Growth and the Size Distribution of Firms

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First version: November 12, 2003
This version: May 16, 2005

[1]The views expressed herein are those of the author and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System. I thank Michele Boldrin, Jonathan Eaton, Xavier Gabaix, Tom Holmes, Sam Kortum and Bob Lucas for helpful discussions based on earlier versions of this paper, as well as seminar participants at the 2004 SED meetings, New York University, and the University of Chicago. The usual disclaimer applies.

Abstract

This paper describes an analytically tractable model of balanced growth that is consistent with the observed size distribution of firms. Growth is the result of idiosyncratic firm productivity improvements, selection of successful firms, and imitation by potential entrants. The empirical phenomon of Zipf's law can be interpreted to mean that entry costs are high and that imitation is difficult. Lowering barriers to entry tends to speed up the rate at which selection improves aggregate productivity, and this increases the growth rate of the economy.

1. Introduction

This paper presents an analytically tractable model of growth resulting from firm-specific preference and technology shocks, selective survival of successful firms, and imitation by entering firms. The model generates balanced growth and is consistent with salient features of the firm size distribution.

As many have noted, the size distribution of firms exhibits a striking pattern. Using 1997 data from the U.S. Census, Axtell (2001) finds that the log right tail probabilities of this distribution, with firm size measured by the log of employment, are on a virtual straight line with a slope of -1.059. Figure 1 below shows the data for 2001, together with a curve generated by a version of the model presented in this paper. A straight line fitted using all size categories with at least 5 employees has a slope of -1.053. This evidence suggests that the firm size distribution is well approximated, over much of its range, by a (generalized) Pareto distribution with right tail probabilities of the form $1/S^{\zeta}$, where S represents firm employment and $\zeta \approx 1.053$.[1]

The remarkable fit of this distribution has been documented and interpreted before, perhaps most notably by Simon and Bonini (1958), Steindl (1965), and Ijiri and Simon (1977). As far back as Gibrat (1931), researchers have related the shape of the observed size distribution to models of firm entry, random growth, and exit. The mechanism described in this paper is most like the one proposed for the city size distribution by Gabaix (1999).[2] In contrast to this literature, this paper explains the observed firm size distributions in terms of primitives such as entry and fixed costs, and the ease with which firms can imitate. The explanation is set in the context of a general equilibrium model, and this allows one to predict the effects of changes in various barriers to entry on the level and the growth rate of aggregate output. The model can also be extended in a

[1]The data shown in Figure 1 summarize a population of 5,657,774 U.S. firms in 2001. The largest size category, that of 10,000 employees and over, still contains 930 firms. There is a size category of zero employees (in March of 2001) accounting for 703,837 firms that is not shown. The data are originally from the U.S. Census Bureau, and were obtained from the Small Business Administration internet site, and from the Statistics of U.S. Businesses site of the U.S. Census Bureau (the size categories 5,000-9,999 and 10,000 and over). The fitted curve represents a mixture of Gamma distributions, as discussed in Section 6. Of course, any theoretical distribution with an unbounded support must fail to fit the data for large enough firms.

[2]Sutton (1997) surveys the literature on firm size and Gibrat's law: firm growth is independent of size. Gabaix (1999) contains extensive discussions of the literature on probability models that give rise to Pareto distributions, and their application in economics. Gabaix and Ioannides (2003) survey the literature on Zipf's law for cities.

tractable way to accommodate more extensive forms of heterogeneity (Luttmer [2004]), making it a potentially useful tool for empirical research on the relation between firm heterogeneity and aggregate productivity.

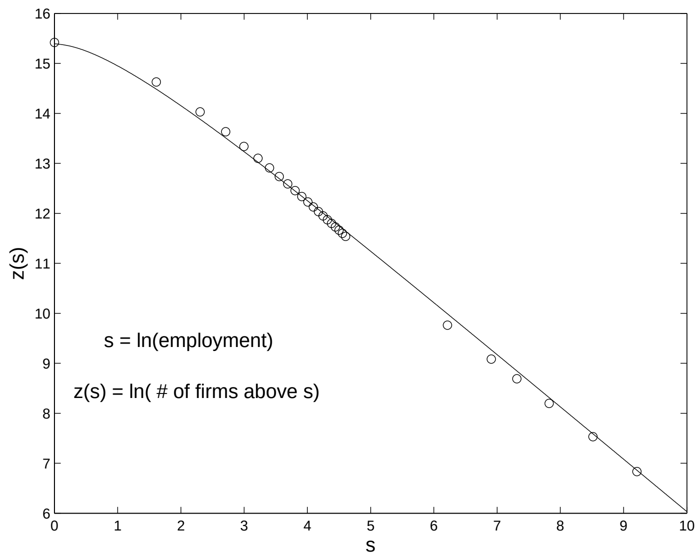

Figure 1: Size Distribution of U.S. Firms in 2001

Firms in this paper are monopolistic competitors producing differentiated goods, as in Dixit and Stiglitz (1977), using a linear technology. There is an entry cost for new firms, and it takes a fixed cost per unit of time to continue an existing firm. A typical firm is subject to shocks to both productivity and the demand for its differentiated good. These shocks are firm-specific and permanent.[3] A stationary firm size distribution arises if the average rate at which these shocks improve the profitability of incumbent firms is not too high relative to the rate at which the technology available to potential entrants improves over time.

One version of this economy is a model of technology adoption in which the technologies available to potential entrants improve at an exogenous rate. This rate determines the growth rate of the economy. If there is not too much heterogeneity among entrants,

[3] See Melitz (2003) for a related model that features firm heterogeneity, monopolistic competition, together with entry and fixed costs. Much of what follows can be shown also in an economy with perfectly competitive final goods markets, decreasing returns at the firm level, and firm-specific technology shocks. This would give rise to an economy similar to Lucas (1978), Hopenhayn (1992), Atkeson and Kehoe (2001), and Hellwig and Irmen (2001). Most data sets show a lot of heterogeneity across firms, even within narrowly defined industries. An advantage of the monopolistic competition formulation is that shocks to the demands for differentiated goods can be a source of firm heterogeneity, above and beyond firm-specific technology shocks.

then the equilibrium size distribution is well approximated, over much of its range, by a Pareto distribution. A tail index ζ slightly above 1 arises if the technologies available to entrants improve at a rate that is only slightly above the rate at which the technologies of incumbents improve. In this economy, a proportional increase in entry and fixed costs lowers the level of aggregate output by reducing the number of firms and thereby the variety of goods produced. This is analogous to results for static economies in Krugman (1979). The shape of the size distribution is not affected by proportional changes in entry and fixed costs. A reduction in the entry cost alone does change the shape of the size distribution, although not its tail index. Lower entry costs lead to more firms and more variety, but the positive effect of this on the level of output is weakened by the fact that more inefficient firms will enter and survive.[4]

A second version of this economy is a model of endogenous growth in which entering firms can imperfectly imitate incumbent firms. A potential entrant can pay an entry cost to sample at random from the population of incumbent firms. The entrant can then attempt to imitate the incumbent drawn from the population by introducing a new good with an initial productivity and market size that are scaled down relative to the productivity and market size of the incumbent. This spillover ensures that the technologies available to potential entrants are never so far behind those of incumbent firms that entry of new firms is not feasible. The economy has a continuum of stationary size distributions that are consistent with balanced growth. One possibility is for log firm size to follow a Gamma distribution. All possible size distributions have a tail similar to that of a Pareto distribution, with an analogous tail index ζ that must be slightly above 1 to fit the data shown in Figure 1. The main result for this economy is that ζ converges to 1 from above as the cost of entry becomes large relative to the fixed cost of operating a firm, and as the extent to which new entrants lag behind incumbents in terms of productivity and market size becomes large.

To see why the asymptote $\zeta = 1$ arises, note that the mean of a distribution with right tail probabilities of the order $1/S^{\zeta}$ grows without bound as ζ approaches 1 from above. The combination of fixed costs and constant returns implies that firm profitability is tied to size, and the fact that potential entrants imitate a randomly sampled incumbent ties the expected gains from entry to the average size of the incumbent population. In equilibrium, the higher the entry cost, the higher must be the expected gains from entry, and therefore the larger must be the size of the average firm. A size distribution such as

[4] See Parente and Prescott (1999) for an alternative model of technology adoption in which lowering barriers to entry can have large positive effects on the level of output.

shown in Figure 1 thus means that entry is very difficult, because of high entry costs, or because entrants lag so far behind incumbents in terms of productivity and market size.

As in the version with exogenous growth, a proportional reduction in entry and fixed costs increases the level of output in this economy. The effect of lowering entry costs alone is to lower the average size and profitability of firms. This is achieved in equilibrium by an increase in the turnover rate of firms, and this speeds up the selection mechanism by which aggregate productivity improves over time. As a result, the growth rate of the economy increases. A reduction in barriers to entry will, over time, have large effects on output when entrants can imitate incumbents. This is in sharp contrast to the level effect that arises when the technologies available to entrants are exogenous.

Related Literature Incumbent firms in this paper are engaged in a form of learning-by-doing, and imitation by entering firms creates an externality, two features of growth emphasized by Arrow (1962).[5] Following Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992), technological progress is embodied in firms, and firms have some market power. As in Romer (1990), this takes the form of monopolistic competition.[6] The current paper differs in two important respects from Romer (1990). First, firms experience idiosyncratic permanent shocks to their technologies and to the demands for their differentiated commodities. This introduces selection as a mechanism by which the economy-wide distribution of productivity improves over time. Random growth and selection are crucial for matching the observed firm size distribution. Second, the mechanism that allows potential entrants to make use of the existing stock of ideas is made explicit. This yields an economic interpretation of the size distribution shown in Figure 1: imitation is imperfect and must be very costly.[7]

In Jovanovic (1982), the effects of selection on the evolution of an industry eventually die out because firms are not subject to ongoing technology shocks. A key feature of the industry equilibrium studied by Hopenhayn (1992) is the assumption that firm

[5] The more recent literature making use of these features includes Boldrin and Scheinkman (1988), Lucas (1988), Stokey (1988) and Young (1991)

[6] Jones and Manuelli (1990) and Boldrin and Levine (2000) construct models of endogenous growth that do not rely on imperfect competition or externalities.

