# Nominal Government Securities*

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**Abstract**

Nominal discount bonds issued by the government can be interpreted as government shares that pay dividends in terms of new shares. The associated dividend yield is the nominal interest rate. We describe policies that allow the government to uniquely implement any sequence of taxes and purchases that, following any history, has a non-negative present value when evaluated at the marginal utilities implied by endowments and government purchases. The response of the government to low off-equilibrium-path prices of its shares is to raise nominal interest rates in surplus periods, and to reduce spending in deficit periods.

## 1. Introduction

The US Treasury does not define the payoff of a T-bill in terms of some fixed bundle of commodities. Instead, it defines a T-bill as a claim to deposits at the Federal Reserve, or perhaps as a claim to new T-bills. If the Federal Reserve were to eliminate the rate-of-return differential between T-bills and deposits held at the Federal Reserve, then it would probably pay interest simply by issuing new deposits, or printing new Federal Reserve Notes. Government securities have the striking feature that their payoffs appear not to be defined in terms of commodities that provide utility or can be used in production, but using a recursion—a T-bill today is a claim to more T-bills tomorrow, and so on. What determines the value of such securities?

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In this paper, we study an exchange economy with a sequence of competitive markets and a general class of government securities. These securities are defined, in part implicitly, by the government policies that describe taxes, government purchases, and trading strategies for government securities. These policies can be complicated functions of history and the new information revealed in every period.

We assume that the government is a price taker in the sequence of markets in which it trades. In every period, the government must choose a demand schedule that is budget feasible at any vector of prices. As a large agent, the government can manipulate prices using the shape of its demand schedules. Government policy is also required to satisfy a set of legal restrictions that define planned taxes and government purchases. We only allow policy rules that are consistent with a competitive equilibrium following every possible history, and we assume the government can commit to follow these rules. We are interested in policies that implement the legally mandated taxes and government purchases as the outcome of a unique competitive equilibrium.

We concentrate on government policies that do not have the government lend to other agents in the economy. This means that the government cannot build up reserves to finance future deficits. In particular, it is not possible for the government to issue, in the first period, a single security backed by all its future tax revenues, invest the proceeds in state-contingent consumption claims on other agents in the economy, and then finance all subsequent government purchases by selling these consumption claims. Instead, in every period in which the government intends to run a primary deficit, it has to sell securities to the public. Market expectations about future government policy will determine the amount of revenue that can be raised in this way.

Despite these restrictions, we show that the government can construct policies so that the unique competitive equilibrium yields the planned sequence of taxes and government purchases, as long as these sequences imply that the present value of taxes minus purchases is non-negative following any history—where the present values are evaluated using the marginal utilities implied by aggregate endowments and planned government purchases.

There are a great many policies that can accomplish this. A very simple one is for the government to sell "government shares" when it plans to run a deficit, and pay dividends or repurchase some of these shares when it has a surplus. In any equilibrium, the present value of current and future primary surpluses will be equal to the market value of government shares. But how the government plans to trade at off-equilibrium prices is crucial for determining the primary surpluses that will be realized.

In a deficit period, a policy of selling more shares can run into trouble only if the price of government shares is actually zero. According to the policy we describe, the government starts to reduce spending when the market value of its shares drops far below the present value of its planned primary surpluses. This response to out-of-equilibrium prices raises the present value of primary surpluses and helps to rule out very low prices of government shares as possible equilibria. If the present value of planned primary surpluses is zero, then the government cannot sell more of its outstanding shares at a positive price. Instead, the government must issue a new type of shares to finance its planned current deficit. These shares are, in equilibrium, a claim against planned future primary surpluses with a strictly positive present value, and they will therefore trade at a strictly positive price. The price of the old shares will be zero. One can interpret this as a "currency reform."

In a surplus period, the government can choose to pay dividends or repurchase shares. If government share prices are too low, a policy of using the surplus to repurchase shares could lead the government to repurchase all of its shares. This would then force the government to implement primary surpluses that have a zero present value in the next period, even though the original plan may have called for a strictly positive present value. The result can be a self-fulfilling equilibrium in which the government spends less than planned. To rule out this possibility, the government must switch to paying dividends instead of repurchasing shares when the price of its shares is too low.

If we use the cum-dividend price of government shares as the numeraire, then the dividend yield on government shares —defined as the ratio of the dividend paid over the ex-dividend price of government shares— is just the nominal interest rate in the economy. An ex-dividend share is just a claim to one cum-dividend share in the next period. Since this cum-dividend share will be the numeraire, this makes an ex-dividend share into a one-period nominal discount bond. By altering the split between dividends and share-repurchases, the government can regulate the nominal interest rate in this economy.

With this change of numeraire, the policy we described for periods in which the government plans to run a surplus can be translated as: the government should raise nominal interest rates when the price level —the price of consumption goods in terms of cum-dividend government shares— is too high. This gives a possible interpretation of the common policy prescription of raising nominal interest rates to "fight inflation" or to defend the value of a currency against a "speculative attack." Note however that the high-interest rate policies described here are only applied at off-equilibrium prices.

The policies we have described lead to a unique equilibrium in which the government implements its planned primary surpluses. It is easy to construct examples of policies that lead to multiple equilibria. These policies make the primary surpluses of the government a function of the market value of its outstanding securities in ways that can lead to self-fulfilling adjustments of these primary surpluses. Roughly, if the market puts a high (low) value on future primary surpluses, and government policy then makes these surpluses increase (decrease) as a result, then the market valuations will be justified automatically. In our examples, a policy of committing to repurchase some fraction of the supply of outstanding securities is what leads to the indeterminacy.

**Related Literature** Traditionally, nominal assets are defined as assets that have payoffs defined in terms of fiat money. The economy described in this paper is a frictionless economy without fiat money, and we need another definition. Our setup borrows from a literature in general equilibrium theory that defines nominal securities as securities with payoffs defined in terms of some numeraire. Nominal assets can easily lead to an indeterminacy of equilibrium. When markets are incomplete, this indeterminacy includes real allocations, and this has been the focus of much of the work in general equilibrium theory (Geanakoplos and Mas-Colell (1989), Balasko and Cass (1989), Werner (1990)).

Somewhat different indeterminacy issues are at the heart of the debate about the fiscal theory of the price level, as proposed by Leeper (1991), Sims (1994) and Woodford (1995). This theory states that the price level is uniquely determined when the government commits to a sequence of real primary surpluses and deficits. One way to interpret this theory is to suppose that the government is the Walrasian auctioneer and selects prices such that the implied aggregate net trade by agents in the economy matches its own desired net trade. When consumers have strictly convex preferences this will lead to a unique competitive equilibrium. Kocherlakota and Phelan (1999) suggest that this may amount to selecting a unique equilibrium out of the many that could arise if the government must take prices as given. Buiter (2002) has argued, similarly, that the government should be understood as taking prices as given and formulating policies that are well defined at all prices.

In an important contribution to this debate, Bassetto (2002a) uses the market game of Shubik (1973) and describes strategies for the government so that the unique subgame perfect equilibrium outcome is equal to the competitive equilibrium outcome that a government intends to implement. The notion of implementing plans of government purchases and taxes used in this paper is related to ideas in Bassetto (2002b).

The analogy between nominal securities issued by the government and shares issued by firms follows Cochrane (2001a). But we specify government policy for all non-negative prices of government shares, rather than only for a specific sequence of equilibrium prices. The specification of policy at out-of-equilibrium prices is crucial for a government that cannot set prices.

A government policy in this paper is a set of rules that describes how the government trades in a sequence of markets at whatever prices it faces. We specify these rules such that an equilibrium exists following every history. The government is committed to follow these rules, and the origin of these rules is taken to be exogenous. It is important to determine how these rules come about, and to what extent governments can indeed commit to follow them. The literature on fiscal and monetary policy under alternative assumptions about commitment and government objectives is large. This literature is surveyed in Chari and Kehoe (1999). The tight connection between fiscal policy and inflation dates back to Sargent and Wallace (1981).

Section 2 describes the economy. A basic result on nominal dividend neutrality is discussed in Section 3. Real dividend policies are discussed in Section 4. Policies with zero dividends on the equilibrium path are presented in Section 5. Section 6 gives examples of policies that lead to multiple equilibria.

## 2. The Economy

The economy is a discrete-time exchange economy with many consumers and a government. Time is indexed by $t = 0, 1, 2, \dots$ , and the information revealed over time is described by a non-decreasing sequence of sigma-algebra's $\{\mathcal{F}_t\}_{t=0}^{\infty}$ defined on some underlying probability space. All agents in the economy have access to the same information. The expectation operator conditional on $\mathcal{F}_t$ is denoted by $\mathrm{E}_t[\cdot]$. Random variables or functions indexed by $t$ are $\mathcal{F}_t$-measurable.

