqrt{a}}\right)^2 + \frac{1}{2}\left(\frac{[\mu+\zeta\sigma^2]a}{\sigma\sqrt{a}}\right)^2\right)$$

and:

$$-\frac{1}{2}\left(\frac{\mu a}{\sigma\sqrt{a}}\right)^2 + \frac{1}{2}\left(\frac{[\mu+\zeta\sigma^2]a}{\sigma\sqrt{a}}\right)^2 = \left(\mu + \frac{1}{2}\zeta\sigma^2\right)\zeta a$$

so that:

$$\frac{\phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right)}{\phi\left(\frac{[\mu+\zeta\sigma^2]a}{\sigma\sqrt{a}}\right)} = \exp\left(\left[\mu + \frac{1}{2}\zeta\sigma^2\right]\zeta a\right) \quad (36)$$

1.1 Exponential Initial Conditions

The survivor function is:

$$\Lambda_e(a) = \int_0^\infty \zeta e^{-\zeta x} \Phi\left(\frac{x+\mu a}{\sigma\sqrt{a}}\right) dx - \frac{\zeta}{\chi} \int_0^\infty \chi e^{-\chi x} \Phi\left(\frac{-x+\mu a}{\sigma\sqrt{a}}\right) dx$$

Integration-by-parts gives:

$$\int_0^\infty \zeta e^{-\zeta x} \Phi\left(\frac{x+\mu a}{\sigma\sqrt{a}}\right) dx = \Phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right) + \int_0^\infty \frac{e^{-\zeta x}}{\sigma\sqrt{a}} \phi\left(\frac{x+\mu a}{\sigma\sqrt{a}}\right) dx$$

Note that:

$$-\zeta x - \frac{1}{2}\left(\frac{x+\mu a}{\sigma\sqrt{a}}\right)^2 = \left(\mu + \frac{1}{2}\zeta\sigma^2\right)\zeta a - \frac{1}{2}\left(\frac{x+(\mu+\zeta\sigma^2)a}{\sigma\sqrt{a}}\right)^2$$

Therefore:

$$\int_0^\infty \frac{e^{-\zeta x}}{\sigma\sqrt{a}} \phi\left(\frac{x+\mu a}{\sigma\sqrt{a}}\right) dx = \exp\left(\left[\mu + \frac{1}{2}\zeta\sigma^2\right]\zeta a\right) \Phi\left(-\frac{[\mu+\zeta\sigma^2]a}{\sigma\sqrt{a}}\right)$$

Hence:

$$\int_0^\infty \zeta e^{-\zeta x} \Phi\left(\frac{x+\mu a}{\sigma\sqrt{a}}\right) dx = \Phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right) + \exp\left(\left[\mu + \frac{1}{2}\zeta\sigma^2\right]\zeta a\right) \Phi\left(-\frac{[\mu+\zeta\sigma^2]a}{\sigma\sqrt{a}}\right) \quad (37)$$

Integration-by-parts gives:

$$\int_0^\infty \chi e^{-\chi x} \Phi\left(\frac{-x + \mu a}{\sigma\sqrt{a}}\right) dx = \Phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right) - \int_0^\infty \frac{e^{-\chi x}}{\sigma\sqrt{a}} \phi\left(\frac{-x + \mu a}{\sigma\sqrt{a}}\right) dx$$

Note that:

$$-\chi x - \frac{1}{2} \left(\frac{-x + \mu a}{\sigma\sqrt{a}}\right)^2 = \left(-\chi\mu + \frac{1}{2}\chi^2\sigma^2\right) a - \frac{1}{2} \left(\frac{x - (\mu - \chi\sigma^2)a}{\sigma\sqrt{a}}\right)^2$$

Therefore:

$$\int_0^\infty \frac{e^{-\chi x}}{\sigma\sqrt{a}} \phi\left(\frac{-x + \mu a}{\sigma\sqrt{a}}\right) dx = \exp\left(\left[-\chi\mu + \frac{1}{2}\chi^2\sigma^2\right] a\right) \Phi\left(\frac{[\mu - \chi\sigma^2]a}{\sigma\sqrt{a}}\right)$$

Hence:

$$\int_0^\infty \chi e^{-\chi x} \Phi\left(\frac{-x + \mu a}{\sigma\sqrt{a}}\right) dx = \Phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right) - \exp\left(\left[-\chi\mu + \frac{1}{2}\chi^2\sigma^2\right] a\right) \Phi\left(\frac{[\mu - \chi\sigma^2]a}{\sigma\sqrt{a}}\right) \quad (38)$$

Combining (37) and (38) gives:

$$\begin{aligned} \Lambda_e(a) &= \int_0^\infty \zeta e^{-\zeta x} \Phi\left(\frac{x + \mu a}{\sigma\sqrt{a}}\right) dx - \frac{\zeta}{\chi} \int_0^\infty \chi e^{-\chi x} \Phi\left(\frac{-x + \mu a}{\sigma\sqrt{a}}\right) dx \\ &= \Phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right) + \exp\left(\left[\mu + \frac{1}{2}\zeta\sigma^2\right] \zeta a\right) \Phi\left(-\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right) \\ &\quad - \frac{\zeta}{\chi} \left[\Phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right) - \exp\left(\left[-\chi\mu + \frac{1}{2}\chi^2\sigma^2\right] a\right) \Phi\left(\frac{[\mu - \chi\sigma^2]a}{\sigma\sqrt{a}}\right) \right] \end{aligned}$$