[7] Jovanovic (1982) emphasizes the role of selection in the evolution of an industry. Nelson and Winter (1982) relate selection, imitation, and growth, but their model is not analytically tractable. Jovanovic and MacDonald (1994) consider industry growth with very general forms of imitation. Other models of imitation and growth include Segerstrom (1991), Aghion, Harris, Howitt and Vickers (2001), and Eeckhout and Jovanovic (2002). Barro and Sala-i-Martin (2004) present models of growth that rely on cross-country imitation.

productivity is stationary. This guarantees a stationary industry equilibrium, but there is no reason for the resulting size distribution to look like the one displayed in Figure 1. In this paper, instead, all shocks to preferences and technology are permanent. Stationarity of the cross-sectional size distribution is a consequence of the spillover that relates the productivity of entrants to the distribution of productivity among incumbents.

Gabaix (1999) shows how a geometric Brownian motion with a reflecting barrier gives rise to a power law and shows the precise circumstances under which this will lead to Zipf's law. He uses this to construct a model of cities and explain evidence on the size distribution of cities. In the presence of entry and fixed costs, the process of firm entry and exit does not lead to a reflecting barrier, but to a "return process" according to which firms exit below some barrier and enter at a point above this barrier. The two processes are closely related, and the limiting argument used by Gabaix (1999) will be discussed below.

The economy described here has many elements in common with Klette and Kortum (2004), who build on Grossman and Helpman (1991) to construct a quality ladder model in which firm growth is the result of research and development choices made by firms. The economy grows because firms are able to improve on the quality of existing producers. As in this paper, entrants in Klette and Kortum (2004) are small relative to the average firm, firm growth satisfies Gibrat's law, and firms are eventually driven out of business with probability one. The resulting size distribution, where size is measured by the number of goods produced by the firm, is logarithmic. This distribution is highly skewed, with a monotonically decreasing density. But a plot as in Figure 1 generates a curve that is concave and that does not asymptote to a straight line for large firm sizes. The right tail of the distribution is too thin.

Rossi-Hansberg and Wright (2004) solve for the firm size distribution in an economy with many industries and many identical firms in each industry. Firms face a fixed cost in every period and operate Cobb-Douglas technologies that exhibit decreasing returns. Human capital is industry specific, and the number and size of firms in a particular industry at a point in time is determined by a static free-entry condition. It does not matter which of the infinitesimal firms in an industry exit when net exit from a particular industry is required. As a result, the model has no determinate implications for the joint age-size distribution of firms. In equilibrium, the industry-specific human capital stock exhibits mean reversion, and this generates a stationary firm size distribution. If shocks to the human capital accumulation technology are log-normal, then the size distribution is log-normal. This seems to be at odds with Figure 1. The log-normal distribution has many fewer large firms than are observed in the data.

Outline of the Paper The model of technology adoption is set up in Section 2. The size distribution is characterized in Section 3 and the balanced growth path is determined in Section 4. Imitation is introduced in Section 5, and the relations between entry costs, the size distribution, and the growth rate of the economy are described. Section 6 allows for multiple industries with different cost structures and growth rates, and shows how Figure 1 can be interpreted in such an economy. Section 7 contains concluding remarks.

2. Technology Adoption

2.1 Consumers

Time is continuous and indexed by t. There is a continuum of consumers alive at any point in time. The population size at time t is $He^{\eta t}$, and the population growth rate η is non-negative. During their lifetimes, consumers supply one unit of labor at every point in time. There is a representative consumer with preferences over rates of dynastic consumption $\{C_t\}_{t\geq 0}$ of a composite good, defined by the utility function:

$$\left(\mathrm{E}\left[\int_0^\infty \rho e^{-\rho t}\left[C_t e^{-\eta t}\right]^{1-\gamma}\mathrm{d}t\right]\right)^{1/(1-\gamma)}$$

The discount rate ρ and the intertemporal elasticity of substitution $1/\gamma$ are positive. The composite good is made up of a continuum of differentiated commodities. Preferences over these commodities are additively separable with weights that define the type of a commodity. This implies that all commodities of the same type and trading at the same price are consumed at the same rate. Let $c_t(u,p)$ be consumption at time t of a commodity of type u that trades at a price p. In equilibrium, there will be a measure M_t of commodities that are available at time t, defined on the set of commodity types and prices. The composite good is a version of the one specified in Dixit and Stiglitz (1977). For some $\omega \in (0,1)$:

$$C_t = \left[\int u^{1-\omega} c_t^{\omega}(u,p)\mathrm{d}M_t(u,p)\right]^{1/\omega} \tag{1}$$

The type u of a commodity can be viewed as measure of its quality. The level of $c_t(u,p)$ is chosen to minimize the cost of acquiring C_t. This implies that:

$$pc_t(u,p) = P_t\left(uC_t\right)^{1-\omega} c_t^{\omega}(u,p) \tag{2}$$

where P_t is the price index:

$$P_t = \left[\int u p^{-\omega/(1-\omega)} \mathrm{d}M_t(u,p) \right]^{-(1-\omega)/\omega} \tag{3}$$

The price elasticity of the demand for commodity (u, p) is $-1/(1-\omega)$, and the implied expenditure share is $u(p/P_t)^{-\omega/(1-\omega)}$.

The representative consumer faces a standard present-value budget constraint. The consumer's wealth consists of claims to firms and labor income. Along the balanced growth path constructed below, per capita consumption and wages grow at a common rate κ. The paths of per capita consumption and wages are denoted by $C_t e^{-\eta t} = C e^{\kappa t}$ and $w_t = w e^{\kappa t}$. When the composite good is used as the numeraire, the interest rate is constant and given by $r = \rho + \gamma\kappa$. The following assumption ensures that the present value of aggregate consumption and labor income is finite.

Assumption 1 *The growth rates η and κ satisfy $\eta \geq 0$ and $\rho + \gamma\kappa > \kappa + \eta$.*

This assumption implies that $\rho > (1-\gamma)\kappa$, and thus utility is finite.

2.2 Firms

A firm is defined by its unique access to a technology for producing a particular differentiated commodity. At age a, a firm that was set up at time t uses labor $L_{t,a}$ to produce $z_{t,a}L_{t,a}$ units of a differentiated commodity of quality $u_{t,a}$. Given a price $p_{t,a}$, the revenues of the firm are given by $R_{t,a} = p_{t,a}z_{t,a}L_{t,a}/P_t$, in units of the composite good. The demand function for type-$u_{t,a}$ commodities implies that these revenues can be written as:

$$R_{t,a} = C_{t+a}^{1-\omega} \left(Z_{t,a} L_{t,a} \right)^{\omega} \tag{4}$$

where $Z_{t,a} = (u_{t,a}^{1-\omega} z_{t,a}^{\omega})^{1/\omega}$ combines the state of preferences and technology. Firm revenues vary with aggregate consumption, the weight $u_{t,a}$ of its output in the utility function, and its productivity level $z_{t,a}$. With some abuse of terminology, the combination of quality and quantity measured by $Z_{t,a}$ will be referred to simply as productivity. The productivities $Z_{t,a}$ are assumed to evolve independently across firms, according to:

$$Z_{t,a} = Z \exp\left(\theta t + \vartheta a + \varsigma W_{t,a} \right) \tag{5}$$

where $\{W_{t,a}\}_{a \geq 0}$ is a standard Brownian motion and Z is an initial condition.[8] Note that $Z_{t,0} = Z e^{\theta t}$ is the initial productivity of a new firm at time t. Thus θ is the rate at

[8]This productivity process will result, for example, if both $u_{t,a}$ and $z_{t,a}$ are geometric Brownian motions.

which the productivity of entering firms grows over time. After entry, the trend of log productivity is determined by ϑ. The difference between θ and ϑ is a key determinant of the firm size distribution. It will be made endogenous in Section 5.

An existing firm can be continued only at a cost equal to λ_{F} units of labor per unit of time. The firm must exit if this fixed cost is not paid, and exit is irreversible. One interpretation is that it is costly to preserve the information accumulated as a result of past firm-specific shocks to preferences and technology, and that this information is lost as soon as the required costs are not incurred.[9] Measured in units of the composite good, the value $V_t[Z]$ at time t of a firm with initial productivity $Ze^{\theta t}$ is given by:

$$V_t[Z] = \max_{L,\tau} \mathrm{E}_t \left[\int_0^{\tau} e^{-ra} \left(R_{t,a} - w_{t+a} \left[L_{t,a} + \lambda_{\mathrm{F}} \right] \right) \mathrm{d}a \right]$$

The maximization is subject to (4) and (5), and subject to the restriction that production and exit decisions only depend on the available information.

The aggregate supply of labor grows at a rate η, and every firm must use at least λ_{F} units of labor to stay in business. Along the balanced growth path, the number of firms grows at the rate η. Entry and exit generates time-t cross-sectional distributions of labor inputs $L_{t-a,a}$ and productivities relative to trend $Z_{t-a,a}e^{-\theta t}$ that are time invariant. Since the number of firms grows at a rate η, the growth rate κ of per capita consumption must also be the growth rate of average revenues per firm. Together with (4) this gives:

$$\kappa = \theta + \left(\frac{1-\omega}{\omega} \right) \eta \tag{6}$$

Population growth implies growth in the number of differentiated commodities. This adds to the growth rate θ of productivity, with a slope that is large when substitution between these commodities is difficult.