### 2.1. Consumers

There is a continuum of identical, infinitely-lived consumers who can consume a single type of good in every period. A typical consumer in period $t$ has preferences over consumption processes $\{c_v\}_{v=t}^{\infty}$ given by:

$$\mathrm{E}_t\left[\sum_{v=t}^{\infty} \beta^v u(c_v)\right] \tag{1}$$

for some subjective discount factor $\beta > 0$ and a period utility function $u$. We maintain the following assumption.

**Assumption 1:** *The utility function* $u : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ *is strictly increasing, strictly concave, differentiable on* $\mathbb{R}_{++}$, *with a marginal utility that is unbounded near zero.*

Consumers receive endowments $\{e_t\}_{t=0}^{\infty}$ and pay lump-sum taxes $\{\theta_t\}_{t=0}^{\infty}$.

There is a sequence of markets. In every period consumers can trade in one-period ahead state-contingent claims, and government securities. The number of government securities traded in period $t$ is denoted by $n_t$. Before trade takes place in period $t$, the government announces an $n_t \times n_{t-1}$ matrix $a_t$ that describes how the set of securities is adjusted. This matrix will be discussed below. For now it is sufficient to know that $a_t \geq 0$, and that the government can only close down markets for a particular government security if the outstanding supply is zero. The government securities traded in period $t$ pay non-negative dividends given by an $n_t$-vector $d_t$. In any period, trade takes place, and then dividends are paid. The cum-dividend prices of government securities are denoted by the $n_t$-vector $s_t$, measured in units of consumption.

The prices of state-contingent claims can be used to compute the probability-weighted price of state-contingent consumption in period $t$ in terms of consumption in period 0. This price is denoted by $\pi_t/\pi_0$, where $\pi_0$ is some arbitrary positive number. Consumers take the stochastic process of prices $\{\pi_t, s_t\}_{t=0}^{\infty}$ as given and choose consumption and holdings of contingent claims and government securities subject to the following sequence of budget constraints:

$$c_t + \frac{1}{\pi_t}\mathrm{E}_t[\pi_{t+1}b_{t+1}] + (s_t - d_t)'k_t \leq e_t - \theta_t + b_t + s_t' a_t k_{t-1}, \tag{2}$$

together with the present-value borrowing constraints:

$$\pi_{t+1}(b_{t+1} + s_{t+1}' a_{t+1} k_t) + \mathrm{E}_{t+1}\left[\sum_{v=t+1}^{\infty} \pi_v(e_v - \theta_v)\right] \geq 0, \tag{3}$$

and the non-negativity constraint $k_t \geq 0$, for $t = 0, 1, 2, \ldots$ . The initial values $b_0$ and $k_{-1}$ are given by 0 and some non-negative $n_{-1}$-vector, respectively.[1]

Consumers are not allowed to counterfeit government securities. That is, they are not allowed to print their own government securities, and sell them at a price $s_t$, even if they deliver the same dividends as the government does. As we shall see, government securities may trade at a positive price even if the government never pays any dividends, essentially because the government is committed to use primary surpluses to repurchase government securities. A consumer who could print government securities without being committed to buy them back would have an arbitrage opportunity.[2] Here we only allow consumers to issue one-period claims to consump-

[1] The contingent claims are redundant in this representative agent economy. They are not when agents are heterogeneous. In an heterogeneous-agent economy it can also be important to allow consumers to trade claims that are contingent on the prices of government securities. Otherwise certain government policies would have to be excluded simply because they would bankrupt some consumers and lead to non-existence of a competitive equilibrium.

[2] The constraint (3) actually rules this out, but it is more transparent to impose $k_t \geq 0$. Suppose there is only one government security, and no uncertainty. If the government does not pay dividends, then it is going to be the case that $s_t$ grows at the real interest rate. If a consumer chose, for example, $b_t = 0$ and $k_t = -1$ throughout, then (3) would eventually be violated, since $\pi_{t+1}s_{t+1}$ would be constant, while the discounted present value in (3) would go to zero. Although $k_t \geq 0$, one can still allow for short sales defined as forward contracts contingent on the price of government securities. This is how short sales (as opposed to new issues) of securities work in actual economies. The arbitrage transaction described in the text would arise if we had replaced $s_{t+1}k_t$ in (3) by $\frac{1}{\pi_{t+1}}\mathrm{E}_{t+1}\left[\sum_{v=t+1}^{\infty}\pi_v d_v\right]k_t$.

tion, and we assume that consumers are forced to honor these claims. The borrowing constraint (3) makes sure that they always can. We could allow consumers to issue their own long-lived securities, but then we would have to specify what dividend and repurchase policies consumers were committed to conduct.

In any equilibrium, the prices of government securities must satisfy:

$$\begin{aligned} \pi_t(s_t - d_t)' - \mathrm{E}_t[\pi_{t+1}s'_{t+1}a_{t+1}] &\geq 0 \\ \left(\pi_t(s_t - d_t)' - \mathrm{E}_t[\pi_{t+1}s'_{t+1}a_{t+1}]\right) k_t &= 0 \end{aligned} \tag{4}$$

If the inequality ran in the opposite direction, then a consumer could make unbounded profits by selling contingent claims and investing in government securities. If the complementary slackness condition did not hold, then a consumer could sell certain government securities, and buy state-contingent claims that yield the same period-$t+1$ payoff, but at a lower cost.

Because consumer borrowing is constrained only by (3), it must be that the present value of after-tax endowments is finite in any equilibrium. Combining (2), (3) and (4), one can verify that an optimal consumption process must maximize (1) subject to the present-value budget constraint:

$$\mathrm{E}_t\left[\sum_{v=t}^{\infty}\pi_v c_v\right] \leq \mathrm{E}_t\left[\sum_{v=t}^{\infty}\pi_v(e_v - \theta_v)\right] + \pi_t(b_t + s'_t a_t k_{t-1}) \tag{5}$$

as of any period $t$. Furthermore, because preferences are strictly increasing in consumption, it must be that this inequality holds as an equality. One can verify that therefore:

$$\lim_{T\to\infty} \mathrm{E}_t[\pi_T(b_T + s_T a'_T k_{T-1})] = 0 \tag{6}$$

almost surely, in any equilibrium.

### 2.2. The Government

We consider a government that only trades in consumption goods and its own securities. It does not lend to consumers. This means that the only prices that show up in the budget constraints of the government are the prices of its own securities.

A government policy is a sequence of functions that describes, for every history, government purchases, lump-sum taxes, securities trades and the dividends paid by the government. Government policy must be consistent with a set of legal restrictions, and satisfy a sequence of budget constraints. In particular, in every period, the government must take the prices of its securities as given and government policy must be feasible for every possible vector of prices.

**Histories** Let $\sigma_{-1} = k_{-1}$ and define $\Sigma_{-1} = \{\sigma_{-1}\}$. For periods $t = 0, 1, 2, \dots$, the sets of possible histories are defined recursively by a sequence of random functions:

$$N_t : \Sigma_{t-1} \to \mathbb{N}$$

and:

$$\Sigma_t = \left\{ (s, \sigma) : s \in \mathbb{R}_+^{N_t(\sigma)}, \sigma \in \Sigma_{t-1} \right\}$$

The functions $N_t$ are the outcome of government policy, as described below. Note that if $N_0(\sigma_{-1})$ is random, then $\Sigma_0$ will be a random set, and similarly for subsequent sets of histories. Every $\Sigma_t$ is simply the non-negative orthant of some Euclidean space, but the dimension may be stochastic. In period $t$, and following a history $\sigma \in \Sigma_{t-1}$, $N_t(\sigma)$ represents the number of government securities traded. Given a period-$t$ history $(s, \sigma) \in \Sigma_t$, $s$ represents the $N_t(\sigma)$-vector of prices of government securities in period $t$, and $\sigma \in \Sigma_{t-1}$. We let $s_n$ denote the $n^{\text{th}}$ component of $s$. For a random variable $(s_t, \sigma_{t-1}) \in \Sigma_t$, the period-$t$ price of the $n^{\text{th}}$ security is written as $s_{n,t}$.