Using (9) and (10):

$$\Lambda_e(a) = \left(1 - \frac{\zeta}{\chi}\right) \Phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right) + \left(1 + \frac{\zeta}{\chi}\right) \exp\left(\left[\mu + \frac{1}{2}\zeta\sigma^2\right] \zeta a\right) \Phi\left(-\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right)$$

Using (36):

$$\Lambda_e(a) = \left[\left(1 - \frac{\zeta}{\chi}\right) \frac{\Phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right)}{\phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right)} + \left(1 + \frac{\zeta}{\chi}\right) \frac{\Phi\left(-\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right)}{\phi\left(\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right)} \right] \phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right)$$

Note that:

$$1 - \frac{\zeta}{\chi} = \frac{\mu}{\mu + \frac{1}{2}\sigma^2\zeta}, \quad 1 + \frac{\zeta}{\chi} = \frac{\mu + \sigma^2\zeta}{\mu + \frac{1}{2}\sigma^2\zeta}$$

This yields:

$$\Lambda_e(a) = \frac{1}{\mu + \frac{1}{2}\sigma^2\zeta} \left[\frac{\mu\Phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right)}{\phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right)} + \frac{[\mu + \sigma^2\zeta]\Phi\left(-\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right)}{\phi\left(\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right)} \right] \phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right)$$

Another way to normalize this, as long as $\mu \neq 0$, is:

$$\Lambda_e(a) = \frac{\mu}{\mu + \frac{1}{2}\sigma^2\zeta} \left[\frac{\frac{[\mu + \sigma^2\zeta]a}{\sigma\sqrt{a}} \Phi\left(-\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right)}{\phi\left(\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right)} - \frac{-\frac{\mu a}{\sigma\sqrt{a}} \Phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right)}{\phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right)} \right] \frac{1}{\frac{\mu a}{\sigma\sqrt{a}}} \phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right)$$

or, using the definitions of u and v :

$$\Lambda_e(a) = \frac{-\mu}{\mu + \frac{1}{2}\sigma^2\zeta} [\Psi(v) - \Psi(u)] \frac{\phi(u)}{u} \quad (39)$$

Another way to write this is:

$$\Lambda_e(a) = \frac{\mu}{\mu + \frac{1}{2}\sigma^2\zeta} \left\{ 1 + [\Psi(v) - \Psi(w)] \frac{\phi(w)}{w} \right\} \quad (40)$$

since $w = -u$.

1.2 Gamma Initial Conditions

The survivor function is:

$$\Lambda_g(a) = \int_0^\infty \zeta^2 x e^{-\zeta x} \Phi\left(\frac{x + \mu a}{\sigma\sqrt{a}}\right) dx - \left(\frac{\zeta}{\chi}\right)^2 \int_0^\infty \chi^2 x e^{-\chi x} \Phi\left(\frac{-x + \mu a}{\sigma\sqrt{a}}\right) dx$$

Integration-by-parts gives:

$$\begin{aligned} & \int_0^\infty \zeta^2 x e^{-\zeta x} \Phi\left(\frac{x + \mu a}{\sigma\sqrt{a}}\right) dx \\ &= -(1 + \zeta x) e^{-\zeta x} \Phi\left(\frac{x + \mu a}{\sigma\sqrt{a}}\right) \Big|_0^\infty + \int_0^\infty \frac{(1 + \zeta x) e^{-\zeta x}}{\sigma\sqrt{a}} \phi\left(\frac{x + \mu a}{\sigma\sqrt{a}}\right) dx \\ &= \Phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right) + \int_0^\infty \frac{e^{-\zeta x}}{\sigma\sqrt{a}} \phi\left(\frac{x + \mu a}{\sigma\sqrt{a}}\right) dx + \int_0^\infty \frac{\zeta x e^{-\zeta x}}{\sigma\sqrt{a}} \phi\left(\frac{x + \mu a}{\sigma\sqrt{a}}\right) dx \end{aligned}$$

Now:

$$\begin{aligned} & \int_0^\infty \frac{x e^{-\zeta x}}{\sigma\sqrt{a}} \phi\left(\frac{x + \mu a}{\sigma\sqrt{a}}\right) dx \\ &= \exp\left(\left[\mu + \frac{1}{2}\zeta\sigma^2\right]\zeta a\right) \int_0^\infty \frac{x}{\sigma\sqrt{a}} \phi\left(\frac{x + [\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right) dx \\ &= \exp\left(\left[\mu + \frac{1}{2}\zeta\sigma^2\right]\zeta a\right) \left[\sigma\sqrt{a} \phi\left(\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right) - [\mu + \zeta\sigma^2]a \Phi\left(\frac{-[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right) \right] \end{aligned}$$

Hence:

$$\begin{aligned}
& \int_0^\infty \zeta^2 x e^{-\zeta x} \Phi\left(\frac{x + \mu a}{\sigma\sqrt{a}}\right) dx \\
&= \Phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right) + \exp\left(\left[\mu + \frac{1}{2}\zeta\sigma^2\right]\zeta a\right) \Phi\left(-\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right) + \\
& \quad + \zeta \exp\left(\left[\mu + \frac{1}{2}\zeta\sigma^2\right]\zeta a\right) \left[\sigma\sqrt{a}\phi\left(\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right) - [\mu + \zeta\sigma^2]a\Phi\left(\frac{-[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right)\right]
\end{aligned}$$