2.2.1 Production Decisions

Firms choose variable labor to maximize variable profits $R_{t,a} - w_{t+a}L_{t,a}$, subject to (4). The optimal choice is:

$$\begin{bmatrix} R_{t,a} \\ w_{t+a}L_{t,a} \end{bmatrix} = \begin{bmatrix} 1 \\ \omega \end{bmatrix} \left(\frac{\omega Z_{t,a}}{w_{t+a}} \right)^{\omega/(1-\omega)} C_{t+a} \tag{7}$$

[9] Atkeson and Kehoe (2002) assume perfect competition together with decreasing returns to variable inputs and interpret λ_{F} as the cost of a managerial fixed factor, along the lines of Lucas (1978). Much of what follows continues to hold for such an alternative model.

Together with (5) and (6) this implies that, along the balanced growth path, labor and revenues measured in units of labor do not depend on calendar time. In particular, the revenues net of fixed and variable costs can be written as:

$$R_{t,a} - w_{t+a}\left(L_{t,a} + \lambda_{\mathrm{F}}\right) = w_{t+a}\lambda_{\mathrm{F}}\left(e^{s_a} - 1\right)$$

where s_a equals:

$$s_a = S[Z] + \frac{\omega}{1-\omega}\left[\ln\left(\frac{Z_{t,a}}{Z_{t,0}}\right) - \theta a\right] \tag{8}$$

and where $S[Z]$ is defined by:

$$e^{S[Z]} = \frac{1-\omega}{\lambda_{\mathrm{F}}}\frac{C}{w}\left(\frac{\omega Z}{w}\right)^{\omega/(1-\omega)} \tag{9}$$

Both revenues and variable labor inputs are proportional to $w_{t+a}\lambda_{\mathrm{F}}e^{s_a}$. The variable s_a can thus be viewed as a measure of firm size relative to fixed costs. If $s_a = 0$, then variable revenues just cover fixed costs. It follows from (5) and (8) that firm size evolves with age according to $\mathrm{d}s_a = \mu \mathrm{d}a + \sigma \mathrm{d}W_{t,a}$, where:

$$\begin{bmatrix} \mu \\ \sigma \end{bmatrix} = \frac{\omega}{1-\omega}\begin{bmatrix} \vartheta - \theta \\ \varsigma \end{bmatrix} \tag{10}$$

Firm size has a negative drift when productivity inside the firm is expected to grow more slowly than the productivity of new entrants. Note that the differences in these growth rates and the variance of productivity shocks are greatly magnified when the differentiated goods are close substitutes.

The function $S[Z]$ defined in (9) plays an important role in the rest of the paper. Along the balanced growth path, where (6) holds, it relates the de-trended productivity of any firm to its size. More precisely, $e^{S[Z]}$ is the size of any firm with productivity $Ze^{\theta t}$ at time t, relative to its fixed costs at time t. In particular, it is the size relative to fixed costs of a new firm entering with a de-trended initial productivity Z.

2.2.2 The Exit Decision

The presence of fixed costs implies a minimum size. Firms with very low productivity choose to exit since they face only a small probability of ever recovering the fixed costs required to continue the firm. The value of a firm of size s relative to its current fixed costs is equal to:

$$V(s) = \max_{\tau} \mathrm{E}\left[\int_0^{\tau} e^{-(r-\kappa)a}\left(e^{s_a} - 1\right)\mathrm{d}a \,\middle|\, s_0 = s\right]$$

The value of a firm entering at time t with initial productivity Z is equal to $V_t[Z] = w_t \lambda_{\mathrm{F}} V(S[Z])$. This depends on the level of wages directly via w_t, and indirectly via $S[Z]$.

Assumption 2 *Preference and technology parameters satisfy* $\rho + \gamma\kappa > \kappa + \mu + \frac{1}{2}\sigma^2$.

Assumption 1 implies that $r > \kappa$, and thus the fixed cost of operating a firm forever is finite. Assumption 2 means that $r > \kappa + \mu + \sigma^2/2$, and this implies that the revenues of such a policy are also finite. Together, these assumptions are sufficient to ensure that the value of a firm is finite. The value function $V(s)$ must satisfy the following Bellman equation in the range of s where a firm is not shut down:

$$rV(s) = \kappa V(s) + \mathcal{A}V(s) + e^s - 1$$

where:

$$\mathcal{A}V(s) = \mu \mathrm{D}V(s) + \frac{1}{2}\sigma^2 \mathrm{D}^2 V(s)$$

The return to owning a firm consists of a capital gain $\kappa + \mathcal{A}V(s)/V(s)$ and a dividend yield $(e^s - 1)/V(s)$. It is optimal to shut down a firm when its size s falls below some threshold b. Given that the firm is shut down at b, it must be that the value of a firm is zero at that point. This implies the boundary condition $V(b) = 0$. The optimal threshold must be such that V is differentiable at b, and so $\mathrm{D}V(b) = 0$. A further boundary condition follows from the fact that the value function cannot exceed the value of a firm that operates without fixed costs. This implies that $V(s)$ must lie below $e^s/(r - [\kappa + \mu + \sigma^2/2])$.

With these boundary conditions, the Bellman equation has only one solution:[10]

$$V(s) = \frac{1}{r-\kappa}\left(\frac{\xi}{1+\xi}\right)\left[e^{s-b} - 1 - \frac{1 - e^{-\xi(s-b)}}{\xi}\right]^+ \tag{11}$$

The exit barrier b is determined by:

$$e^b = \left(\frac{\xi}{1+\xi}\right)\left(1 - \frac{\mu + \sigma^2/2}{r-\kappa}\right), \quad \xi = \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{r-\kappa}{\sigma^2/2}} \tag{12}$$

Assumptions 1 and 2 imply that $\xi > 0$ and that b is well defined. As expected, $V(s)$ is strictly increasing on $[b, \infty)$. It will be useful to note that $V(x + b)$ is increasing in ξ, and that $V(x + b)$ goes to zero as ξ goes to zero, for all x. This will happen when μ becomes large and negative. If the productivity of new entrants grows very quickly, then the value of being an incumbent at any given distance x away from the exit barrier will be very small.

[10] See Dixit and Pindyck (1994) for a detailed treatment of closely related stopping problems.

2.3 Entry

New firms can be set up at a cost that is linear in the entry rate. Entry at a rate of l firms per unit of time costs $\lambda_{\mathrm{E}} l$ units of labor per unit of time. Entry results in a draw of Z from a distribution G. At time t, a draw Z yields an initial productivity $Ze^{\theta t}$ and thus an initial size $S[Z]$. Along the balanced growth path, entry takes place at all times. This means that the profits from entry must be zero:

$$\lambda_{\mathrm{E}} = \lambda_{\mathrm{F}} \int V(S[Z])\mathrm{d}G(Z) \tag{13}$$

The distribution G is taken to be exogenous until imitation is introduced in Section 5. The only assumption needed here is that the implied value of entry is finite.

Assumption 3 *The initial productivity distribution G satisfies:*

$$\int Z^{\omega/(1-\omega)}\mathrm{d}G(Z) < \infty.$$

The value of entry depends on steady-state wages and aggregate consumption via $S[Z]$. Recall from (9) that $S[Z]$ is proportional to $(C/w)/w^{\omega/(1-\omega)}$. The returns to entry can therefore be made arbitrarily small or large by taking $(C/w)/w^{\omega/(1-\omega)}$ to be small or large, respectively. Thus the zero-profit condition (13) implies a unique equilibrium value for $(C/w)/w^{\omega/(1-\omega)}$, and therefore also for $S[Z]$. It is not difficult to see that $S[Z]$ is increasing in λ_{E}. In equilibrium, the initial size and productivity of firms must be high when entry is costly.

3. The Distribution of Firm Characteristics

There is a continuum of infinitesimal firms. The underlying stochastic structure is assumed to be such that probability distributions for individual firm size can be interpreted as cross-sectional size distributions for the whole continuum of firms.

Along the balanced growth path to be constructed, there is a time-invariant cross-sectional distribution of firm size. Firms enter and exit at constant aggregate rates in such a way that the aggregate measure of firms expands at the rate η. A time-invariant size distribution will result if η is positive, or if η is zero and μ is negative. In any equilibrium, the distribution of firm size, measured by e^s, must also have a finite mean.

The following assumption will turn out to be necessary and sufficient for this to be the case, given that η is non-negative.

Assumption 4 *The productivity parameters satisfy* $\eta > \mu + \frac{1}{2}\sigma^2$.

Note that $\mu + \sigma^2/2$ is the drift of the size variable e^{s_a}. Thus Assumption 4 means that the size of a typical incumbent firm is not expected to grow faster than the population growth rate. If η is zero then μ must be negative, but otherwise it can be positive.

Although age does not directly affect firm behavior, it is convenient to include age with size as a state variable. Age increases deterministically with a unit drift, and size has drift μ and diffusion coefficient σ. The measure of firms, defined on the set of possible ages a and firm sizes s, grows at a rate η. The density of this measure at date t can be written as $m(a,s)Ie^{\eta t}$, where $Ie^{\eta t}$ is the number of new firms entering per unit of time. The market clearing conditions that will determine the balanced growth path are linear in m, and this makes it convenient not to normalize m to be a probability density. The density $m(a,s)Ie^{\eta t}$, viewed as a function of the state (a,s) and time t, must satisfy the Kolmogorov forward equation.[11] The resulting partial differential equation for m is given by:

$$\mathrm{D}_a m(a,s) = -\eta m(a,s) - \mu \mathrm{D}_s m(a,s) + \frac{1}{2}\sigma^2 \mathrm{D}_{ss} m(a,s) \tag{14}$$

for all $a > 0$ and $s > b$. The first term on the right-hand side of (14) reflects the fact that the measure of firms grows over time. The remaining two terms describe how $m(a,s)$ evolves as a result of stochastic changes in the sizes of individual firms.