**Budget Constraints** At the beginning of any period $t$, the government uses the information revealed by $\mathcal{F}_t$ to adjust the set of traded government securities. Given a history $(s, \sigma) \in \Sigma_{t-1}$, this adjustment is described by an $N_t(s, \sigma) \times N_{t-1}(\sigma)$ random matrix $A_t(s, \sigma)$. The government's choice of $A_t$ implicitly determines $N_t$. The supply of government securities in period $t$ is defined by an $\mathcal{F}_t$-measurable random function $K_t$ that maps price histories contained in $\Sigma_t$ into a non-negative vector of dimension $N_t$. For every $\sigma \in \Sigma_{t-1}$, the matrix $A_t(\sigma)$ has zero entries everywhere, except that the $n^{\text{th}}$ column has a 1 in some row if $K_{n,t-1}(\sigma)$ is positive. That is, markets for securities that are still held by consumers cannot be closed down.

Given the markets that are open in period $t$, government policy specifies taxes $T_t(\sigma)$, purchases $G_t(\sigma)$, a supply of securities $K_t(\sigma)$, and dividends $D_t(\sigma)$, for every period-$t$ history $\sigma \in \Sigma_t$. These policies are random functions that depend on the information revealed in $\mathcal{F}_t$, and they must satisfy the period-$t$ budget constraint:

$$D_t(s, \sigma)' K_t(s, \sigma) = s' \left[ K_t(s, \sigma) - A_t(\sigma) K_{t-1}(\sigma) \right] + T_t(s, \sigma) - G_t(s, \sigma) \tag{7}$$

and:

$$0 \leq D_t(s, \sigma), K_t(s, \sigma) \tag{8}$$

for all period-$t$ histories $(s, \sigma) \in \Sigma_t$. The budget constraint (7) says that dividend payments have to be financed out of the primary surplus of the government, or out of sales of additional cum-dividend securities. By (8), consumers cannot be forced to pay dividends on the securities they hold. The non-negativity constraint on the supply of government securities prevents the government from lending to consumers and implies that the government cannot finance primary deficits out of assets accumulated in the past. It is important to note that, for a given set of traded securities, all government actions in (7) are allowed to depend on the current price of cum-dividend government securities.

For securities that are outstanding at the beginning of period $t$ following a history $\sigma \in \Sigma_{t-1}$, the government must compete with the consumers who hold $K_{t-1}(\sigma)$ if it tries to sell additional quantities of these securities. The government can avoid this competition by issuing new types of securities and sell them at whatever is the market clearing price. The government can issue new securities by letting $A_t(\sigma)$ have extra rows of zeros. Note that this does not necessarily mean that the government will actually sell these securities: $K_t(s,\sigma)$ may be zero in the corresponding rows. To retire the $n^{\text{th}}$ security, the government can set $K_{n,t}(s,\sigma) = 0$ if $s_n \leq D_{n,t}(s,\sigma)$ and $K_{n,t}(s,\sigma) = K_{n,t-1}(\sigma)$ otherwise, and then never pay dividends or trade in the security again.

**Legal Restrictions** Government purchases and taxes are mandated by law. The law specifies a desired stochastic process of purchases $\{g_t\}_{t=0}^{\infty}$ and taxes $\{\tau_t\}_{t=0}^{\infty}$. Government policy has to satisfy:

$$G_t(\sigma) \in [0, g_t] \tag{9}$$

$$T_t(\sigma) \leq \tau_t \tag{10}$$

for every $\sigma \in \Sigma_t$.

**Commitment** We are going to assume that the government decides on a particular policy at the beginning of period 0, and commits to follow it at all times. The restrictions (7)-(10) leave a lot of flexibility in the design of government policy rules. Furthermore, the mere fact that government policy is feasible following any history says nothing about whether a policy is consistent with a competitive equilibrium or not. The possibility of non-existence of equilibrium following a particular history gives the government a lot of power to rule out equilibria. All the government needs to do is design a policy that rules out equilibrium following a particular history, and then this history cannot be part of any equilibrium. Since the government is a large player, it is easy to construct such policies.

But this stretches the assumption of commitment. If, following some history, no equilibrium exists under the pre-specified government policy, will it really be the case that government policy is not adjusted if this history is realized anyway? And if so, what happens then?

To avoid this, we will only consider policy rules that are consistent with the existence of an equilibrium following every history.

## 2.3. Competitive Equilibrium

A competitive equilibrium is defined by an $\mathcal{F}_t$-adapted stochastic process $\{\pi_t, s_t\}_{t=0}^{\infty}$ of state prices and securities prices. The process of securities prices implies a stochastic process of histories $\{\sigma_t\}_{t=0}^{\infty}$ via the recursion $\sigma_t = (s_t, \sigma_{t-1})$. Using these histories, one can define $\{a_t, c_t, d_t, n_t, \theta_t\}_{t=0}^{\infty}$ by $a_t = A_t(\sigma_{t-1})$, $c_t = e_t - G_t(\sigma_t)$, $d_t = D_t(\sigma_t)$, $n_t = N_t(\sigma_{t-1})$ and $\theta_t = T_t(\sigma_t)$. The state prices and securities prices are equilibrium prices if the consumption process $\{c_t\}_{t=0}^{\infty}$ is optimal for consumers given $\{\pi_t, s_t\}_{t=0}^{\infty}$ and the implied $\{a_t, d_t, n_t, \theta_t\}_{t=0}^{\infty}$.

A first requirement for optimality is the absence of arbitrage opportunities (4) between state-contingent claims and government securities. Furthermore, it must be that after-tax wealth of consumers is finite, and that their present-value budget constraints (5) hold with equality at all times. Together these observations imply that the period-0 market value $s_0' a_0 k_{-1}$ of cum-dividend government shares is equal to the present value of the primary surpluses $\{T_t(\sigma_t) - G_t(\sigma_t)\}_{t=0}^{\infty}$. Assumption 1 implies that $c_t$ must be positive in all periods, as long as consumer wealth is strictly positive. A standard first-order condition then implies that the $\pi_t$-process must be proportional to the process of marginal utilities. This motivates the following definition.

**Definition 1:** *Given $k_{-1}$ and government policy functions $\{(A_t, D_t, G_t, K_t, T_t)\}_{t=0}^{\infty}$ that satisfy (7)-(10), a competitive equilibrium consists of a stochastic process of non-negative prices $\{\pi_t, s_t\}_{t=0}^{\infty}$ together with histories constructed recursively by $\sigma_t = (s_t, \sigma_{t-1})$, such that:*

$$\pi_t = \beta^t \mathrm{D}u(e_t - G_t(\sigma_t)) \tag{11}$$

$$\begin{aligned} &\pi_t s_t' - \pi_t D_t(\sigma_t)' - \mathrm{E}_t[\pi_{t+1} s_{t+1}' A_{t+1}(\sigma_t)] \geq 0 \\ &\left(\pi_t s_t' - \pi_t D_t(\sigma_t)' - \mathrm{E}_t[\pi_{t+1} s_{t+1}' A_{t+1}(\sigma_t)]\right) K_t(\sigma_t) = 0 \end{aligned} \tag{12}$$

*and:*

$$s_t' A_t(\sigma_{t-1}) K_{t-1}(\sigma_{t-1}) = \frac{1}{\pi_t} \mathrm{E}_t \left[ \sum_{v=t}^{\infty} \pi_v [T_v(\sigma_v) - G_v(\sigma_v)] \right] \tag{13}$$

*in all periods $t = 0, 1, 2, \ldots$, almost surely.*

Implicit in condition (13) of Definition 1 is the requirement that the sum on the right-hand side is well-defined and finite. Note that (13) together with $c_t = e_t - G_t(\sigma_t)$ and $b_t = 0$ implies that the period-$t$ present-value budget constraint (5) of consumers holds with equality. Together with (11)-(12) this implies that $c_t = e_t - G_t(\sigma_t)$ defines an optimal consumption process, and justifies calling the process $\{\pi_t, s_t\}_{t=0}^{\infty}$ a competitive equilibrium.

The fact that the right-hand side of (13) converges implies that:

$$\lim_{T \to \infty} \mathrm{E}_t \left[\pi_T s_T' A_T(\sigma_{T-1}) K_{T-1}(\sigma_{T-1})\right] = 0$$

almost surely, in any equilibrium. It is important to note that this need not be true for $\mathrm{E}_t[\pi_T s_T]$ as $T$ gets large. In particular, we shall see examples in which equilibrium dividends $D_t(\sigma_t)$ are zero, and $\pi_t s_t = \mathrm{E}_t[\pi_T s_T] > 0$.

The following assumption characterizes the stochastic processes of government purchases $\{g_t\}_{t=0}^{\infty}$ and taxes $\{\tau_t\}_{t=0}^{\infty}$ for which we want to design government policies.