Integration-by-parts gives:

$$\begin{aligned}
& \int_0^\infty \chi^2 x e^{-\chi x} \Phi\left(\frac{-x + \mu a}{\sigma\sqrt{a}}\right) dx \\
&= -(1 + \chi x)e^{-\chi x} \Phi\left(\frac{-x + \mu a}{\sigma\sqrt{a}}\right) \Big|_0^\infty - \int_0^\infty \frac{(1 + \chi x)e^{-\chi x}}{\sigma\sqrt{a}} \phi\left(\frac{-x + \mu a}{\sigma\sqrt{a}}\right) dx \\
&= \Phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right) - \int_0^\infty \frac{e^{-\chi x}}{\sigma\sqrt{a}} \phi\left(\frac{x - \mu a}{\sigma\sqrt{a}}\right) dx - \int_0^\infty \frac{\chi x e^{-\chi x}}{\sigma\sqrt{a}} \phi\left(\frac{x - \mu a}{\sigma\sqrt{a}}\right) dx
\end{aligned}$$

Now:

$$\begin{aligned}
& \int_0^\infty \frac{x e^{-\chi x}}{\sigma\sqrt{a}} \phi\left(\frac{x - \mu a}{\sigma\sqrt{a}}\right) dx \\
&= \exp\left(\left[-\mu + \frac{1}{2}\chi\sigma^2\right]\chi a\right) \int_0^\infty \frac{x}{\sigma\sqrt{a}} \phi\left(\frac{x - [\mu - \chi\sigma^2]a}{\sigma\sqrt{a}}\right) dx \\
&= \exp\left(\left[-\mu + \frac{1}{2}\chi\sigma^2\right]\chi a\right) \left[\sigma\sqrt{a}\phi\left(\frac{[\mu - \chi\sigma^2]a}{\sigma\sqrt{a}}\right) + [\mu - \chi\sigma^2]a\Phi\left(\frac{[\mu - \chi\sigma^2]a}{\sigma\sqrt{a}}\right)\right]
\end{aligned}$$

Hence:

$$\begin{aligned}
& \int_0^\infty \chi^2 x e^{-\chi x} \Phi\left(\frac{-x + \mu a}{\sigma\sqrt{a}}\right) dx \\
&= \Phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right) - \exp\left(\left[-\chi\mu + \frac{1}{2}\chi^2\sigma^2\right]a\right) \Phi\left(\frac{[\mu - \chi\sigma^2]a}{\sigma\sqrt{a}}\right) \\
& \quad - \chi \exp\left(\left[-\mu + \frac{1}{2}\chi\sigma^2\right]\chi a\right) \left[\sigma\sqrt{a}\phi\left(\frac{[\mu - \chi\sigma^2]a}{\sigma\sqrt{a}}\right) + [\mu - \chi\sigma^2]a\Phi\left(\frac{[\mu - \chi\sigma^2]a}{\sigma\sqrt{a}}\right)\right]
\end{aligned}$$

Combining these two results gives:

$$\begin{aligned}
& \Lambda_g(a) \\
&= \Phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right) + \exp\left(\left[\mu + \frac{1}{2}\zeta\sigma^2\right]\zeta a\right) \Phi\left(-\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right) \\
& \quad + \zeta \exp\left(\left[\mu + \frac{1}{2}\zeta\sigma^2\right]\zeta a\right) \left[\sigma\sqrt{a}\phi\left(\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right) - [\mu + \zeta\sigma^2]a\Phi\left(-\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right)\right]
\end{aligned}$$

$$\begin{aligned}
& - \left(\frac{\zeta}{\chi}\right)^2 \left[\Phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right) - \exp\left(\left[-\chi\mu + \frac{1}{2}\chi^2\sigma^2\right]a\right) \Phi\left(\frac{[\mu - \chi\sigma^2]a}{\sigma\sqrt{a}}\right) \right] \\
& + \left(\frac{\zeta}{\chi}\right)^2 \chi \exp\left(\left[-\mu + \frac{1}{2}\chi\sigma^2\right]\chi a\right) \left[\sigma\sqrt{a}\phi\left(\frac{[\mu - \chi\sigma^2]a}{\sigma\sqrt{a}}\right) + [\mu - \chi\sigma^2]a\Phi\left(\frac{[\mu - \chi\sigma^2]a}{\sigma\sqrt{a}}\right) \right]
\end{aligned}$$

or:

$$\begin{aligned}
& \Lambda_g(a) \\
& = \left(1 - \left(\frac{\zeta}{\chi}\right)^2\right) \Phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right) + \left(1 + \left(\frac{\zeta}{\chi}\right)^2\right) \exp\left(\left[\mu + \frac{1}{2}\zeta\sigma^2\right]\zeta a\right) \Phi\left(-\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right) \\
& + \zeta \left(1 + \frac{\zeta}{\chi}\right) \exp\left(\left[\mu + \frac{1}{2}\zeta\sigma^2\right]\zeta a\right) \left[\sigma\sqrt{a}\phi\left(\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right) - [\mu + \zeta\sigma^2]a\Phi\left(-\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right) \right]
\end{aligned}$$