Firms use at least λ_{F} units of labor, and so the measure of firms has to be finite in any equilibrium. This amounts to a first boundary condition for (14). As age goes to zero, the size distribution implied by m must approach the size distribution among entrants. This distribution, denoted by F, follows from the productivity distribution G at entry via $G(Z) = F(S[Z])$. This gives a further boundary condition:

$$\lim_{a \downarrow 0} \int_b^s m(a,x)\mathrm{d}x = F(s) - F(b) \tag{15}$$

for all $s > b$. The remaining boundary condition is given by the requirement that:

$$m(a,b) = 0 \tag{16}$$

for all $a > 0$. This condition arises from the fact that firms exit at b while none enter starting with a size below b.

[11] See Feller (1971), and Dixit and Pindyck (1994) for applications to industry equilibrium.

Lemma 1 *The solution to* (14) *subject to the boundary conditions* (15)-(16) *is:*

$$m(a,s) = \int_b^\infty e^{-\eta a}\psi(a,s|x)\mathrm{d}F(x)$$

for all $a > 0$ *and all* $s > b$, *where:*

$$\psi(a,s|x) = \frac{1}{\sigma\sqrt{a}}\left[\phi\left(\frac{s-x-\mu a}{\sigma\sqrt{a}}\right) - e^{-\frac{\mu(x-b)}{\sigma^2/2}}\phi\left(\frac{s+x-2b-\mu a}{\sigma\sqrt{a}}\right)\right]$$

and where ϕ *is the standard normal density.*

The two terms that define $e^{-\eta a}\psi(a,s|x)$ both satisfy (14). For small values of a, the first term approximates a normal probability density that puts almost all probability close to $s = x$. The second term converges to zero as a goes to zero, since $s + x > 2b$. This implies the boundary condition (15). The fact that $\psi(a,b|x) = 0$ for $a > 0$ implies (16). Together with $\eta \geq 0$, Assumption 4 suffices to ensure that $e^{-\eta a}\psi(a,s|x)$ can be integrated over all $a > 0$ and $s > b$ so that the overall measure of firms is finite. The following remark will be used to further characterize m.

Remark *The roots of the characteristic polynomial* $\eta + \mu z - z^2\sigma^2/2$ *of* (14) *are:*

$$\zeta = -\frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}, \quad \zeta_* = \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}$$

Since $\eta \geq 0$, *both roots are real, and Assumption 4 is equivalent to* $\zeta > 1$. *If* $\eta = 0$, *then* ζ *simplifies to* $\zeta = -\mu/(\sigma^2/2)$. *The root* ζ_* *is non-negative, and positive if and only if* $\eta > 0$. *If* $\mu < 0$, *then* ζ_*/η *converges to* $1/(-\mu)$ *as* η *goes to zero.*

Observe that $m(a,s)$ reduces to $e^{-\eta a}\psi(a,s|x)$ if F is replaced by a distribution concentrated at x. This means that $e^{-\eta a}\psi(a,s|x)$ is the density of firm age and size among all firms with the same initial size x. Let $\pi(a,s|x)$ be the associated probability density. Integrating $e^{-\eta a}\psi(a,s|x)$ to obtain the normalizing constant yields:

$$\pi(a,s|x) = \left(\frac{1 - e^{-\zeta_*(x-b)}}{\eta}\right)^{-1} e^{-\eta a}\psi(a,s|x)$$

Combining this with the above solution for $m(a,s)$ gives:

$$m(a,s) = \int_b^\infty \pi(a,s|x)\left(\frac{1 - e^{-\zeta_*(x-b)}}{\eta}\right)\mathrm{d}F(x) \tag{17}$$

Thus $m(a, s)$ is a weighted sum of the densities $\pi(a, s|x)\mathrm{d}F(x)$, with weights that are increasing in the distance of initial size x from the exit barrier b. In the special case of $\eta = 0$, these weights reduce to $(x - b)/(-\mu)$, which is the expected life span of a new firm entering with size x. Relatively large entering firms stay around longer, and appear more often in the population than suggested by the size distribution of entrants.

3.1 The Age Distribution

The shape of the age distribution of firms implied by m depends on the size distribution F of entering firms. If heterogeneity among entrants is small relative to heterogeneity in the overall population, then the age distribution will look much like the one obtained by conditioning on a typical $x > b$. Integrating $\pi(a, s|x)$ over s gives the age density among firms with the same size at entry. The result is:

$$\pi(a|x) = \left(\frac{1 - e^{-\zeta_*(x-b)}}{\eta}\right)^{-1} e^{-\eta a}\left[\Phi\left(\frac{x - b + \mu a}{\sigma\sqrt{a}}\right) - e^{-\frac{\mu(x-b)}{\sigma^2/2}}\Phi\left(\frac{\mu a - (x - b)}{\sigma\sqrt{a}}\right)\right]$$

where Φ is the standard normal distribution function. If there is no population growth, then the density $\pi(a|x)$ can also be interpreted as the survivor function of a cohort of firms, scaled by the average life span of a firm.[12] It can be shown that this survivor function has a hump-shaped hazard rate. Any entrant with $x > b$ stays around for some time, and then the negative drift μ will start to generate a high rate of exit. Among older firms, more will be large as a result of a string of positive productivity shocks, and this implies a lower exit rate. If x is close to b, then the hazard rate will be decreasing in age over much of its domain. This is consistent with the declining hazard rates in the US Census of Manufactures reported by Dunne, Roberts and Samuelson (1988, 1989).[13]

3.2 The Size Distribution

It follows from (17) that the firm size density is a weighted average of the density $\pi(s|x)$ of size given initial size. For any $x > b$, integrating $\pi(a, s|x)$ over all ages gives:

$$\pi(s|x) = \left(\frac{e^{\zeta_*(x-b)} - 1}{\zeta_*}\frac{e^{\zeta(s-b)}}{\zeta}\right)^{-1} \min\left\{\frac{e^{[\zeta+\zeta_*](s-b)} - 1}{\zeta + \zeta_*}, \frac{e^{[\zeta+\zeta_*](x-b)} - 1}{\zeta + \zeta_*}\right\} \tag{18}$$

[12] The size density at age a of firms of the same cohort and initial size x then satisfies (14) with $\eta = 0$, and the age-zero boundary condition is a point mass at x. From this the result follows.

[13] The results of Dunne, Roberts and Samuelson (1988, 1989) are based on five-year census cohorts observed at five-year intervals. Caves (1998) discusses the literature on firm exit rates and cites additional studies documenting hazard rates that decline with age, as well as a study by Brüderl, Preisendörfer and Ziegler (1992) based on monthly observations of a cohort of new firms in the Munich (Germany) area that reports a hump-shaped hazard function.

for all $s \geq b$. This is a well-defined density for any $\zeta > 0$ and $\zeta_* \geq 0$. The mean of firm size, when size is measured by e^s, is finite if and only if $\zeta > 1$. As noted earlier, this is guaranteed by Assumption 4. An example of $\pi(s|x)$ is given in Figure 2. The kink at $s = x$ is a result of the entry that takes place at x. Conditional on $s \geq x$, the density of e^s implied by (18) is a Pareto density with tail probabilities of the form $e^{-\zeta(s-x)}$. The parameter ζ is the tail index of the size distribution.[14] [15]

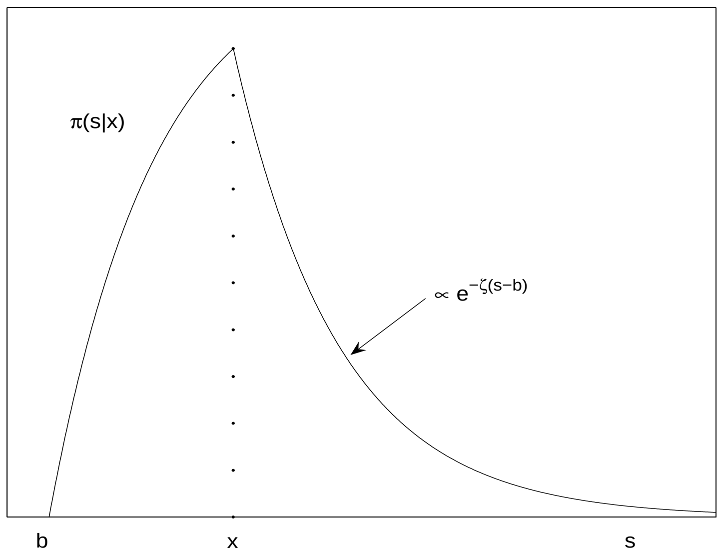

Figure 2: Size Density Conditional on Initial Size

If all new firms enter with the same initial productivity, then F is a point mass at some initial size x. In that case, (17) implies that $\pi(s|x)$ is the firm size density. This density matches the data presented in Figure 1 if $x - b$ is small and $\zeta \approx 1.05$. More generally, suppose that F is a distribution with few firms that are much larger than the exit barrier. Then $m(s)$ will inherit the exponentially declining tail common to all $\pi(s|x)$ over most of the support (b, ∞). The deviations from linearity seen in Figure 1 occur for small firms: there are fewer of them than would be the case if the size distribution was Pareto.