**Assumption 2:** *Planned government purchases* $\{g_t\}_{t=0}^{\infty}$ *and taxes* $\{\tau_t\}_{t=0}^{\infty}$ *are such that* $\max\{\tau_t, g_t\} < e_t$, *together with:*

$$\mathrm{E}_0 \left[ \sum_{t=0}^{\infty} \beta^t u(e_t - g_t) \right] > -\infty \tag{14}$$

*and:*

$$\mathrm{E}_0 \left[ \sum_{t=0}^{\infty} \beta^t \mathrm{D}u(e_t - g_t) e_t \right] < \infty. \tag{15}$$

*Furthermore:*

$$V_t = \frac{1}{\mathrm{D}u(e_t - g_t)} \mathrm{E}_t \left[ \sum_{v=t}^{\infty} \beta^{v-t} \mathrm{D}u(e_v - g_v)(\tau_v - g_v) \right] \geq 0, \tag{16}$$

*for all non-negative* $t$.

Part (16) imposes that the present value of intended primary surpluses is non-negative starting from any period. The need for this restriction to hold in every period arises from the fact that we do not allow the government to accumulate claims on consumers.

As argued earlier, it is easy to construct examples of government policy rules for which no equilibrium can exist. There are also policy rules that give rise to multiple competitive equilibria, and it can be the case that some equilibria imply that $G_t(\sigma_t) = g_t$ and $T_t(\sigma_t) = \tau_t$ , while others do not. We are interested in specifying policy rules that ensure that the government achieves its desired spending.

**Definition 2:** *The government policy* $\{(A_t, D_t, G_t, K_t, T_t)\}_{t=0}^{\infty}$ *implements* $\{g_t, \tau_t\}_{t=0}^{\infty}$ *if it satisfies (7)-(10), and if in every competitive equilibrium prices* $\{\pi_t, s_t\}_{t=0}^{\infty}$ *are such that the recursion* $\sigma_t = (s_t, \sigma_{t-1})$ *yields* $g_t = G_t(\sigma_t)$ *and* $\tau_t = T_t(\sigma_t)$ *for all* $t$.

The following easy proposition indicates that $D_t(0, \sigma)$ must be positive if the government is to have any chance of implementing $\{g_t, \tau_t\}_{t=0}^{\infty}$.

**Proposition 1:** *Suppose government policy specifies* $D_t(0, \sigma) = 0$ *for any* $\sigma \in \Sigma_{t-1}$. *Then there is a competitive equilibrium in which* $s_t = 0$ *at all times. Government purchases in period* $t$ *are given by* $\min\{g_t, \tau_t\}$. *Taxes are equal to government purchases in every period.*

If, following a history $\sigma_{t-1}$, consumers in period $t$ believe that $s_{t+1} = 0$, then they will not want to purchase any government securities at a positive ex-dividend price in period $t$. Thus $D_t(s_t, \sigma_{t-1}) = s_t$, and this equation is solved by $s_t = 0$, by assumption. Given $s_t = 0$, the government budget constraint (7) implies that $T_t(0, \sigma_{t-1}) = G_t(0, \sigma_{t-1})$. The spending rule (9) and the limitation on taxes (10) then imply the taxes and government purchases described in the proposition. For consumers, no trade is optimal if $s_t = 0$ at all times.

Note that Proposition 1 does not say that $s_t = 0$ is necessarily the only equilibrium. It is not difficult to construct examples in which $D_t$ is identically zero, and in which the equilibrium prices of government securities are positive at all times. But equilibrium indeterminacy is inevitable in such examples.

### 2.4. Some Examples

**Nominal Dividends** Let $\delta_t$ be an $N_t$-vector of non-negative, real-valued random functions defined on $\Sigma_t$. An important special type of dividend policy is the following "nominal dividend policy:"

$$D_{n,t}(s, \sigma) = \frac{\delta_{n,t}(s, \sigma) s_n}{1 + \delta_{n,t}(s, \sigma)} \tag{17}$$

for any $(s, \sigma) \in \Sigma_t$. Note the obvious fact that $D_{n,t}(0, \sigma) = 0$ in (17). By construction, $s_n \geq D_{n,t}(s, \sigma)$, and this inequality is strict if $s_n$ and $\delta_{n,t}(s, \sigma)$ are strictly positive. In that case, $D_{n,t}(s, \sigma)/[s_n - D_{n,t}(s, \sigma)]$ is equal to $\delta_{n,t}(s, \sigma)$, and for this reason we call $\delta_n(s, \sigma)$ the dividend yield on the $n^{\text{th}}$ security.

One way the government can implement a nominal dividend policy with dividend yield $\delta_{n,t}(s, \sigma)$ is simply to hand out $\delta_{n,t}(s, \sigma)$ new units of security $n$, per unit held by consumers at the end of period $t$. In particular, if the government does not otherwise change the supply of this security, then:

$$K_{n,t}(s, \sigma) = [1 + \delta_{n,t}(s, \sigma)] K_{n,t-1}(\sigma) \tag{18}$$

and thus:

$$D_{n,t}(s, \sigma) K_{n,t}(s, \sigma) = s_n \left[ K_{n,t}(s, \sigma) - K_{n,t-1}(s, \sigma) \right]$$

for all $(s, \sigma) \in \Sigma_t$. Under this policy, the $n^{\text{th}}$ security is not used to finance a deficit or pay out a surplus in period $t$, following $\sigma \in \Sigma_{t-1}$.

**An Alternative Numeraire** Suppose that $\{\pi_t, s_t\}_{t=0}^{\infty}$ is a stochastic process of equilibrium prices, and write $\{\sigma_t\}_{t=0}^{\infty}$ for the associated histories. Suppose that $s_{1,t}$ is positive at all times. Then one can use the first government security as the numeraire in all periods. Let $p_t$ be the price of one unit of consumption in terms of this numeraire, and let $q_t$ be the price of a one-period nominal discount bond in this numeraire. Then:

$$\begin{aligned} p_t &= \frac{1}{s_{1,t}} \\ q_t &= 1 - \frac{1}{s_{1,t}} D_{1,t}(\sigma_t) \end{aligned}$$

The price of a nominal discount bond follows from the fact that an ex-dividend unit of security 1 costs $s_{1,t} - D_{1,t}(\sigma_t)$ units of consumption, and this translates into $q_t$ units of cum-dividend government security 1. The non-negativity of $D_{1,t}(\sigma_t)$ implies that nominal interest rates are non-negative. The no-arbitrage condition (4) implies that the price of a nominal discount bond is non-negative in any equilibrium. Observe that for a nominal dividend policy $D_{1,t}(s,\sigma) = \delta_{1,t}(s,\sigma)s_1/(1+\delta_{1,t}(s,\sigma))$ one obtains $q_t = 1/(1+\delta_{1,t}(\sigma_t))$. Thus $\delta_{1,t}(\sigma_t)$ is simply the nominal interest rate. Note that while $D_{1,t}(\sigma_t)$ is the dividend paid in period $t$, the associated dividend yield $\delta_{1,t}(\sigma_t)$ is the nominal interest rate that applies to the holding period from $t$ to $t+1$.

A dividend policy $D_t(s,\sigma) \geq \underline{s}_t$ for all $s$ below $\underline{s}_t$ can keep $s$ from falling below $\underline{s}_t$, provided the government has the required primary surpluses to finance these dividends. The expression for $q_t$ shows that this policy amounts to committing to pay agents to hold nominal discount bonds in case $s_{1,t} < \underline{s}_{1,t}$, and thus $s_{1,t} \geq \underline{s}_{1,t}$ in any equilibrium. If government policy is such that $s_1 > D_{1,t}(s,\sigma) \geq \underline{s}_{1,t}$ also holds for $s_1$ slightly above $\underline{s}_{1,t}$, then government policy amounts to raising the nominal interest rate to very high levels as $s_1$ gets close to the intended lower bound $\underline{s}_{1,t}$.

**Multi-Period Nominal Discount Bonds** Suppose that starting from some period $t$ and history $\sigma_t \in \Sigma_{t-1}$, security 2 is characterized by:

$$\begin{aligned} D_{2,t}(x_0, \sigma_{t-1}) &= 0 \\ D_{2,t+1}(x_1, \sigma_{t-1}) &= 0 \\ &\vdots \\ D_{2,t+N-1}(x_{N-1}, \sigma_{t-1}) &= 0 \\ D_{2,t+N}(s, x_{N-1}, \sigma_{t-1}) &= s_1 \end{aligned}$$

for all $(x_v, \sigma_{t-1}) \in \Sigma_{t+v+1}$, $v = 0, \ldots, N-1$ and $(s, x_{N-1}, \sigma_{t-1}) \in \Sigma_{t+N}$. Following this, the security is retired. Thus, if security 1 is the numeraire, then security 2 is an $N$-period nominal discount bond in period $t$. At the beginning of period $t+N$, owning one unit of security 2 is equivalent to owning one unit of security 1. In either case, one unit of the security represents one unit of a maturing nominal discount bond. In equilibrium, it will be the case that $s_{1,t+N} = s_{2,t+N}$. At these prices, the government can repurchase all of the supply of security 2 in exchange for an equal number of units of security 1.

