Convergence in a Stochastic Dynamic Heckscher-Ohlin Model

Partha Chatterjee
University of Minnesota
partha@econ.umn.edu

Malik Shukayev
University of Minnesota
maliksh@econ.umn.edu

Abstract

We characterize the equilibrium for a small economy in a dynamic Heckscher-Ohlin model with uncertainty. We show that when trade is balanced period-by-period, the per capita output and consumption of a small open economy converge to an invariant distribution that is independent of the initial wealth. Further, at the invariant distribution, with probability one there are some periods in which the small economy diversifies. These results are in sharp contrast with those of deterministic dynamic Heckscher-Ohlin models, in which permanent specialization and non-convergence occur. One key feature of our model is the presence of market incompleteness as a result of the period-by-period trade balance. The importance of market incompleteness, and not just uncertainty, in achieving our results is illustrated through an analytical example. Further, numerical simulations show that the speed of convergence is increasing in the size of the shocks. Thus, our results extend the predictions of income convergence, standard in one sector neoclassical growth models, to the dynamic multi-country Heckscher-Ohlin environment.

[*] We are extremely grateful to Michele Boldrin for his guidance and constant support. We thank V.V.Chari, Larry Jones and Tim Kehoe for valuable suggestions. We also thank Soma Dey, Urvi Neelakantan, Han Ozsoylev and Shino Takayama for useful discussions.

[†] Contact Address: 1035 Heller Hall, 271 19th Avenue S, Minneapolis, MN 55455.

1 Introduction

Will income levels in two countries, which start from different conditions, converge? Traditionally, deterministic closed economy neoclassical growth models were used to answer this question. These models predict that, as long as countries have same preferences, same technologies and same population dynamics, they will converge to the same level of per-capita income from any positive initial wealth. So, initial conditions do not matter for the long-run income levels. Brock and Mirman (1972) extended this result to stochastic environment by showing that countries will converge to the same invariant distribution of income irrespective of their positive initial wealth. More recently Chen (1992), Ventura (1997), Atkeson and Kehoe (2000) have shown that in deterministic dynamic Heckscher-Ohlin models - that is models with two or more tradeable commodities which are produced using neoclassical production functions differing in capital intensities - convergence may not occur. This is despite all countries being identical up to their initial conditions, and all the production functions being strictly concave. Although the models vary in details, all of them rely on trade-induced factor-price-equalization which leads to existence of multiple steady states. Initial conditions determine to which steady state a particular economy will converge. This result has led to a surge of interest in dynamic Heckscher-Ohlin models, with the view that such models can potentially account for the observed income differences across countries without resorting to non-convexities or structural differences between countries.

It is natural to wonder if the results from a deterministic model will carry over to an uncertain world. We introduce technological uncertainty in a dynamic Heckscher-Ohlin model and find that, just as in the one sector neoclassical growth model, we obtain income convergence across countries. We show that when trade is balanced period-by-period, a standard assumption in deterministic Heckscher-Ohlin models, the per capita output and consumption of a small trading economy converge to an invariant distribution that is independent of the (positive) initial wealth. Furthermore, introduction of uncertainty overturns another prediction of the deterministic model: that countries may permanently specialize, that is, may never produce all tradeable commodities. We find that in an uncertain environment, when income of an economy is within the invariant distribution, there will surely be some periods in which the small economy diversifies. It is important to state that in our modelling strategy we are following Atkeson and Kehoe (2000) in concentrating on the dynamics of a small trading economy that has no effect on world prices of tradable goods.

Why are the results so different in the stochastic version of the Heckscher-Ohlin model from those in the deterministic version? There are two ingredients in our model which are crucial for our results: uncertainty and market incompleteness arising from period-by-period trade balance constraint. To understand the role played by these two factors, first consider the deterministic version. In such models when countries diversify (i.e., when their aggregate capital-labor ratios are within the diversification cone), factor price equalization means that the countries face same rates of return to capital. Thus, when preferences are identical across countries, there is no incentive for agents in one country to accumulate capital if there is no incentive for agents in the other country to do so. This implies that if one country is in a steady state, so is the other. In particular, if the world economy is in steady state, all capital-labor ratios within the diversification cone can be sustained as steady states. Consequently, any country that starts with a capital-labor ratio within the diversification cone will remain at that level of capital forever. Countries that start outside the diversification cone, with a low capital-labor ratio, will grow till they reach the lower boundary of the diversification cone. Such countries will never enter the diversification cone and will permanently specialize in production of less-capital-intensive tradable goods. Therefore, in this case, the initial conditions determine the fate of the country in the long run.