Using (36):

$$\begin{aligned}
& \Lambda_g(a) \times \left[\frac{1}{\chi^2} \frac{1}{\frac{\mu a}{\sigma\sqrt{a}}} \phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right) \right]^{-1} \\
& = (\zeta^2 - \chi^2) \left[\frac{-\frac{\mu a}{\sigma\sqrt{a}} \Phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right)}{\phi\left(\frac{\mu a}{\sigma\sqrt{a}}\right)} \right] + (\zeta^2 + \chi^2) \left(\frac{\mu}{\mu + \zeta\sigma^2} \right) \left[\frac{\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}} \Phi\left(-\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right)}{\phi\left(\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right)} \right] \\
& + \zeta\chi(\zeta + \chi)\mu a \left[1 - \frac{\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}} \Phi\left(-\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right)}{\phi\left(\frac{[\mu + \zeta\sigma^2]a}{\sigma\sqrt{a}}\right)} \right]
\end{aligned}$$

Using the definitions of u , v and λ :

$$\begin{aligned}
& \Lambda_g(a) \times \left[-\frac{1}{u} \phi(u) \right]^{-1} \\
& = (\lambda^2 - 1) \left[\frac{u\Phi(-u)}{\phi(u)} \right] + (\lambda^2 + 1) \left(\frac{\mu}{\mu + \zeta\sigma^2} \right) \left[\frac{v\Phi(-v)}{\phi(v)} \right] + \lambda(\zeta + \chi)\mu a \left[1 - \frac{v\Phi(-v)}{\phi(v)} \right]
\end{aligned}$$

By (13), $(\zeta + \chi)\mu a = -2uv$ and hence:

$$\Lambda_g(a) = \left((\lambda^2 - 1) \left[\frac{-\Phi(-u)}{\phi(u)} \right] + (\lambda^2 + 1) \left[\frac{\Phi(-v)}{\phi(v)} \right] + 2\lambda v \left[1 - \frac{v\Phi(-v)}{\phi(v)} \right] \right) \phi(u)$$

Observe:

$$\begin{aligned}
\Lambda_g(a) & = \left((\lambda^2 - 1) \left[\frac{-\Phi(-u)}{\phi(u)} \right] + (\lambda^2 + 1) \left[\frac{\Phi(-v)}{\phi(v)} \right] + 2\lambda v \left[1 - \frac{v\Phi(-v)}{\phi(v)} \right] \right) \phi(u) \\
& = \left((\lambda^2 - 1) \left[\frac{-u\Phi(-u)}{\phi(u)} \right] + (\lambda^2 + 1) \left[\frac{u}{v} \right] \left[\frac{v\Phi(-v)}{\phi(v)} \right] + 2\lambda \left[\frac{u}{v} \right] v^2 \left[1 - \frac{v\Phi(-v)}{\phi(v)} \right] \right) \frac{\phi(u)}{u}
\end{aligned}$$

$$\begin{aligned}
&= \left(-(\lambda^2 - 1) \Psi(u) + (\lambda^2 + 1) \left[\frac{u}{v} \right] \Psi(v) + 2\lambda \left[\frac{u}{v} \right] v^2 [1 - \Psi(v)] \right) \frac{\phi(u)}{u} \\
&= \left[-(\lambda^2 - 1) \Psi(u) + (\lambda + 1)^2 \left[\frac{u}{v} \right] \Psi(v) - 2\lambda \left[\frac{u}{v} \right] (1 - (1 + v^2) [1 - \Psi(v)]) \right] \frac{\phi(u)}{u} \\
&= \left[(\lambda^2 - 1) [\Psi(v) - \Psi(u)] - 2\lambda \left[\frac{u}{v} \right] (1 - (1 + v^2) [1 - \Psi(v)]) \right] \frac{\phi(u)}{u}
\end{aligned}$$

By (12) $2\lambda u/v = (1 - \lambda^2) \omega$, and hence:

$$\Lambda_g(a) = (1 - \lambda^2) [\Psi(u) - \Psi(v) - \omega (1 - (1 + v^2) [1 - \Psi(v)])] \frac{\phi(u)}{u} \quad (41)$$

where ω is the constant defined in (8). Another way to write this is:

$$\Lambda_g(a) = (1 - \lambda^2) \left\{ 1 - [\Psi(w) - \Psi(v) - \omega (1 - (1 + v^2) [1 - \Psi(v)])] \frac{\phi(w)}{w} \right\} \quad (42)$$

since $w = -u$.

B CALCULATING UNCONDITIONAL HAZARD RATES

2.1 Exponential Initial Conditions

The definitions of u and v immediately imply $u/v = -\mu/(\mu + \zeta\sigma^2)$, and thus:

$$[\Psi(v) - \Psi(u)] \frac{1}{u} \phi(u) = \frac{\mu + \zeta\sigma^2}{-\mu} \frac{\Phi(-v)}{\phi(v)} \times \phi(u) - \Phi(-u)$$

Note that $D\phi(v) = -v\phi(v)$, and hence:

$$D \left[\frac{\Phi(-v)}{\phi(v)} \right] = - \left[1 - \frac{v\Phi(-v)}{\phi(v)} \right]$$

Therefore:

$$\begin{aligned}
&\frac{\partial}{\partial a} \left[\frac{\mu + \zeta\sigma^2}{-\mu} \frac{\Phi(-v)}{\phi(v)} \times \phi(u) - \Phi(-u) \right] \\
&= \frac{\mu + \zeta\sigma^2}{-\mu} \frac{\partial}{\partial a} \left[\frac{\Phi(-v)}{\phi(v)} \times \phi(u) \right] - \frac{\partial}{\partial a} \Phi(-u) \\
&= \frac{v}{u} \left[\frac{\Phi(-v)}{\phi(v)} [-u\phi(u)] \frac{\partial u}{\partial a} - \left[1 - \frac{v\Phi(-v)}{\phi(v)} \right] \frac{\partial v}{\partial a} \phi(u) \right] + \phi(u) \frac{\partial u}{\partial a} \\
&= \left[1 - \frac{v\Phi(-v)}{\phi(v)} \right] \phi(u) \frac{1}{u} \left(u^2 \left[\frac{1}{u} \frac{\partial u}{\partial a} \right] - v^2 \left[\frac{1}{v} \frac{\partial v}{\partial a} \right] \right) \\
&= \left[1 - \frac{v\Phi(-v)}{\phi(v)} \right] \phi(u) (u^2 - v^2) \frac{1}{u} \frac{1}{2a}
\end{aligned}$$