[14] Suppose population growth rates are zero. Consider the limiting distribution obtained by letting x go to b. This turns the profitability process of a dynasty of firms into a Brownian motion with a negative drift and a reflecting barrier at b. The resulting distribution for e^s is a Pareto distribution on $e^s \geq e^b$ with mean $e^b\zeta/(\zeta - 1)$. In Gabaix (1999), e^s is the size of a city relative to the average city size. This must have mean 1, and so $\zeta = 1/(1 - e^b)$. The explanation given in Gabaix (1999) for Zipf's law for relative city sizes is that b must be very small.

[15] In independent work Miao (2004) derives the same size distribution in a related model of industry equilibrium in which the entry distribution F is assumed to be uniform over an interval.

Since $\pi(s|x)$ is upward-sloping on the interval (b, x), this is exactly what is predicted when F tends to have most of its mass close to the exit barrier.

Random Growth and Selection To emphasize the importance of randomness in shaping the firm size distribution, it is instructive to consider what happens as the variance of productivity shocks goes to zero. For simplicity, suppose that $\eta = 0$. Assumption 4 then requires $\mu < 0$ and at $\sigma^2 = 0$ one obtains $\xi = (r - \kappa)/|\mu|$ and $b = 0$. Firms exit immediately when they no longer break even. There is no option value that would justify continuing to operate a loss-making firm. An entering firm starts with size x, and this size will then decline linearly to 0, at which point the firm exits. As σ^2 goes to 0, the tail index ζ grows without bound. Using (18) one can verify that the size distribution converges to a uniform distribution on $(0, x)$. In this limiting economy, the largest and most profitable firm conditional on initial size is the most recent entrant. This is in sharp contrast to what is found in the data (Dunne, Roberts and Samuelson [1988, 1989], Caves [1998]). The randomness in productivity growth generates a selection mechanism by which the typical firm can be much larger and productive than recent entrants.

4. The Balanced Growth Path

Per capita consumption and wages grow at the rate κ determined in (6). The resulting interest rate is $r = \rho + \gamma\kappa$, and together with the κ this pins down the value function $V(s)$. As noted earlier, the zero-profit condition then determines $(C/w)/w^{\omega/(1-\omega)}$ and the function $S[Z]$ defined in (9). The preceding section shows how this determines the size distribution of firms.

To complete the construction of the balanced growth path, it remains to determine the levels of per capita consumption and wages, as well as the level of firm entry I. These variables are determined by goods and labor market clearing conditions. Let $L_{\mathrm{E}}e^{\eta t}$, $L_{\mathrm{F}}e^{\eta t}$ and $Le^{\eta t}$ denote the amounts of labor assigned to, respectively, setting up new firms, fixed costs to operate existing firms, and production. It follows from the firm decision rules (7)-(9) that:

$$\begin{bmatrix} L_{\mathrm{E}} & L_{\mathrm{F}} & L \end{bmatrix} = \begin{bmatrix} \lambda_{\mathrm{E}} & \lambda_{\mathrm{F}} \int_b^\infty m(s)\mathrm{d}s & \lambda_{\mathrm{F}} \left(\frac{\omega}{1-\omega}\right) \int_b^\infty e^s m(s)\mathrm{d}s \end{bmatrix} I \qquad (19)$$

Together with the labor market clearing condition $L_E + L_F + L = H$, this determines the rate of entry I. Aggregate output is the sum of firm revenues. The decision rules

(7)-(9) imply that aggregate output $Ye^{(\kappa+\eta)t}$ satisfies:

$$\frac{Y}{w} = \frac{\lambda_{\mathrm{F}} I}{1-\omega} \int_b^\infty e^s m(s) \mathrm{d}s \tag{20}$$

In combination with the goods market clearing condition $C = Y$, this determines the ratio C/w. Since $(C/w)/w^{\omega/(1-\omega)}$ is determined by the zero-profit condition, this pins down C and w. This leads to the first part of the following proposition.

Proposition 1 *If Assumptions 1-4 hold, then there exists a balanced growth path. A proportional reduction in the entry and fixed cost parameters* $(\lambda_{\mathrm{E}}, \lambda_{\mathrm{F}})$ *raises the level of output with an elasticity* $(1-\omega)/\omega$.

At $t = 0$, the distribution of productivities available to potential entrants is $G(Z)$. At that same time, there will be some measure of incumbent firms with given levels of productivity. The balanced growth path of Proposition 1 will be an equilibrium if at $t = 0$ the density of productivity among incumbent firms is $m(S[Z]) \, |\mathrm{D}S[Z]|$. What happens for different initial conditions is not known.

To see the second part of Proposition 1, observe that a proportional reduction in $(\lambda_{\mathrm{E}}, \lambda_{\mathrm{F}})$ does not affect the zero-profit condition. The function $S[Z]$ and the size density $m(s)$ therefore do not change. It follows from (19) and the labor market clearing condition that I increases in such a way that $(\lambda_{\mathrm{E}}, \lambda_{\mathrm{F}})I$ remains constant. Together with (20) and $C = Y$ this implies that C/w remains unchanged. Since $S[Z]$ is proportional to $(1/\lambda_{\mathrm{F}})(C/w)/w^{\omega/(1-\omega)}$, it follows that $1/w$ must increase with an elasticity $(1-\omega)/\omega$. This is also the effect on consumption. Lower setup and fixed costs imply a larger number of firms. Since firms are identified with distinct differentiated goods, this means a larger number of goods. The elasticity $(1-\omega)/\omega$ measures the increase in composite consumption arising from this increase in variety.

Note that (19) and (20) depend on $(\lambda_{\mathrm{E}}, \lambda_{\mathrm{F}})/H$ when labor and output are expressed in per capita terms. Also, the function $S[Z]$ can be written in terms of C/H and λ_{F}/H. Thus an increase in the size of the population is equivalent to a proportional reduction in the setup and fixed costs. The resulting elasticity $(1-\omega)/\omega$ of per capita consumption with respect to H corresponds to the one obtained for the growth rate κ in (6). The benefits of lower setup and fixed costs and larger population sizes derived here replicate those obtained for a static economy by Krugman (1979).

5. Imperfect Imitation

The equilibrium constructed in Proposition 1 is well defined only if $\zeta > 1$, and the data in Figure 1 suggest that ζ should be close to one. The parameter ζ is a function of the population growth rate η, the curvature parameter ω of the utility function, and the technology parameters $[\theta, \vartheta, \varsigma]$. So far, these parameters have been taken as exogenous, and the model cannot explain Figure 1 unless they happen to be of just the right magnitude to imply $\zeta \approx 1.05$. Recall from (5) and (8) that θ is the growth rate of new firm productivity, and also the rate at which the cross-sectional distribution of productivity trends up over time. This section makes θ endogenous and gives conditions under which the distribution shown in Figure 1 will arise.

By paying fixed costs, incumbent firms can continue production and generate stochastic productivity improvements. The productivity of surviving firms will tend to grow forever as long as ϑ is not too small. If new firms had to start from the same level of productivity as existing firms entered with in the past, then the value of entry would eventually become too small to justify the cost of entry. The high productivity of successful survivors would drive up wages beyond the level at which it would be profitable for new firms to enter. The size distribution of firms would be non-stationary.

To avoid this outcome, some mechanism is needed that allows potential entrants to benefit from the productivity improvements obtained by incumbents. The mechanism proposed here is imitation. Suppose potential entrants can pay the entry cost λ_{E} to select a random incumbent firm and then adopt a scaled-down version of its technology. More precisely, if the randomly selected firm at time t has a productivity $Xe^{\theta t}$, then the potential entrant obtains a technology capable of producing a new good with productivity $Ze^{\theta t} = Xe^{\theta t - \delta(1-\omega)/\omega}$. The parameter δ measures how much the productivity of the potential entrant will be below that of the incumbent. It is taken to be non-negative, so that imitation is imperfect, and imitation is difficult if δ is large. The implied initial size of the potential entrant is $S[Z] = S[X] - \delta$, and the entry attempt will be successful if this exceeds b. Observe how random sampling and imitation tie the expected size and profitability of a potential entrant to the average size and profitability of incumbents. This sets up strong incentives for entry when incumbents become large and profitable on average. The result is a stationary size distribution with a well defined and finite average firm size.[16]

[16]In Eaton and Eckstein (1997), knowledge spillovers across existing cities provide the mechanism by which the size distribution of cities is prevented from spreading out. Jovanovic and MacDonald (1994) and Eeckhout and Jovanovic (2002) allow all firms to copy, imperfectly, from the whole population of

5.1 Stationary Size Distributions

Assume the cross-sectional distribution of productivity is stationary when productivity is de-trended by some growth rate θ. Suppose the resulting size distribution has a probability density $f(s)$. The mechanism by which potential entrants obtain a new technology implies a size density for entering firms equal to $\mathrm{D}F(x) = f(x+\delta)$. Combined with $\pi(s|x)$, this initial size density in turn implies a size density for all firms via (17). It follows that the probability density $f(s)$ must satisfy the fixed point condition:

$$f(s) = \frac{\int_b^\infty \pi(s|x)[1 - e^{-\zeta_*(x-b)}]f(x+\delta)\mathrm{d}x}{\int_b^\infty [1 - e^{-\zeta_*(x-b)}]f(x+\delta)\mathrm{d}x} \tag{21}$$

for all $s \geq b$. Observe that this condition only depends on the parameters ζ, ζ_* and δ. By using a series expansion of f to evaluate the two sides of (21) and equating coefficients one can construct the set of sufficiently smooth densities that satisfy (21). The resulting fixed points are described in the following lemma.