## 3. Nominal Dividend Neutrality

We want to be able to analyze the effects of alternative nominal interest rate policies. This is easy when the government sells only one type of government security. In this section, therefore, suppose that $N_t = 1$, and thus $\Sigma_t = \mathbb{R}_+^t$, and ignore the $A_t$ matrices.

Consider a feasible policy $\{(D_t^*, G_t^*, K_t^*, T_t^*)\}_{t=0}^{\infty}$, and let $\{\delta_t\}_{t=0}^{\infty}$ be some sequence of non-negative real-valued random functions, with $\delta_t(\cdot)$ defined on $\Sigma_t$. Let $\Delta_{-1} = 1$, and define recursively:

$$\Delta_t(s, \sigma) = [1 + \delta_t(s, \sigma)]\Delta_{t-1}(\sigma) \tag{19}$$

for all histories $(s, \sigma)$. Construct the following histories of scaled prices:

$$\mathrm{S}_t^*(s, \sigma) = (s\Delta_{t-1}(\sigma), \mathrm{S}_{t-1}^*(\sigma)) \tag{20}$$

for all $(s, \sigma)$. Note that given $\sigma$ one can recover $s$ from $\mathrm{S}_t^*(s, \sigma)$ by dividing the first component of $\mathrm{S}_t^*(s, \sigma)$ by $\Delta_{t-1}(\sigma)$. Thus there is a one-to-one mapping between the two types of histories. We can now define the following alternative government policy:

$$\begin{aligned} G_t(s, \sigma) &= G_t^*(\mathrm{S}_t^*(s, \sigma)) \\ T_t(s, \sigma) &= T_t^*(\mathrm{S}_t^*(s, \sigma)) \\ D_t(s, \sigma) &= \left[D_t^*(\mathrm{S}_t^*(s, \sigma)) + s\Delta_{t-1}(\sigma)\delta_t(s, \sigma)\right] / \Delta_t(s, \sigma) \\ K_t(s, \sigma) &= K_t^*(\mathrm{S}_t^*(s, \sigma))\Delta_t(s, \sigma) \end{aligned} \tag{21}$$

for all $(s, \sigma) \in \Sigma_t$. The policy (19)-(21) is constructed so that the government budget constraint (7) holds, and such that government purchases and taxes are the same for appropriately re-scaled prices.

**Proposition 2:** *Let $\{\delta_t\}_{t=0}^{\infty}$ be some sequence of non-negative real-valued random functions, with $\delta_t(\cdot)$ defined on $\Sigma_t$. Define $\{\Delta_t\}_{t=0}^{\infty}$ and $\{\mathrm{S}_t^*\}_{t=0}^{\infty}$ as in (19)-(20). Then $\{\pi_t, s_t\}_{t=0}^{\infty}$ defines a competitive equilibrium for the policy $\{(D_t, G_t, K_t, T_t)\}_{t=0}^{\infty}$ constructed in (21) and the associated histories $\sigma_t = (s_t, \sigma_{t-1})$ if and only if $\{\pi_t^*, s_t^*\}_{t=0}^{\infty}$ is a competitive equilibrium for the policy $\{(D_t^*, G_t^*, K_t^*, T_t^*)\}_{t=0}^{\infty}$, and:*

$$\pi_t^* = \pi_t \tag{22}$$

$$s_t^* = s_t\Delta_{t-1}(\sigma_{t-1}) \tag{23}$$

*for all $t = 0, 1, 2, \ldots$ .*

The proof follows from feasibility and the fact that $\{D_t, K_t\}_{t=0}^{\infty}$ implies the same real returns for consumers at prices $\{s_t\}_{t=0}^{\infty}$ as $\{D_t^*, K_t^*\}$ does at prices $\{s_t^*\}_{t=0}^{\infty}$.

Note that the equilibrium supplies $\{k_t\}_{t=0}^{\infty}$ and $\{k_t^*\}_{t=0}^{\infty}$ of government securities for the two economies in Proposition 2 are related via $k_t = k_t^* \Delta_t(\sigma_t)$. Note also that the equilibrium price $s_t$ constructed in (23) depends on the policies $\{\delta_u\}_{u=0}^{t-1}$, but not on $\{\delta_u\}_{u=t}^{\infty}$. In particular, a surprise in $\delta_t$ has no effect on the period-$t$ value of cum-dividend shares.

It should be emphasized that Proposition 2 allows for very general interest rate rules $\{\delta_t\}_{t=0}^{\infty}$. One can allow for complicated forms of feedback between current nominal interest rates and current and past inflation. The much-discussed Taylor rules are included. It is easy to construct examples of rules that generate unstable or chaotic dynamics in prices and nominal interest rates.

As we shall see later, Proposition 2 depends crucially on the assumption that there are no other securities with payoffs that could be affected by the price of the security for which the dividend policy is adjusted.

## 4. Government Shares with Real Dividends

In this section we suppose initially that the present value of planned primary surpluses is always strictly positive. For this case, we assume there is only one type of government security, and we refer to it as a government share. When the present value of primary surpluses can occasionally hit zero, additional securities may be required.

### 4.1. Strictly Positive Present Values

We assume that the process $\{V_t\}_{t=0}^{\infty}$ of present values defined in (16) is strictly positive at all times. This will ensure that the price of government shares never has to be zero in an equilibrium in which the government tries to implement $\{g_t, \tau_t\}_{t=0}^{\infty}$. The initial supply of government shares is $k_{-1} = 1$. In the policies described below, the government never repurchases shares, and so $K_t(\sigma) \geq 1$ under all circumstances.

Consider first states of the world in which $\tau_t \geq g_t$. That is, the government intends to run a primary surplus. We assume the government simply pays the surplus as a dividend:

$$\begin{aligned} G_t(s,\sigma) &= g_t \\ T_t(s,\sigma) &= \tau_t \\ D_t(s,\sigma) &= (\tau_t - g_t)/K_{t-1}(\sigma) \\ K_t(s,\sigma) &= K_{t-1}(\sigma) \end{aligned} \tag{24}$$

for any $(s,\sigma) \in \Sigma_t$. Since $K_{t-1}(\sigma)$ will be no less than 1, this policy is well defined.

Next consider states of the world in which $\tau_t < g_t$, so that the government needs to finance a deficit. As long as $s > 0$, the government could simply sell as many new shares as needed to finance a deficit. At $s = 0$, this is no longer possible, and the government will have to adjust purchases. We specify the following class of policies, defined for $\alpha_t \in (0, 1]$:

$$\begin{aligned} G_t(s,\sigma) &= \tau_t + (g_t - \tau_t)\min\{1, sK_{t-1}(\sigma)/\alpha_t V_t\} \\ T_t(s,\sigma) &= \tau_t \\ D_t(s,\sigma) &= 0 \\ K_t(s,\sigma) &= K_{t-1}(\sigma) + \lim_{x\downarrow s}\tfrac{1}{x}\left[G_t(s,\sigma) - T_t(s,\sigma)\right] \end{aligned} \tag{25}$$

for any $(s, \sigma) \in \Sigma_t$. Observe that (25) is a continuous function of $s$. The policies defined by (25) fix taxes, and implement the planned purchases when $s \geq \alpha_t V_t / K_{t-1}(\sigma)$. The government sells just as many shares as it needs to. For $s \leq \alpha_t V_t / K_{t-1}(\sigma)$, the government continues to sell as many shares as it would have at a price $\alpha_t V_t / K_{t-1}(\sigma)$, but it reduces spending as much as needed to satisfy its budget constraint. In the extreme case of $s = 0$, spending is reduced all the way down to tax revenues. If $\alpha_t = 1$, then the government starts to reduce spending as soon as the price of its shares falls below the price that is consistent with its intended primary surpluses. For small but positive values of $\alpha_t$, the government adjusts spending only for very low share prices.