The hazard is thus:

$$-\frac{D\Lambda_e(a)}{\Lambda_e(a)} = -\frac{\left[1 - \frac{v\Phi(-v)}{\phi(v)}\right] \phi(u) (u^2 - v^2) \frac{1}{u} \frac{1}{2a}}{[\Psi(v) - \Psi(u)] \frac{1}{u} \phi(u)} = \frac{u^2 - v^2}{2a} \frac{1 - \Psi(v)}{\Psi(u) - \Psi(v)}$$

The expressions (19) and (20) follow from the definition of ω and $v/\sqrt{a} = (\mu + \zeta\sigma^2)/\sigma$.

2.2 Gamma Initial Conditions

Define:

$$\Xi(u, v) = [\Psi(u) - \Psi(v) - \omega (1 - (1 + v^2) [1 - \Psi(v)])] \frac{\phi(u)}{u}$$

We need to evaluate $d\Xi(u, v)/da$, which equals:

$$\begin{aligned} D_1\Xi(u, v) \frac{\partial u}{\partial a} + D_2\Xi(u, v) \frac{\partial v}{\partial a} &= \frac{1}{a} \left[u D_1\Xi(u, v) \frac{a}{u} \frac{\partial u}{\partial a} + v D_2\Xi(u, v) \frac{a}{v} \frac{\partial v}{\partial a} \right] \\ &= \frac{1}{2a} [u D_1\Xi(u, v) + v D_2\Xi(u, v)] \end{aligned}$$

Write:

$$\Xi(u, v) = \Phi(-u) - \frac{\phi(u)}{u} [\omega (1 - (1 + v^2) [1 - \Psi(v)]) + \Psi(v)]$$

Note that:

$$u D \left[\frac{\phi(u)}{u} \right] = - (1 + u^2) \frac{\phi(u)}{u}$$

and therefore:

$$u D_1\Xi(u, v) = (-u^2 + (1 + u^2) [\omega (1 - (1 + v^2) [1 - \Psi(v)]) + \Psi(v)]) \frac{\phi(u)}{u}$$

Next, note that:

$$\begin{aligned} D\Psi(v) &= D \left[\frac{v\Phi(-v)}{\phi(v)} \right] \\ &= \frac{D[v\Phi(-v)] \phi(v) - v\Phi(-v) D[\phi(v)]}{\phi^2(v)} \\ &= -\frac{1}{v} \left[v^2 - (1 + v^2) \frac{v\Phi(-v)}{\phi(v)} \right] \\ &= \frac{1}{v} (1 - (1 + v^2) [1 - \Psi(v)]) \end{aligned}$$

and:

$$\begin{aligned} D[(1 + v^2) [1 - \Psi(v)]] &= 2v [1 - \Psi(v)] - (1 + v^2) D\Psi(v) \\ &= \frac{1}{v} [2v^2 [1 - \Psi(v)] - (1 + v^2) (1 - (1 + v^2) [1 - \Psi(v)])] \end{aligned}$$

and therefore:

$$\begin{aligned} vD_2\Xi(u, v) &= \omega [2v^2 [1 - \Psi(v)] - (1 + v^2) (1 - (1 + v^2) [1 - \Psi(v)])] \frac{\phi(u)}{u} \\ &\quad - (1 - (1 + v^2) [1 - \Psi(v)]) \frac{\phi(u)}{u} \end{aligned}$$

Adding these two expressions gives:

$$\begin{aligned} &\left[\frac{\phi(u)}{u} \right]^{-1} [uD_1\Xi(u, v) + vD_2\Xi(u, v)] \\ &= -u^2 + (1 + u^2) [\omega (1 - (1 + v^2) [1 - \Psi(v)]) + \Psi(v)] \\ &\quad + \omega [2v^2 [1 - \Psi(v)] - (1 + v^2) (1 - (1 + v^2) [1 - \Psi(v)])] \\ &\quad - (1 - (1 + v^2) [1 - \Psi(v)]) \\ &= \omega (u^2 - v^2) (1 - (1 + v^2) [1 - \Psi(v)]) + 1 - (1 + u^2) [1 - \Psi(v)] \\ &\quad + \omega 2v^2 [1 - \Psi(v)] - (1 - (1 + v^2) [1 - \Psi(v)]) \\ &= \omega (u^2 - v^2) (1 - (1 + v^2) [1 - \Psi(v)]) + 2v^2 \left[\omega - \frac{u^2 - v^2}{2v^2} \right] [1 - \Psi(v)] \end{aligned}$$

The definition of ω implies that the second term in this expression is zero, and therefore:

$$uD_1\Xi(u, v) + vD_2\Xi(u, v) = \omega (u^2 - v^2) (1 - (1 + v^2) [1 - \Psi(v)]) \frac{\phi(u)}{u}$$