Lemma 2 *Suppose $\delta \geq 0$, and let $\pi(s|x)$ be defined by (18), for some $\zeta > 0$ and $\zeta_* \geq 0$. If $\zeta > \zeta_*/(1 + \delta\zeta_*)$, then (21) is solved by:*

$$f(s) = \left(\frac{\alpha\beta}{\beta - \alpha}\right)\left[e^{-\alpha(s-b)} - e^{-\beta(s-b)}\right] \tag{22}$$

for any $\alpha > 0$ and $\beta \geq \alpha$ that satisfy:

$$\zeta = \frac{\alpha(\alpha + \zeta_*)e^{\alpha\delta} - \beta(\beta + \zeta_*)e^{\beta\delta}}{(\alpha + \zeta_*)e^{\alpha\delta} - (\beta + \zeta_*)e^{\beta\delta}} \tag{23}$$

The condition $\zeta > \zeta_*/(1 + \delta\zeta_*)$ is necessary and sufficient for (23) to have a solution. The constraint $\beta \geq \alpha$ is without loss of generality since α and β enter symmetrically in (22) and (23). Given this constraint, the right tail probabilities of $f(s)$ behave like $e^{-\alpha s}$ for large s, and α will again be referred to as the tail index of the distribution. The mean of e^s implied by $f(s)$ is finite if and only if $\alpha > 1$.

Consider a density of the form (22) with $\alpha = \beta$. This means that the size density is given by the Gamma density $\alpha^2(s-b)e^{-\alpha(s-b)}$. The tail index that solves (23) is then given by $\alpha = -\mu/\sigma^2$ when $\delta = 0$, and by:

$$\alpha = -\left(\frac{\mu}{\sigma^2} + \frac{1}{\delta}\right) + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{1}{\delta^2} + \frac{\eta}{\sigma^2/2}} \tag{24}$$

firms. Here, the spillover is only from incumbents to potential entrants. Incumbents are locked into their idiosyncratic productivity processes and are not assumed to be able to imitate the successes of other incumbent firms. The result is that incumbents eventually exit with probability one.

when $\delta > 0$. For large δ this tail index is essentially the same as the tail index ζ of $\pi(s|x)$. The expression for α shown in (24) is decreasing in μ, and thus increasing in the growth rate θ. The higher the average growth rate θ of productivity in the population, relative to the drift ϑ of surviving incumbents, the more aggregate productivity growth must be due to selection, and this implies a size distribution with a thinner tail. It will be useful to note that a Gamma distribution with a thick tail stochastically dominates a Gamma distribution with a thin tail in a first-order sense.

5.2 The Balanced Growth Path and Zipf's Law

The size density $f(s)$ constructed in Lemma 2 is a function of the assumed productivity growth rate θ, through its dependence on $\pi(s|x)$ and the parameters ζ and ζ_*. The value function $V(s)$ is also a function of θ, via the drift parameter μ, as well as via the equilibrium interest rate r and the growth rate κ of per capita consumption and wages. Taken together, this means that the expected profits from entry are a function of θ. The only values of θ that are consistent with balanced growth are those for which these profits are zero:

$$\lambda_{\mathrm{E}} = \lambda_{\mathrm{F}} \int_b^\infty V(x) f(x+\delta) \mathrm{d}x \tag{25}$$

Observe that the equilibrium conditions (21) and (25) only depend on θ and f, and equilibrium variables that are themselves functions only of θ and f.

To complete the construction of a balanced growth path, fix some growth rate θ and a density f that satisfy (21) and (25). As before, the relation between firm size and productivity is summarized by the function $S[Z]$. Recall from (9) that $e^{S[Z]}$ is proportional to $(C/w)/w^{\omega/(1-\omega)}$. The location of the productivity density $f(S[Z])\,|\mathrm{D}S(Z)|$ is therefore determined by the log of $(C/w)/w^{\omega/(1-\omega)}$. On a balanced growth path, the density $f(S[Z])\,|\mathrm{D}S(Z)|$ must correspond to the density of productivity among incumbent firms at the initial date. This requirement determines the equilibrium value of $(C/w)/w^{\omega/(1-\omega)}$, provided that the distribution of productivity at the initial date is consistent with balanced growth. As in the case of exogenous growth, goods and labor market clearing conditions determine the ratio C/w and the rate I at which firms attempt to enter. Together with $(C/w)/w^{\omega/(1-\omega)}$ this gives C and w separately, and the economy will be on a balanced growth path if the number of firms at the initial date corresponds to the number implied by f and I.

The following proposition shows that this construction works if consumers discount the future enough. Precise conditions and a proof are given in the appendix.

Proposition 2 *Suppose the population growth rate* η *and the drift* ϑ *of technological progress among incumbents are non-negative. If the discount rate* ρ *is large enough, then there exists a continuum of balanced growth paths with size distributions of the form (22)-(23). The tail index* α *of the size distribution converges to one —Zipf's Law— as the ratio* $\lambda_{\mathrm{E}}/\lambda_{\mathrm{F}}$ *of entry over fixed costs grows without bound.*

The existence of a balanced growth path and the circumstances in which Zipf's law arises are most transparent in the special case of logarithmic utility. This case implies that $r = \rho + \kappa$, simplifying the dependence of the value of a firm on θ. For fixed x, the value $V(x + b)$ is then unambiguously decreasing in θ. Higher productivity growth in the population drives incumbents at a given distance from the exit barrier out of business more quickly, and this implies a low firm value. Consider the case $\alpha = \beta$. As noted earlier, higher productivity growth then also implies a size distribution with thinner tails, or a downward shift (in the sense of first-order stochastic dominance) in the distribution implied by $f(x + b + \delta)$. Since $V(x + b)$ is an increasing function of x, it follows that the right-hand side of the zero-profit condition (25) is decreasing in θ.[17] Equivalently, the expected value of entry is decreasing in the tail index α. It is not difficult to show that the value of entry goes to zero for very large α. Finally, the dominant term in the value function $V(x+b)$ is the firm size variable e^x, and this implies that the expected value of entry grows without bound as the tail index α approaches 1 from above. The right-hand side of the zero-profit condition is therefore as shown in Figure 3, with a vertical asymptote at $\alpha = 1$ and a horizontal asymptote at 0. From this the results of Proposition 2 follow.

If the utility function exhibits more curvature than logarithmic utility, then the value function continues to be monotone in α for high enough discount rates. But if $\gamma < 1$, then the discount factor $1/(r - \kappa)$ is increasing in κ and thus also in θ and α. This can outweigh the negative effect on the value function of a larger gap $\theta - \vartheta$ between productivity growth in the population and the drift of incumbent productivity. The value of a firm may, over some range, increase with the growth rate of productivity in the population. This can make the expected value of entry non-monotone in θ and α. The proof given in the appendix shows that a balanced growth path does nevertheless exist for high enough discount rates ρ.

[17] Faster growth increases the exit barrier b and this tends to shift the size distribution to the right. But, because entrants sample from the population of incumbents, what matters for the value of entry is the distribution of size relative to the exit barrier.

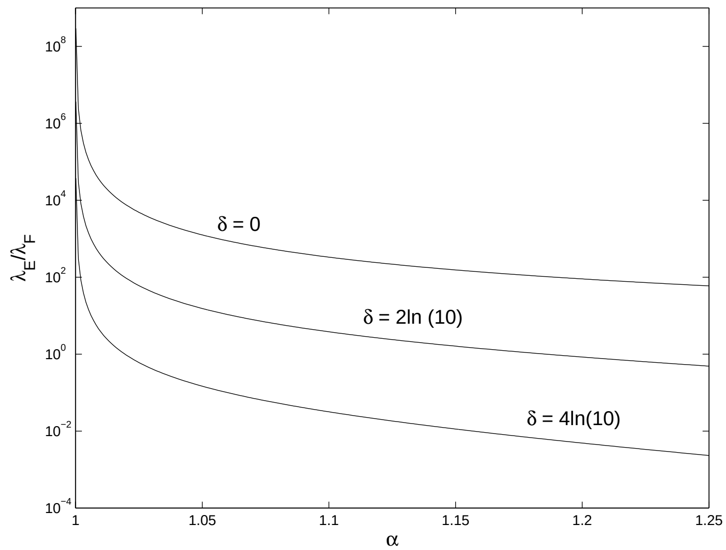

Figure 3: Entry Costs, Fixed Costs and the Tail Index

5.3 Barriers to Entry and Growth

The equilibrium conditions (21) and (25), and therefore the growth rate θ, are independent of the scale of the entry and fixed costs $(\lambda_{\mathrm{E}}, \lambda_{\mathrm{F}})$. As in the case of exogenous growth, lowering both costs at the same time simply increases the level of output with an elasticity $(1-\omega)/\omega$. The effects of changing only barriers to entry —the entry cost λ_{E} or the difficulty of imitation δ— are described in the following corollary of Proposition 2.

Corollary *Suppose the conditions of Proposition 2 hold and consider equilibria with $\alpha = \beta$. The growth rate θ of productivity in the population is decreasing in the entry cost parameter λ_{E} and the imitation parameter δ when $\gamma \geq 1$, and for sufficiently large entry costs when $\gamma < 1$.*

For $\gamma \geq 1$, this result follows from the fact that the value of entry, as illustrated in Figure 3, is decreasing in the tail index α. A higher entry cost λ_{E} implies a higher equilibrium value of entry, and thus a lower equilibrium value of α, and a lower θ. Similarly, a larger δ implies a lower equilibrium value of α since the expected value of entry is lower when imitation is more difficult.[18] Given that the right-hand side of (24) is increasing in δ and decreasing in μ, this implies a lower growth rate θ. For $\gamma < 1$ these conclusions continue to hold provided entry costs are high. High entry costs imply that α must be close to 1

[18] That is, the right-hand side of (25) is decreasing in δ. This can be shown by differentiating (31) in the appendix.

and the expected value of entry can be shown to be monotone for all α close enough to the asymptote $\alpha = 1$.