To examine the policy (24)-(25), note that:

$$\tau_t - G \leq 0 \text{ implies } \frac{\partial}{\partial G}\left[\mathrm{D}u(e_t - G)\,(\tau_t - G)\right] \leq 0 \tag{26}$$

That is, when the government runs a deficit, lowering government purchases a little bit can only bring the shadow value of the deficit closer to zero. The policy (24)-(25) therefore implies that:

$$\frac{\mathrm{D}u(e_v - G_v(\sigma_v))}{\mathrm{D}u(e_t - G_t(\sigma))}\left[T_v(\sigma_v) - G_v(\sigma_v)\right] \geq \frac{\mathrm{D}u(e_v - g_v)}{\mathrm{D}u(e_t - g_t)}\left[\tau_v - g_v\right] \tag{27}$$

for any history $\sigma \in \Sigma_t$ and any sequence of histories $\sigma_v$, $v \geq t$, such that $\mathrm{proj}_{\Sigma_t}\sigma_v = \sigma$. Together with the equilibrium condition (13) this implies that $s_t K_{t-1}(\sigma_{t-1}) \geq V_t \geq \alpha_t V_t$ for any equilibrium price process $\{\pi_t, s_t\}_{t=0}^{\infty}$ with associated histories $\sigma_t = (s_t, \sigma_{t-1})$. And then (25) implies that $s_t K_{t-1}(\sigma_{t-1}) = V_t$ is the only equilibrium. This gives the following proposition.

**Proposition 3:** *If Assumptions 1 and 2 hold, and $V_t > 0$ at all times, then the policy (24)-(25) implements $\{g_t, \tau_t\}_{t=0}^{\infty}$.*

**Equilibrium Nominal Interest Rates** In equilibrium, the policy (24)-(25) implies:

$$q_t = 1 - \max\{\pi_t(\tau_t - g_t), 0\}\left(\mathrm{E}_t\left[\sum_{v=t}^{\infty}\pi_v(\tau_v - g_v)\right]\right)^{-1}$$

Thus, nominal interest rates are zero in periods in which the government runs a primary deficit. In periods in which it runs a surplus, the nominal interest rate is just the share of the current primary surplus out of the present value of all current and future primary surpluses. Thus, nominal interest rates will bounce around quite a bit if $\tau_t - g_t$ does. If aggregate consumption $e_t - g_t$ is stable, then inflation bounces around as well. The dividend neutrality result described in Proposition 2 can be used to smooth nominal interest rates and expected inflation. Note however that all of the period-$t$ surprise in the present value of expected primary surpluses must be reflected in a surprise in $s_t$. The initial supply $K_{t-1}(\sigma_{t-1})$ is given, and there is only one security price that can respond to news.

## 4.2. Occasionally Zero Present Values

The assumption that present values of planned primary surpluses are strictly positive at all times is crucial to make the policy (24)-(25) work. In particular, (25) is not well defined at $V_t = 0$ and $s = 0$. Furthermore, it would have to be the case that $s_t K_{t-1}(\sigma_{t-1}) = 0$ if $V_t = 0$. Since $K_t(\sigma_{t-1}) \geq 1$ it follows that $s_t = 0$ in any equilibrium. But then the government has no chance to raise any revenue by issuing more of this security to finance a deficit $\tau_t - g_t < 0$.

The solution to this problem is simply for the government to issue a new security and stop trading in the old one when $V_t$ hits zero. Note that it follows from Assumption 2 that $V_t = 0$ implies $\tau_t - g_t \leq 0$. Furthermore, $V_{t+1} > 0$ with positive probability if $\tau_t - g_t < 0$. The policy for the new security is:

$$\begin{aligned}
G_t(s, \sigma) &= \tau_t + (g_t - \tau_t) \min\{1, s\} \\
T_t(s, \sigma) &= \tau_t \\
D_t(s, \sigma) &= 0 \\
K_t(s, \sigma) &= (g_t - \tau_t) \min\{1/s, 1\}
\end{aligned} \tag{28}$$

where $s$ is now the price of the new security. The policy (24)-(25) then applies to the new security, as long as the present value of planned primary surpluses is strictly positive. If this present value hits zero again, then another new security is issued, and (28) is used again in that period.

In the special case in which $V_t = 0$ is known in period $t-1$ while $V_{t-1} > 0$, the government could also try to use its primary surplus in period $t-1$ to repurchase all its shares. But if the government does not commit to issue a new security in period $t$ in case it was not able to repurchase all its shares in period $t-1$, then there can be a self-fulfilling equilibrium in which the primary surplus in period $t-1$ is not enough to repurchase all shares, and the government ends up having to reduce spending in period $t$ or at some later date, if only marginally, to make sure that the present value of its outstanding securities is strictly positive as of period $t$.

## 5. Government Shares with Zero Dividends in Equilibrium

We know from Proposition 1 that a policy that never pays any dividends cannot guarantee that the government implements its fiscal mandate $\{g_t, \tau_t\}_{t=0}^{\infty}$. In this section we show that it is possible to implement $\{g_t, \tau_t\}_{t=0}^{\infty}$ using a policy that pays dividends only at non-equilibrium prices. It then follows from Proposition 2 that the government can choose any stochastic process of equilibrium nominal interest rates.

The policy (24)-(25) already specifies no dividends when the government intends to implement a primary deficit. The policy described here is simply (25) in that event. If $V_t$ happens to be zero, and $\tau_t - g_t < 0$, then the government issues a new security. Note that the supply of government securities can only increase in a deficit period.

Consider next a state of the world in which the intended primary surplus $\tau_t - g_t$ is non-negative. We replace (24) by a class of policies parameterized by some $\gamma_t \in (0, 1]$:

$$\begin{aligned} G_t(s,\sigma) &= g_t \\ T_t(s,\sigma) &= \tau_t \\ D_t(s,\sigma) &= s + (\tau_t - g_t - sK_{t-1}(\sigma))/K_t(s,\sigma) \\ K_t(s,\sigma) &= \lim_{x \downarrow s} \max\left\{K_{t-1}(\sigma) + \tfrac{1}{x}(g_t - \tau_t), \gamma_t K_{t-1}(\sigma)\right\} \end{aligned} \tag{29}$$

for any $(s, \sigma) \in \Sigma_t$. Observe that this policy is a continuous function of $s$, provided $K_{t-1}(\sigma) > 0$. Note that $K_t(s, \sigma) \geq \gamma_t K_{t-1}(\sigma)$. Since we take $\gamma_t > 0$, there will always be some shares that are held by consumers after trading. Because of this, the government can commit to pay as a real dividend the part of its primary surplus that was not spent on repurchasing shares. The resulting dividend satisfies:

$$s - D_t(s,\sigma) = \frac{1}{\gamma_t} \min\left\{\gamma_t s, s - \gamma_t \left(\frac{\tau_t - g_t}{K_{t-1}(\sigma)}\right)\right\}$$

This will be non-negative if and only if $sK_{t-1}(\sigma) \geq \gamma_t(\tau_t - g_t)$. By setting $\gamma_t$ small but positive, the government offers an arbitrage opportunity for very low values of $sK_{t-1}(\sigma)$. If $\gamma_t < 1$, then the dividend on government shares is zero whenever $sK_{t-1}(\sigma) \geq (\tau_t - g_t)/(1 - \gamma_t)$. If $\gamma_t$ is small, then the government starts to pay a dividend only if the price of its cum-dividend shares falls to a level that is marginally above its current primary surplus. Since we assume that the present value of primary surpluses is strictly positive at all times, it is possible to set $\gamma_t$ small enough so that no dividends are paid in equilibrium.

In states of the world in which $\tau_t - g_t < 0$, government policy is exactly as described before, in (25). As a result, government policy again satisfies (27), and this leads to the following proposition.

**Proposition 4:** *If Assumptions 1 and 2 hold, and $V_t > 0$ at all times, then the policy defined by (25) and (29) implements $\{g_t, \tau_t\}_{t=0}^{\infty}$.*

If we choose the stochastic process $\{\gamma_t\}_{t=0}^{\infty}$ so that:

$$\mathrm{E}_t\left[\sum_{v=t+1}^{\infty}\pi_v(\tau_v - g_v)\right] \geq \left(\frac{\gamma_t}{1-\gamma_t}\right)\pi_t(\tau_t - g_t) \tag{30}$$

then dividends are zero along the equilibrium path. This is also true at high off-equilibrium prices. At positive but low enough off-equilibrium prices, the government starts paying a dividend if it has a primary surplus. Instead of using the surplus to repurchase shares, it uses the surplus to pay a dividend.