Hence:

$$\begin{aligned} -\frac{uD_1\Xi(u, v) + vD_2\Xi(u, v)}{\Xi(u, v)} &= \frac{\omega (u^2 - v^2) (1 - (1 + v^2) [1 - \Psi(v)])}{\omega (1 - (1 + v^2) [1 - \Psi(v)]) + \Psi(v) - \Psi(u)} \\ &= \frac{u^2 - v^2}{1 + \frac{1}{\omega} \frac{\Psi(v) - \Psi(u)}{1 - (1 + v^2) [1 - \Psi(v)]}} = \frac{2v^2\omega}{1 + \frac{1}{\omega} \frac{\Psi(v) - \Psi(u)}{1 - (1 + v^2) [1 - \Psi(v)]}} \end{aligned}$$

Therefore:

$$-\frac{D_1\Xi(u, v) \frac{\partial u}{\partial a} + D_2\Xi(u, v) \frac{\partial v}{\partial a}}{\Xi(u, v)} = \frac{v^2\omega}{a} \left[1 + \frac{1}{\omega} \frac{\Psi(v) - \Psi(u)}{1 - (1 + v^2) [1 - \Psi(v)]} \right]^{-1} \quad (43)$$

C CONTINUED FRACTION BOUNDS

3.1 Bounds on the Normal Integral

The following bounds are based on:

Lee, C.C. (1992): "On Laplace Continued Fraction for the Normal Integral," *Annals of the Institute of Statistical Mathematics*, vol. 44, no. 1, pp. 107-120.

Define:

$$\begin{aligned}
f_1(x; a, b) &= \frac{1}{x + \frac{b}{x+a}} = \frac{x+a}{x^2 + b + ax} \\
f_2(x; a, b) &= \frac{1}{x + \frac{1}{x + \frac{b}{x+a}}} = \frac{x^2 + ax + b}{x^3 + ax^2 + (b+1)x + a} \\
f_3(x; a, b) &= \frac{1}{x + \frac{1}{x + \frac{2}{x + \frac{b}{x+a}}}} = \frac{x^3 + ax^2 + (b+2)x + 2a}{x^4 + ax^3 + (b+3)x^2 + 3ax + b} \\
&\vdots \\
f_n(x; a, b) &= \frac{1}{x + \frac{1}{x + \frac{2}{x + \frac{3}{x + \frac{\ddots}{x + \frac{n-1}{x + \frac{b}{x+a}}}}}}} = \frac{1}{x+} \frac{1}{x+} \frac{2}{x+} \frac{3}{x+} \dots \frac{n-1}{x+} \frac{b}{x+} a
\end{aligned}$$

and:

$$L_n = f_n(x; 0, n)$$

The results presented in Lee (1992) show that:

$$L_1 < L_3 < \dots < R(x) < \dots < L_4 < L_2$$

and:

$$f_{2k}\left(x; \sqrt{2k+1}, 2k\right) < R(x) < f_{2k+1}\left(x; \sqrt{2(k+1)}, 2k+1\right)$$

for $k = 0, 1, 2, \dots$

3.2 The First Bound

We want to show:

$$R(x) = e^{x^2/2} \int_x^\infty e^{-z^2/2} dz \geq \frac{1}{4x^3} \left(1 + 5x^2 - \sqrt{1 + 10x^2 + x^4}\right) \quad (44)$$

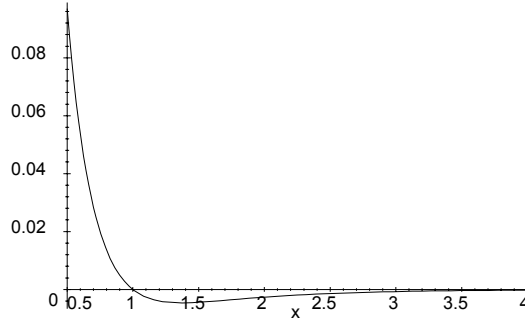
Take $k = 1$. Then the lower bound $f_2(x; \sqrt{3}, 2)$ yields:

$$e^{x^2/2} \int_x^\infty e^{-z^2/2} dz - \frac{x^2 + x\sqrt{3} + 2}{x^3 + x^2\sqrt{3} + 3x + \sqrt{3}} \geq 0$$

Observe that:

$$\frac{x^2 + x\sqrt{3} + 2}{x^3 + x^2\sqrt{3} + 3x + \sqrt{3}} - \frac{1}{4x^3} \left(1 + 5x^2 - \sqrt{1 + 10x^2 + x^4}\right)$$

looks like:



Note that:

$$\frac{1 + \sqrt{3} + 2}{1 + \sqrt{3} + 3 + \sqrt{3}} - \frac{1}{4} \left(1 + 5 - \sqrt{1 + 10 + 1}\right) = 0$$

So the bound works for all $x \in [0, 1]$. More explicitly:

$$\frac{x^2 + x\sqrt{3} + 2}{x^3 + x^2\sqrt{3} + 3x + \sqrt{3}} \geq \frac{1}{4x^3} \left(1 + 5x^2 - \sqrt{1 + 10x^2 + x^4}\right)$$

corresponds to:

$$1 + 10x^2 + x^4 \geq \left(1 + 5x^2 - 4x^3 \left(\frac{x^2 + x\sqrt{3} + 2}{x^3 + x^2\sqrt{3} + 3x + \sqrt{3}}\right)\right)^2 \geq 0$$

Now:

$$1 + 10x^2 + x^4 - \left(1 + 5x^2 - 4x^3 \left(\frac{x^2 + x\sqrt{3} + 2}{x^3 + x^2\sqrt{3} + 3x + \sqrt{3}}\right)\right)^2 = \frac{16\sqrt{3}x^3(1 - x^2)}{(x^3 + x^2\sqrt{3} + 3x + \sqrt{3})^2}$$

and this is indeed non-negative for $x \in [0, 1]$.