The corollary says that lowering entry costs will tend to increase the growth rate of the economy. If imitation is very difficult, then (24) implies that $\alpha \approx \zeta$, and if there is also no population growth, then $\zeta = -2\mu/\sigma^2$. Using the definitions of μ and σ^2 given in (10), this yields:

$$\theta \approx \vartheta + \frac{\alpha\omega}{1-\omega}\frac{\varsigma^2}{2} \tag{26}$$

The drift of incumbent productivity is ϑ, and the second term in (26) captures the effect of selection on productivity growth in the population of firms. The effect of selection on θ can be substantial if the differentiated commodities are close substitutes. For example, if the elasticity of substitution between differentiated commodities is 10, or $\omega = .9$, then $\varsigma = .05$ ($\varsigma = .10$) implies that selection adds at least 1.125% (4.5%) to the growth rate of productivity in the population. Lower barriers to entry imply smaller firms and this corresponds to higher values of α. By (26), this means faster productivity growth in the population. Incumbent productivity drifts up at a rate ϑ in any case, but the lower barriers to entry generate more firm turnover and this increases the effect of selection.

6. Heterogeneity Across Industries

All firms discussed so far face demand curves with the same elasticity $-1/(1-\omega)$, and productivity processes with the same drift and diffusion parameters ϑ and ς. No doubt, the degree to which the differentiated commodities produced in an industry are substitutable differs across industries, as do the typical rates of technological progress. Entry and fixed costs and the difficulty of imitation are unlikely to be the same across industries either.[19] Given such heterogeneity, does the model still predict a size distribution for all firms that looks like the one shown in Figure 1?

Suppose consumption consists of N different goods, each of which is a composite of a continuum of differentiated commodities. Industries are identified with different composite goods. As before, different firms in an industry produce distinct differentiated commodities. Specifically, suppose consumption is given by the Cobb-Douglas aggregate $C_t = \prod_{n=1}^{N} C_{n,t}^{\nu_n}$, where $C_{n,t}$ satisfies (1) with ω replaced by an industry-specific curvature parameter ω_n. The share parameters ν_n are between zero and one and add up to one.

[19] Luttmer (2004) allows for additional sources of within-industry heterogeneity by incorporating within-industry variation in fixed and entry costs, as well as in the technologies used to combine physical capital and labor to produce differentiated goods.

Idiosyncratic firm productivity in industry n is assumed to follow (5), with $[\theta, \vartheta, \varsigma]$ replaced by $[\theta_n, \vartheta_n, \varsigma_n]$.

Along any balanced growth path, aggregate consumption of the composite good produced by industry n will be $C_{n,t} = C_n e^{(\kappa_n+\eta)t}$, where κ_n is defined in terms of θ_n and ω_n as in (6). Aggregate consumption will be $C_t = Ce^{(\kappa+\eta)t}$ and the growth rate κ of per capita consumption is simply the average of the industry growth rates weighted by expenditure-shares:

$$\kappa = \sum_{n=1}^{N} \nu_n \kappa_n \tag{27}$$

Wages also grow at this rate. The price index for aggregate consumption is $P_t = \prod_{n=1}^{N} (P_{n,t}/\nu_n)^{\nu_n}$, where $P_{n,t}$ is the price index for the composite good of industry n, defined as in (3). The relative prices $P_{n,t}/P_t$ must be given by $(P_n/P)e^{(\kappa-\kappa_n)t}$, since expenditure shares are constant. Let $\lambda_{\mathrm{F},n}$ be the fixed cost required to continue a firm in industry n. A calculation along the lines of (7)-(9) implies that the relation between productivity and size in industry n is given by:

$$e^{S_n[Z]} = \frac{\nu_n(1-\omega_n)}{\lambda_{\mathrm{F},n}} \left(\frac{\omega_n Z P_n/P}{w} \right)^{\omega_n/(1-\omega_n)} \frac{C}{w}$$

where $P = \prod_{n=1}^{N} (P_n/\nu_n)^{\nu_n}$. The gross revenues at time t of a firm in industry n with a productivity $Ze^{\theta_n t}$ are given by $w_t \lambda_{\mathrm{F},n} e^{S_n[Z]}$. The (logarithmic) size of such a firm follows a Brownian motion with drift μ_n and diffusion coefficient σ_n defined as in (10), using the industry-specific parameters ω_n and $[\theta_n, \vartheta_n, \varsigma_n]$. Firms choose to follow the same stopping rule as before, exiting when size falls below an industry-specific barrier b_n defined as in (12). The size distributions in all industries are therefore of the form derived in Section 3.

Suppose firms can choose which industry to enter, and then, at a cost of $\lambda_{\mathrm{E},n}$ units of labor, attempt to imitate incumbents in that industry along the lines of Section 5. The extent to which entrants lag behind incumbents in industry n is measured by δ_n. Potential entrants can direct their entry attempts to a specific industry, but imitation of firms in the chosen industry is imperfect, as before.

This setup leads to equilibrium conditions for the industry growth rate θ_n and size density f_n that are exactly analogous to (21) and (25). The value functions V_n appearing in equilibrium conditions analogous to (25) depend on μ_n and the difference $r-\kappa$ between the interest rate and the aggregate growth rate κ. Since κ depends on an expenditure-weighted average of the industry growth rates θ_n, this gives a system of N equilibrium conditions in N unknown growth rates θ_n. For general γ, the analysis of this system is more complicated than the analysis that led to Proposition 2. But $r-\kappa=\rho$ when utility is logarithmic, and then the equations uncouple: the zero-profit condition for industry n only depends on the growth rate θ_n of industry productivity and the size density f_n. As a result, the proof of Proposition 2 applies. In particular, industries with high ratios $\lambda_{\mathrm{E},n}/\lambda_{\mathrm{F},n}$ will have tail indices α_n close to 1, and, ceteris paribus, growth rates θ_n that are not far above ϑ_n.

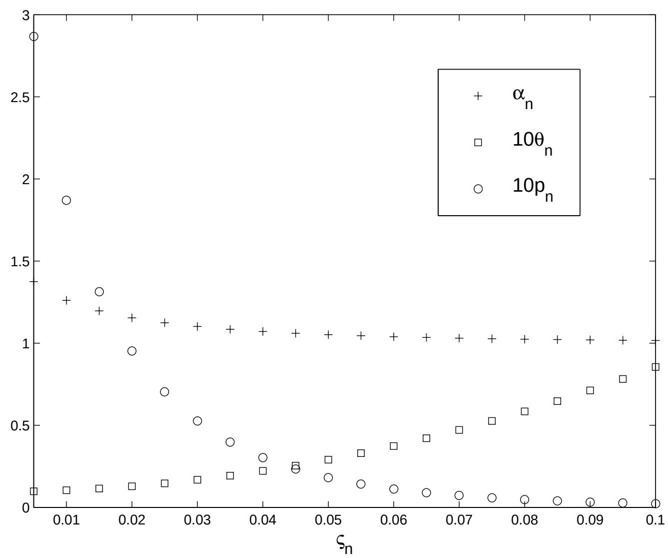

Figure 4: Heterogeneous Firms

The overall size density will be a weighted average of the industry size densities f_n. Since e^s measures net firm revenues relative to fixed costs, and $\lambda_{\mathrm{F},n}$ measures industry-n fixed costs in units of labor, it follows that $e^s\lambda_{\mathrm{F},n}/(1-\omega_n)$ measures gross firm revenues in units of labor, in industry n. Therefore, if size is measured by gross firm revenues, then the overall size density is of the form:

$$f(s)=\sum_{n=1}^{N} p_n f_n\left(s-\ln\left(\frac{\lambda_{\mathrm{F},n}}{1-\omega_n}\right)\right) \tag{28}$$

for weights p_n that add up to one. These weights are proportional to the numbers of firms in each industry. The number of firms in industry n times the average revenues in that industry should equal the value of aggregate consumption of the composite good produced in that industry, or ν_n times the value of aggregate consumption. It follows that the number of firms in industry n is proportional to:

$$p_n \propto \nu_n\left(\frac{\lambda_{\mathrm{F},n}}{1-\omega_n}\int_{b_n}^{\infty} e^s f_n(s)\mathrm{d}s\right)^{-1} \tag{29}$$

Thus, if the goods produced by different industries have similar expenditure shares, then every industry-specific size density is weighted simply by the reciprocal of the average firm size in the industry.

The curve shown in Figure 1 represents the size distribution of an economy with $N = 20$ and industries that differ only in terms of the volatility of the idiosyncratic productivity shocks. The standard deviation for industry n is taken to be $\varsigma_n = .005n$. The population growth rate is $\eta = .01$. Utility is logarithmic and the subjective discount rate is given by $\rho = .05$. In all industries, $\omega_n = .9$, implying a substitution elasticity equal to 9. The entry cost parameter is $\lambda_{\mathrm{E}} = 2$ while the present value of fixed costs incurred in perpetuity is equal to $\lambda_{\mathrm{F}}/\rho = 1.4$. The imitation parameter is $\delta = 2$, and thus $\delta(1-\omega)/\omega = 2/9$. New firms are about 20% less productive than the incumbents they try to imitate. Given that new goods are such close substitutes to existing goods, this leads to initial firm sizes of only about 14% of the size of incumbents. Figure 4 shows how the industry tail indices, growth rates and numbers of firms vary with the standard deviation of productivity growth. The tail indices range from 1.37 to 1.02, and the differences $\theta_n - \vartheta_n$ range from about 1% for industry 1 to around 8.6% for industry N. As a result of a stronger selection effect, industries with the largest productivity shocks grow about 7.6% faster than the ones with smallest productivity shocks.[20] Close to 30% of all firms are in the slow growing industry with tail index 1.37 while only .2% of firms are in the fast growing industry with tail index 1.02. But these firms tend to be larger and dominate the shape of the right tail of the distribution. As a result, the model provides a close fit to the slope -1.05 shown in Figure 1.