The nominal dividend neutrality result of Proposition 2 can be used to modify (25) and (29) to obtain any arbitrary sequence of positive nominal interest rates along the equilibrium path. Suppose there is some desired exogenous process $\{\delta_t\}_{t=0}^{\infty}$ for nominal interest rates. Combining (25) with (29) and (19)-(21) we obtain the following policy:

$$\begin{aligned}
G_t(s,\sigma) &= \min\left\{\tau_t + (g_t - \tau_t)\min\left\{1, sK_{t-1}(\sigma)/\alpha_t V_t\right\}, g_t\right\} \\
T_t(s,\sigma) &= \tau_t \\
D_t(s,\sigma) &= s + \left[T_t(s,\sigma) - G_t(s,\sigma) - sK_{t-1}(\sigma)\right]/K_t(s,\sigma) \\
K_t(s,\sigma) &= (1+\delta_t)\lim_{x\downarrow s}\max\left\{K_{t-1}(\sigma) + \tfrac{1}{x}\left[G_t(s,\sigma) - T_t(s,\sigma)\right], \gamma_t K_{t-1}(\sigma)\right\}
\end{aligned} \tag{31}$$

for all $(s,\sigma) \in \Sigma_t$. The resulting nominal interest rates will indeed be given by $\{\delta_t\}_{t=0}^{\infty}$ if $\{\gamma_t\}_{t=0}^{\infty}$ satisfies (30). If aggregate consumption is smooth, then a stable positive nominal interest rate process will generate stable expected inflation. Off the equilibrium path, (31) specifies a cut in spending if the price of government shares is too low and the government intends to run a deficit. In case of a surplus, there is no cut in spending when the price of government shares is too low. Instead, (31) prescribes an increase in the nominal interest rate above its planned level. This increase in the nominal interest rate is the same as an increase in the dividend paid on shares, and this allows to government to maintain the planned surplus without buying back all of its shares. Raising nominal interest rates is the classic recipe for fighting inflation or fending off an attack on a currency. The policy described here provides an interpretation, but only for off-equilibrium prices.

By choosing $\delta_t$ large enough so that:

$$\frac{\delta_t}{1+\delta_t} \geq \frac{\tau_t - g_t}{V_t}$$

at all times, the government can make sure that the supply of government shares never declines along the equilibrium path. The price of government shares will be positive and uniquely determined, even though the government does not repurchase its shares along the equilibrium path. Government policy at off-equilibrium prices ensures that government shares still trade at a positive price.

### 5.1. The Price of Fiat Money is Zero

To emphasize the importance of government policy at off-equilibrium prices, suppose now that there is a second government security with an initial supply $k_{2,-1} > 0$. Let security 1 be the one described above, and suppose that the government follows the exact same policy for this security as before. For security 2, the policy is as in (17)-(18), at all times. That is, the government pays a nominal dividend on security 2, and it does so by simply selling more of security 2. Because of this, the government's policy for security 2 has no impact on the feasibility of its policy for taxes, government purchases and security 1. The policy (17)-(18) can be used irrespective of prices.

It is not difficult to check that the only possible price for security 2 is $s_{2,t} = 0$ at all times. To be specific, suppose $\{\delta_{2,t}\}_{t=0}^{\infty}$ is some stochastic process, and let $D_{2,t}(\sigma_t) = s_{2,t}\delta_{2,t}/(1+\delta_{2,t})$ and $k_{2,-1} > 0$. Then (12) implies $\pi_t s_{2,t} = (1+\delta_{2,t})\mathrm{E}_t[\pi_{t+1}s_{2,t+1}]$ and (17)-(18) implies $K_{2,t}(\sigma_t) = (1+\delta_{2,t})K_{2,t-1}(\sigma_{t-1})$. This gives:

$$\mathrm{E}_t\left[\pi_{t+1}s'_{t+1}K_t(\sigma_t)\right] \geq \mathrm{E}_t\left[\pi_{t+1}s'_{2,t+1}K_{2,t}(\sigma_t)\right] = \pi_0 s_{2,0}k_{2,-1}$$

If $s_{2,0}$ were positive, then $\mathrm{E}_t\left[\pi_{t+1}s'_{t+1}K_t(\sigma_t)\right]$ would not converge to zero, and this is not consistent with equilibrium.

It might seem that the crucial difference is that security 1 is repurchased, and that security 2 is not. This is not the case. We could use Proposition 2 to construct a unique equilibrium for the one-security economy in which the government is never —in equilibrium— a net purchaser of any of the supply of its one outstanding security. We simply have to choose a dividend yield that is large enough. In the associated two-securities economy with a second security that satisfies (17)-(18), the unique equilibrium will again be the one in which $s_{2,t} = 0$. Neither security is ever repurchased in equilibrium, but the two securities trade at different prices because of the policy to pay a real dividend on security 1, and not on security 2, even if the price of the security drops down to zero.

## 6. Long-Term Nominal Bonds

We assume that the present value $V_t$ of planned primary surpluses is strictly positive at all times.

Suppose there are several government securities that are traded. As before, take $T_t(\sigma) = \tau_t$ for every $\sigma \in \Sigma_t$, and suppose government purchases are determined by:

$$G_t(s,\sigma) = \min\left\{\tau_t + (g_t - \tau_t)\min\left\{1, s'K_{t-1}(\sigma)/\alpha_t V_t\right\}, g_t\right\}$$

for every $(s,\sigma) \in \Sigma_t$. As before, $\alpha_t \in (0,1]$. This policy cuts spending to some level below $g_t$ only if the government intends to run a deficit, and if the market value of all its outstanding securities falls below some fraction $\alpha_t$ of the intended present value

$V_t$. The policy again implies that (27) holds, and therefore that the present value of primary surpluses will be $V_t$ in any equilibrium.

In the following, we examine government policies for trading in long-term nominal bonds. At the outset, we make two assumptions about these trading policies that simplify the presentation. First, government policy is such that the government never repurchases nominal bonds above par. Second, government policy is such that there are always government shares outstanding. If necessary, the government can always pay a high dividend on its shares. Together these two assumptions imply that long-term bonds can never trade above par.

To see this formally, suppose $\{\pi_t, s_t\}_{t=0}^{\infty}$ is an equilibrium price process, and let $\{\sigma_t\}_{t=0}^{\infty}$ represent the associated histories. Suppose the first security is the government share, and the $n^{\text{th}}$ security is a long-term bond that matures in period $t_n$. If consumers hold on to this long-term bond beyond $t_n$, then the long-term bond becomes fiat money, and we know this must have zero value. Therefore $s_{n,t_n} \leq s_{1,t_n}$. The government may repurchase all of the supply of the $n^{\text{th}}$ security before $t_n$. Let $t_n^* \leq t_n$ be the random time at which this happens. The fact that the government never repurchases its bonds above par implies that $s_{n,t_n^*} \leq s_{1,t_n^*}$. Since government shares are always in positive supply, it must be that:

$$\begin{aligned} \pi_t s_{1,t} &= \pi_t D_{1,t}(\sigma_t) + \mathrm{E}_t[\pi_{t+1} s_{1,t+1}] \\ \pi_t s_{n,t} &= \mathrm{E}_t[\pi_{t+1} s_{n,t+1}] \end{aligned}$$

for all $t < t_n^*$. Together, these conditions imply that:

$$s_{n,t} = \frac{1}{\pi_t}\mathrm{E}_t[\pi_{t_n^*} s_{1,t_n^*}] = s_{1,t} - \frac{1}{\pi_t}\mathrm{E}_t\left[\sum_{v=t}^{t_n^*-1} \pi_v D_{1,v}(\sigma_v)\right] \leq s_{1,t} \tag{32}$$

for all $t < t_n^*$, and thus $s_{n,t} \leq s_{1,t}$ for all $t \leq t_n^* \leq t_n$. Given this conclusion, we can restrict attention to long-term bond prices that are not above par. Note that one implication of this restriction on prices is that $s_{1,t} = 0$ implies $s_t = 0$.

The equilibrium conditions $s_0' k_{-1} = V_0$ and (32) imply that:

$$s_{1,0}\left[\sum_{n=1}^{n_{-1}} k_{n,-1}\right] = \frac{1}{\pi_0}\mathrm{E}_0\left[\sum_{n=2}^{n_{-1}} \sum_{t=0}^{t_n^*-1} \pi_t D_{1,t}(\sigma_t)\right] + V_0$$

in any equilibrium. In general, the random times $t_n^*$ and the dividends $D_{1,t}(\sigma_t)$ depend, via the specification of government policy, on the price $s_{1,0}$. How the government finances deficits, and how it uses surpluses to repurchase securities, will affect whether or not long-term bonds of a particular maturity are held by the public. The policy to keep a certain quantity of government shares outstanding can also force the government to raise nominal interest rates, and the quantity outstanding after period 0 is affected by $s_{1,0}$.

## 7. Some Policies That Lead to Indeterminacy

We have already seen that multiple equilibria are possible if the present value of primary surpluses can reach zero and the government cannot commit to issue a new type of security. The equilibria differ in terms of the real allocations implied. Here we give two more examples of policies that lead to indeterminacy, even though the present value of planned primary surpluses is always strictly positive.