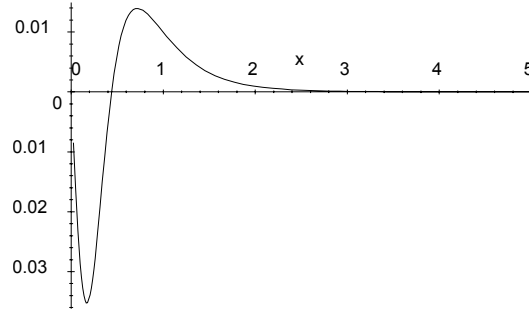
Next, consider the lower bound L_7 :

$$e^{x^2/2} \int_x^\infty e^{-z^2/2} dz - \frac{1}{x + \frac{1}{x + \frac{1}{x + \frac{1}{x + \frac{1}{x + \frac{1}{x + \frac{1}{x}}}}}}} \geq 0$$

To see how this compares with the desired lower bound observe that:

$$\frac{1}{x + \frac{1}{x + \frac{1}{x + \frac{1}{x + \frac{1}{x + \frac{1}{x + \frac{1}{x}}}}}}} - \frac{1}{4x^3} \left(1 + 5x^2 - \sqrt{1 + 10x^2 + x^4}\right)$$

looks like:



This appears to work for all $x \geq 1$. Since we already have a bound that works for all $x < 1$, this will give the desired result. The 5th-order continued fraction does not suffice.

The continued fraction can be written as:

$$\frac{1}{x + \frac{1}{x + \frac{1}{x + \frac{1}{x + \frac{1}{x + \frac{1}{x + \frac{1}{x + \frac{1}{x + \frac{1}{x}}}}}}}}} = x \left(\frac{x^6 + 27x^4 + 185x^2 + 279}{x^8 + 28x^6 + 210x^4 + 420x^2 + 105} \right)$$

We need to show that for $x \geq 1$:

$$\frac{x^6 + 27x^4 + 185x^2 + 279}{x^8 + 28x^6 + 210x^4 + 420x^2 + 105} \geq \frac{1}{4x^4} \left(1 + 5x^2 - \sqrt{1 + 10x^2 + x^4} \right)$$

or:

$$\sqrt{1 + 10x^2 + x^4} \geq 1 + 5x^2 - \frac{4x^4 (x^6 + 27x^4 + 185x^2 + 279)}{x^8 + 28x^6 + 210x^4 + 420x^2 + 105}$$

A sufficient condition is that this holds for the squares of the two sides of this inequality.

Therefore, we need:

$$1 + 10x^2 + x^4 - \left(1 + 5x^2 - \frac{4x^4 (x^6 + 27x^4 + 185x^2 + 279)}{x^8 + 28x^6 + 210x^4 + 420x^2 + 105} \right)^2 \geq 0$$

for $x \geq 1$. This simplifies to:

$$\frac{96x^4 (x^8 + 36x^6 + 414x^4 + 1540x^2 - 315)}{(x^8 + 28x^6 + 210x^4 + 420x^2 + 105)^2} \geq 0$$

or:

$$x^8 + 36x^6 + 414x^4 + 1540x^2 - 315 \geq 0$$

for $x \geq 1$. Equivalently:

$$y^4 + 36y^3 + 414y^2 + 1540y - 315 \geq 0$$

for all $y \geq 1$. This function has a positive slope for all $y \geq 0$. Take $y = .2$. This gives:

$$(.2)^4 + 36 (.2)^3 + 414 (.2)^2 + 1540 (.2) - 315 = 9.8496$$

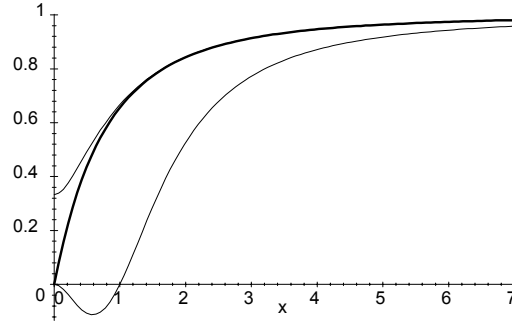
Thus the desired condition holds for all $y \geq .2$, or $x \geq \sqrt{.2} = .44721$.

3.3 The Second Bound

We want to show:

$$1 - \frac{1}{\frac{1}{4}(5 + 3x^2 - \sqrt{1 + 14x^2 + x^4})} \leq M(x) \leq 1 - \frac{1}{\frac{1}{4}(5 + 3x^2 + \sqrt{1 + 14x^2 + x^4})} \quad (45)$$

The function $M(x)$ and its upper and lower bounds look like:



These bounds on $M(x)$ can alternatively be written as:

$$\frac{1}{\frac{1}{4}(5 + 3x^2 + \sqrt{1 + 14x^2 + x^4})} \leq 1 - M(x) \leq \frac{1}{\frac{1}{4}(5 + 3x^2 - \sqrt{1 + 14x^2 + x^4})}$$

or:

$$\frac{1 + 3x^2 - \sqrt{1 + 14x^2 + x^4}}{5 + 3x^2 - \sqrt{1 + 14x^2 + x^4}} \leq M(x) \leq \frac{1 + 3x^2 + \sqrt{1 + 14x^2 + x^4}}{5 + 3x^2 + \sqrt{1 + 14x^2 + x^4}}$$