Given a finite number of industries, the right tail of the size density will be determined by the industry with a value of α_n closest to one. In Figure 1, the slope of the log right tail probabilities with respect to log size is constant and only slightly above -1 over much of the range of the data. This will tend to be the result of (28)-(29) if there are sufficiently many industries in which α_n is close to one, say, because imitation is difficult or entry costs are high.

The only heterogeneity across industries assumed in Figures 1 and 4 is in the variance of idiosyncratic productivity shocks. Because of this, larger firms tend to be in industries with high productivity growth. If, instead, industries only differ in terms of entry costs, then large firms would tend to be in the industries with high entry costs. Other possible sources of variation are the drift of incumbent productivity growth, within-industry

[20] Note that variation in ς_n^2 dominates the equilibrium variation in $\alpha_n \varsigma_n^2$, and the approximation (26) then indicates that industries with more variable productivity shocks must grow faster.

substitutability of the differentiated commodities, fixed costs, and the difficulty of imitation. Rossi-Hansberg and Wright (2004) present evidence that size distributions differ across industries and suggest an interpretation. Further research is needed to see if and how this variation can be accounted for using the model economy described here, possibly augmented with the additional sources of within-industry heterogeneity described in Luttmer (2004).

7. Concluding Remarks

If new entrants can imitate incumbents, then growth is rapid when barriers to entry are low. The engine of growth is experimentation by firms, combined with selection. Lucky firms receive another draw and unlucky ones exit and are replaced by more productive firms. Firms are experiments that can be cut short and replaced by new ones when they do not perform well. Reducing the cost of entry speeds up the rate of economy-wide experimentation and raises the growth rate of the economy. The resulting size distribution is stationary because potential entrants can learn from successes achieved by incumbents. It has a very thick tail when entry is difficult nevertheless.

This model is consistent with three first-order features of the data. The economy grows at a steady rate. Firm exit rates are high for young firms and low for firms that have survived for some time. The predicted size distribution of firms closely approximates Zipf's law if entry is costly. This tends to be true even when the cost of entry is low in some industries.

The closed-form solutions derived in this paper rely heavily on the absence of aggregate uncertainty and on the use of steady states. This precludes an analytical treatment of transitions, and of the possible role of selection and imitation in speeding up transitions. An important abstraction also is that every firm is identified with a technology to produce a single differentiated good. In contrast, the empirical definition of a firm is based on the legal criterion of ownership. Building models of firm dynamics in which the definition of a firm corresponds more closely to the empirical definition remains an important task for further research.

In this paper, the variance of firm growth rates over small intervals of time is the same for firms of all sizes. Many studies have found larger variances for small firms than for large firms. One possible explanation for this phenomenon is the presence of unobservable fixed effects about which young firms learn, as proposed by Jovanovic

(1982).[21] This can be combined with the permanent shocks emphasized in this paper, although the resulting hybrid model does not appear to be analytically tractable. Pakes and Ericson (1998) derive observable implications for such a hybrid model and present evidence that the importance of learning varies across industries.

A Proof of Proposition 2

The following assumptions are maintained throughout:

$$\eta \geq 0,\, \vartheta \geq 0,\, \sigma^2 > 0,\, \delta \geq 0 \tag{30}$$

Population growth is non-negative, the log productivity of incumbents does not decline on average and is stochastic, and potential entrants cannot do better than the incumbents they attempt to imitate.

A.1 Existence

It is convenient to solve for the equilibrium value of μ. The growth rates κ and θ, and the parameter ξ then follow from (6), (10) and (12). The interest rate is given by $r = \rho + \gamma\kappa$. The present value of the aggregate labor endowment must be finite in any equilibrium. Along a balanced growth path, this requires that $r > \kappa + \eta$. Consider stationary densities of the form (22)-(23) with $\alpha = \beta$. The tail index α is given by (24) when δ is positive. The limit as δ approaches zero from above is $\alpha = -\mu/\sigma^2$.

Lemma A1 *If $r > \kappa + \eta$ then $\alpha > 1$ implies $r - \kappa > \mu + \sigma^2/2$.*

This lemma ensures that the value function $V(s)$ given in (11) is well defined whenever $\alpha > 1$ and the present value of the aggregate labor endowment is finite. Under these conditions, the zero-profit condition (25) can be written as:

$$\frac{\lambda_{\mathrm{E}}}{\lambda_{\mathrm{F}}} = \frac{\xi e^{-\alpha\delta}}{r-\kappa} \frac{(\alpha-1)\alpha(\alpha+\xi)\delta + \alpha(\alpha+\xi) + (\alpha-1)(\alpha+\xi) + (\alpha-1)\alpha}{(\alpha-1)^2(\alpha+\xi)^2} \tag{31}$$

Observe that the right-hand side of (31) is decreasing in $r - \kappa > 0$ and $\alpha > 1$, and increasing in $\xi > 0$.

The definition (24) implies that α is strictly decreasing in μ, with a horizontal asymptote at $-1/\delta$ for large μ. Furthermore, α can be made arbitrarily large by taking μ

[21] The fact that the variance of growth rates does not increase one-for-one with size is emphasized in Sutton (2002), who provides an alternative interpretation.

small enough. The condition $\alpha > 1$ corresponds to $\mu < \mu^*$ where:

$$\mu^* = \frac{\delta\eta - \left(1 + \frac{\delta}{2}\right)\sigma^2}{1 + \delta} \tag{32}$$

The parameter ξ defined in (12) depends on μ, both directly and via $r - \kappa$:

$$r - \kappa = \rho + (\gamma - 1)\left[\vartheta + \left(\frac{1-\omega}{\omega}\right)(\eta - \mu)\right] \tag{33}$$

The overall dependence of ξ on μ is characterized in the following lemma.

Lemma A2 *If $r > \kappa$, then ξ is strictly increasing in μ for all $\gamma \in (0, 1]$, and for all $\gamma \in (1, \infty)$ such that:*

$$\rho > (1 - \gamma)\left[\vartheta + \left(\frac{1-\omega}{\omega}\right)\eta\right] + \frac{1}{2}(1-\gamma)^2\varsigma^2 \tag{34}$$

Existence is shown separately for three possible cases, depending on the value of γ.

The Case $\gamma = 1$ This implies $r - \kappa = \rho$, and a necessary condition for a balanced growth path to exist is $\rho > \eta$. This condition is also sufficient. To see this, first note from (12) and (24) that α is decreasing and ξ is increasing in μ. Furthermore, α grows without bound and ξ goes to zero as μ goes to $-\infty$. It follows that the right-hand side of (31) is an increasing function of μ, with a vertical asymptote at μ^* and a horizontal asymptote at 0.

The Case $\gamma > 1$ Note that $r - \kappa$ is decreasing in μ. Assume that (34) holds. Then the right-hand side of (31) is increasing in μ as long as $r > \kappa$. As μ goes to $-\infty$, $r - \kappa$ will become large and the right-hand side of (31) goes to zero. As μ approaches μ^*, the right-hand side will increase without bound as long as $r > \kappa$. It follows that the zero-profit condition will have a unique solution that satisfies $r > \kappa + \eta$ if $\mu < \mu^*$ implies $r > \kappa + \eta$. This is the case if:

$$\rho - \eta > (1 - \gamma)\left(\vartheta + \frac{1-\omega}{\omega}\frac{\eta + \left(1 + \frac{\delta}{2}\right)\sigma^2}{1 + \delta}\right) \tag{35}$$

There is an equilibrium if ρ is large enough to satisfy (34) and (35).

The Case $\gamma \in (0,1)$ Now there is a lower bound μ_* so that $r > \kappa + \eta$ if and only if $\mu > \mu_*$. A necessary condition for the existence of an equilibrium with $\alpha = \beta$ is therefore $\mu_* < \mu^*$. This is guaranteed if (35) holds. As μ approaches μ^* from below, α approaches 1 from above, and the right-hand side of (31) will grow without bound. This means that there exists an equilibrium for large values of λ_E/λ_F. It is not difficult to see that the right-hand side of (31) converges to zero as ρ grows without bound. Thus an equilibrium exists for all large enough ρ.

A.2 The Large λ_E/λ_F Asymptote

For any $\beta \geq \alpha > 1$ satisfying (23), the zero-profit condition is:

$$\frac{\lambda_E}{\lambda_F} = \frac{1}{r-\kappa}\frac{\alpha\beta\xi}{\beta-\alpha}\left(\frac{e^{-\alpha\delta}}{\alpha(\alpha-1)(\alpha+\xi)} - \frac{e^{-\beta\delta}}{\beta(\beta-1)(\beta+\xi)}\right)$$

The right-hand side of this condition is increasing in ξ, and therefore:

$$\frac{\lambda_E}{\lambda_F} \leq \frac{1}{r-\kappa}\frac{\alpha\beta}{\beta-\alpha}\left(\frac{e^{-\alpha\delta}}{\alpha(\alpha-1)} - \frac{e^{-\beta\delta}}{\beta(\beta-1)}\right) \tag{36}$$

in any equilibrium. If λ_E/λ_F goes to infinity, then the right-hand side of (36) must go to infinity as well. In any equilibrium, $r-\kappa > \eta$. If $\eta > 0$, then the only way the right-hand side of (36) can go to infinity subject to $\beta \geq \alpha$ is for α to shrink to 1. If $\eta = 0$, then $\zeta > 0$ if and only $\mu < 0$. This implies $\kappa > \vartheta \geq 0$. Thus $r - \kappa > \rho > 0$. That is, in any equilibrium $r - \kappa$ must be bounded away from zero. Thus α must go to one.

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