### 7.1. Nominal Indeterminacy

Suppose $k_{-1} = 1$, $\tau_t > g_t$, and consider the following policy:

$$\begin{aligned} G_t(s,\sigma) &= g_t \\ T_t(s,\sigma) &= \min\{\tau_t, g_t + \alpha s K_{t-1}(\sigma)\} \\ D_t(s,\sigma) &= \left[\gamma s K_{t-1}(\sigma) + \min\{\tau_t - g_t, \alpha s K_{t-1}(\sigma)\}\right] / K_t(s,\sigma) \\ K_t(s,\sigma) &= (1+\gamma) K_{t-1}(\sigma) \end{aligned} \tag{33}$$

all $(s,\sigma) \in \Sigma_t$, and for some $\alpha \in [0,1]$ and $\gamma \geq 0$. This policy creates a steady growth rate of the supply of government shares. It also incorporates a commitment to use taxes to repurchase a fraction $\alpha$ of the supply of shares, to the extent possible given the tax ceiling $\tau_t$ and the mandated purchases $g_t$. In contrast to all of our earlier examples, taxes are only set at their ceiling $\tau_t$ if this is necessary to implement the intended repurchase of shares.

By construction, state prices must be given by $\pi_t = \beta^t \mathrm{D}u(e_t - g_t)$ in any equilibrium. The remaining equilibrium conditions are then:

$$\begin{aligned} \pi_t s_t &= \pi_t D_t(s_t, \sigma_{t-1}) + \mathrm{E}_t[\pi_{t+1} s_{t+1}] \\ \pi_t s_t K_{t-1}(\sigma_{t-1}) &= \mathrm{E}_t\left[\sum_{v=t}^{\infty} \pi_v \min\{\tau_v - g_v, \alpha s_v K_{v-1}(\sigma_{v-1})\}\right] \end{aligned}$$

for all $t$. Clearly, $s_t = 0$ is one equilibrium.

More generally, suppose we have an equilibrium $\{\pi_t, s_t\}_{t=0}^{\infty}$ with associated histories $\{\sigma_t\}_{t=0}^{\infty}$ so that $\alpha s_t K_{t-1}(\sigma_{t-1}) < \tau_t - g_t$ at all times. Then $\{\pi_t, \phi s_t\}_{t=0}^{\infty}$ will also an equilibrium for any $\phi \in [0,1]$. It remains to verify that there are other equilibria besides the one in which $s_t = 0$ at all times. Note first that $\alpha s_t K_{t-1}(\sigma_{t-1}) < \tau_t - g_t$ implies:

$$s_t - D_t(s_t, \sigma_{t-1}) = \left(\frac{1-\alpha}{1+\gamma}\right) s_t$$

and therefore:

$$\left(\frac{1-\alpha}{1+\gamma}\right) \pi_t s_t = \mathrm{E}_t[\pi_{t+1} s_{t+1}] \tag{34}$$

On the other hand, the present-value condition becomes:

$$\pi_t s_t K_{t-1}(\sigma_{t-1}) = \alpha \mathrm{E}_t \left[ \sum_{v=t}^{\infty} \pi_v s_v K_{v-1}(\sigma_{v-1}) \right] \tag{35}$$

if $\alpha s_t K_{t-1}(\sigma_{t-1}) < \tau_t - g_t$ at all times. Since $K_t(\sigma_t) = (1+\gamma)^{t+1} k_{-1}$, any process $\{\pi_t s_t\}_{t=0}^{\infty}$ that satisfies (34) satisfies (35). Thus we only need to make sure that the price process for government shares satisfies (34) and $\alpha s_t K_{t-1}(\sigma_{t-1}) < \tau_t - g_t$ at all times.

As an example, suppose now that $\mathrm{E}_t[\pi_{t+1}]/\pi_t = \mu$, for some constant $\mu$. Real interest rates are constant. Define:

$$s_t = s_0 \left( \frac{1-\alpha}{\mu(1+\gamma)} \right)^t$$

Then $\{s_t\}_{t=0}^{\infty}$ satisfies (34)-(35). All we need then is that:

$$\tau_t - g_t > \left[ \frac{1-\alpha}{\mu} \right]^t s_0 k_{-1}$$

This will be satisfied for a range of initial share prices $s_0$ if $\tau_t - g_t$ is bounded away from zero and $1 < \alpha + \mu$. A weaker parameter restriction obtains if $\tau_t - g_t$ grows geometrically as well.

Clearly, the indeterminacy in this example arises from the fact that the government's policy rule makes the level of taxes depend on the market value of its outstanding shares. In particular, if the market puts a low value on these shares, then repurchases are cheap, and taxes will be low. The result is a primary surplus that is also low, and this then justifies the low market value of the shares.

Note that the nominal interest rate is given by $(\gamma+\alpha)/(1-\alpha)$ in all these equilibria. For every particular equilibrium obtained here, a policy that commits to the associated taxes, and that prescribes a dividend yield $(\gamma+\alpha)/(1-\alpha)$ together with off-equilibrium path policies as described in Section 5 would have led to a unique equilibrium.

### 7.2. Real Indeterminacy

The following is a variation on the example of nominal indeterminacy given above. Assume again that $g_t < \tau_t$ and suppose $g_t^* \in [0, g_t)$. We modify (33) as follows:

$$\begin{aligned}
G_t(s,\sigma) &= \begin{cases} g_t & \tau_t - g_t > \alpha s K_{t-1}(\sigma) \\ g_t^* & \text{otherwise} \end{cases} \\
T_t(s,\sigma) &= \tau_t \\
D_t(s,\sigma) &= \left[ s\gamma K_{t-1}(\sigma) + \tau_t - G_t(s,\sigma) \right] / (1+\gamma) K_{t-1}(\sigma) \\
K_t(s,\sigma) &= (1+\gamma) K_{t-1}(\sigma)
\end{aligned} \tag{36}$$

for all $(s,\sigma) \in \Sigma_t$, and some $\alpha \in [0,1]$ and $\gamma \geq 0$. Now taxes are set at their ceiling, but the goal of spending tax revenues to repurchase $\alpha s K_{t-1}(\sigma)$ takes priority over planned government purchases. Again, the primary surplus depends on $(s,\sigma) \in \Sigma_t$.

Consider a candidate equilibrium $\{\pi_t^*, s_t^*\}_{t=0}^{\infty}$ in which $G_t(\sigma_t^*) = g_t^*$ and thus $\pi_t^* = \beta^t \mathrm{D}u(e_t - g_t^*)$. This yields:

$$s_t^* K_{t-1}(\sigma_{t-1}^*) = \frac{1}{\pi_t^*} \mathrm{E}_t \left[ \sum_{v=t}^{\infty} \pi_v^* (\tau_v - g_v^*) \right]$$

For this to be an equilibrium, all we need to make sure is that:

$$\pi_t^* (\tau_t - g_t) < \alpha \mathrm{E}_t \left[ \sum_{v=t}^{\infty} \pi_v^* (\tau_v - g_v^*) \right] \tag{37}$$

at all times. Conversely, for a candidate equilibrium $\{\pi_t, s_t\}_{t=0}^{\infty}$ in which $G_t(\sigma_t) = g_t$ and thus $\pi_t = \beta^t \mathrm{D}u(e_t - g_t)$, we must have:

$$\pi_t (\tau_t - g_t) > \alpha \mathrm{E}_t \left[ \sum_{v=t}^{\infty} \pi_v (\tau_v - g_v) \right] \tag{38}$$

Suppose $c\mathrm{D}u(c)$ is increasing. Since $\tau_t - g_t < \tau_t - g_t^*$ at all times, it is easy to construct examples for which both (37) and (38) are satisfied.

A before, the policies discussed in Sections 4 and 5 can be used to implement $\{g_t, \tau_t\}_{t=0}^{\infty}$, and the nominal dividend neutrality result can be used to obtain the desired inflation or nominal interest rates.

## 8. Concluding Remarks

The model presented here backs up the determinacy propositions made in the literature on the fiscal theory of the price level with a complete description of government policy for a government that operates in competitive markets and cannot set prices directly. The government policies obtained for off-equilibrium prices are potentially interesting. In surplus periods, the government should raise nominal interest rates in reaction to inflation that is not consistent with planned taxes and government purchases.

Of course, it is puzzling from the perspective of the model presented here why anyone would hold Federal Reserve Notes instead of T-bills, given the higher return on T-bills. But the supply of Federal Reserve Notes is less than one seventh of the supply of other government securities, and for some purposes it may be convenient to ignore Federal Reserve Notes. By using maturing T-bills as the numeraire, one obtains a potentially useful frictionless benchmark for understanding inflation.

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