3.3.1 The Lower Bound

Note that:

$$e^{x^2/2} \int_x^\infty e^{-z^2/2} dz \geq \frac{1}{x + \frac{1}{x}}$$

It suffices to show that:

$$\frac{x^2}{1 + x^2} \geq \frac{1 + 3x^2 - \sqrt{1 + 14x^2 + x^4}}{5 + 3x^2 - \sqrt{1 + 14x^2 + x^4}}$$

Note that the required inequality is equivalent to:

$$x^2 \left(5 + 3x^2 - \sqrt{1 + 14x^2 + x^4} \right) - \left(1 + 3x^2 - \sqrt{1 + 14x^2 + x^4} \right) (1 + x^2) \geq 0$$

since the denominator is always positive. This simplifies to:

$$x^2 + \sqrt{1 + 14x^2 + x^4} \geq 1$$

which clearly holds for all x .

3.3.2 The Upper Bound

Consider:

$$f_{2k+1} \left(x; \sqrt{2(k+1)}, 2k+1 \right)$$

for $k = 2$. This yields:

$$f_{2k+1} \left(x; \sqrt{2(k+1)}, 2k+1 \right) = f_{2k+1} \left(x; \sqrt{6}, 5 \right) = \frac{1}{x + \frac{1}{x + \frac{2}{x + \frac{3}{x + \frac{4}{x + \frac{5}{x + \sqrt{6}}}}}}}$$

Then:

$$\frac{x}{x + \frac{1}{x + \frac{2}{x + \frac{3}{x + \frac{4}{x + \frac{5}{x + \sqrt{6}}}}}}} - x e^{x^2/2} \int_x^\infty e^{-z^2/2} dz \geq 0$$

Consider:

$$\frac{1 + 3x^2 + \sqrt{1 + 14x^2 + x^4}}{5 + 3x^2 + \sqrt{1 + 14x^2 + x^4}} - \frac{x}{x + \frac{1}{x + \frac{2}{x + \frac{3}{x + \frac{4}{x + \frac{5}{x + \sqrt{6}}}}}}}$$

We need to show that:

$$\begin{aligned} \frac{1 + 3x^2 + \sqrt{1 + 14x^2 + x^4}}{5 + 3x^2 + \sqrt{1 + 14x^2 + x^4}} &\geq \frac{x}{x + \frac{1}{x + \frac{2}{x + \frac{3}{x + \frac{4}{x + \frac{5}{x + \sqrt{6}}}}}}} \\ &= \frac{x^6 + x^5\sqrt{6} + 14x^4 + 9x^3\sqrt{6} + 33x^2 + 8x\sqrt{6}}{x^6 + x^5\sqrt{6} + 15x^4 + 10x^3\sqrt{6} + 45x^2 + 15x\sqrt{6} + 15} \end{aligned}$$

Clearly, the result is true at 0 and at ∞ .

The difference:

$$\frac{1 + 3x^2 + \sqrt{1 + 14x^2 + x^4}}{5 + 3x^2 + \sqrt{1 + 14x^2 + x^4}} - \frac{x^6 + x^5\sqrt{6} + 14x^4 + 9x^3\sqrt{6} + 33x^2 + 8x\sqrt{6}}{x^6 + x^5\sqrt{6} + 15x^4 + 10x^3\sqrt{6} + 45x^2 + 15x\sqrt{6} + 15}$$

is proportional to:

$$(15 - 75x^2 - 19x^4) + (15 + 7x\sqrt{6} + 12x^2 + x^3\sqrt{6} + x^4) \sqrt{1 + 14x^2 + x^4} - (25x\sqrt{6} + 14x^3\sqrt{6} + x^5\sqrt{6} + x^6)$$

Write this as:

$$0 \leq 15 - 25x\sqrt{6} - 75x^2 - 14x^3\sqrt{6} - 19x^4 - x^5\sqrt{6} - x^6 + (15 + 7x\sqrt{6} + 12x^2 + x^3\sqrt{6} + x^4) \sqrt{1 + 14x^2 + x^4}$$

Or:

$$\begin{aligned} & -15 + 25x\sqrt{6} + 75x^2 + 14x^3\sqrt{6} + 19x^4 + x^5\sqrt{6} + x^6 \\ & \leq (15 + 7x\sqrt{6} + 12x^2 + x^3\sqrt{6} + x^4) \sqrt{1 + 14x^2 + x^4} \end{aligned}$$

A sufficient condition is that this inequality holds for the squares of the left and right-hand sides. This amounts to:

$$\begin{aligned} & (15 + 7x\sqrt{6} + 12x^2 + x^3\sqrt{6} + x^4)^2 (1 + 14x^2 + x^4) \\ & \geq (-15 + 25x\sqrt{6} + 75x^2 + 14x^3\sqrt{6} + 19x^4 + x^5\sqrt{6} + x^6)^2 \end{aligned}$$

Subtracting the right-hand side from the left-hand side and dividing the result by $2^6 3$ gives:

$$2x^4 + 12x^2 + (5 - x^2) x\sqrt{6} \geq 0$$

Another way to write this is:

$$y^3 + 36y - 3y^2 + 90 \geq 0$$

where $y = x\sqrt{6}$. The left-hand side has slope:

$$3y^2 - 6y + 36 = 3 [(y - 1)^2 + 11]$$

This is always positive. The value of the difference at zero is positive, and hence the difference is positive everywhere.