In the presence of uncertainty and market incompleteness, however, the initial conditions are eventually irrelevant. In an uncertain world, two small economies starting from different initial conditions will find themselves in a similar situation in the future, in which they will surely diversify their production - produce all tradable goods - in at least some periods. Uncertainty by itself is, however, not enough for convergence. The period-by-period trade balance creates market incompleteness, and that is crucial for the convergence result. In the deterministic model, when all countries have capital-labor ratios within the diversification cone, the rental rates are equated across them and so, there is no incentive to borrow and lend internationally. Thus, the absence of borrowing and lending as a result of the balanced trade in each period is irrelevant. If, however, different countries face different shocks, then rate of return to capital is not the same across countries in each period. Then there can be mutual gains through risk sharing if countries could borrow and lend from each other. Period-by-period trade balance prevents that, forcing countries to self-insure by accumulating more capital when income is higher than expected, and de-accumulating capital when income is lower than expected. This pattern of capital accumulation in economies with uncertainties shifts the policy functions relative to those in economies without uncertainty. This implies that the capital-accumulation policy function in an economy with uncertainty no longer coincides with the 45 degree line, as it does in deterministic Heckscher-Ohlin models, and we no longer get multiplicity of steady states.

The importance of the restriction on risk sharing opportunities imposed by the balanced trade condition is illustrated through an analytical example. In this example, we retain the period-by-period balanced trade constraint but assume away any risk sharing opportunities. We assume that a small economy and the rest of the world economy both face the same realization of shocks each period. In this case, when there are no gains to be made from borrowing and lending from each other, there is no income convergence and the small economy permanently specializes if it starts from outside the diversification cone. Thus, in this example, despite uncertainty, results are very similar to what we observe in deterministic models.

One point to note here is that if, instead of setting borrowing and lending to zero, we allowed for limited borrowing our results would remain unchanged as long as the limit on borrowing was sufficiently low.

We also simulate our model to understand how the speed of convergence depends on the size of the shocks that the economy faces. We find that the bigger the shocks are, the faster is the convergence. In the limit, when uncertainty vanishes the convergence disappears. This suggests that, if uncertainty is small, initial conditions play an important role in the development of a country: it takes a long time for initially capital poor countries to catch up with richer countries. For higher levels of uncertainty, however, initial conditions will quickly cease to have an effect on per-capita income levels across countries. The simulation also lets us see the actual shape of investment policy functions and the location of the support of invariant distribution of capital vis-a-vis the diversification cone.

Our paper encompasses various strands of the literature. First, it generalizes the dynamic Heckscher-Ohlin model in an important way by introducing uncertainty. It shows that deterministic dynamic Heckscher-Ohlin models studied by Chen (1992), Ventura (1997), Atkeson and Kehoe (2000) are very special limiting cases of the stochastic environment. On the other hand this paper also extends to the open economy setting the convergence results of a closed economy stochastic growth model studied by many researchers starting from Brock and Mirman (1972). Our paper also relates to the long literature on income fluctuations problem, which studies savings decisions under market incompleteness in environments with many agents facing idiosyncratic shocks. In particular Clarida (1987), Chamberlain and Wilson (2000) and Aiyagari (1994) are just a few examples that fall into this category.

The paper is organized as follows. In the next section we describe our model's environment. In section 3, we give the equilibrium results for this model, including those on convergence and diversification. In section 4 we simulate this model and discuss the speed of convergence. In section 5 we consider another version of the model with productivity shocks that are economy-wide rather than sector-specific. We show that the convergence and diversification results still hold, but the range of capital-labor ratios observed in the invariant distribution is much bigger. In section 6 we provide an analytical example which shows that the constraint on risk sharing opportunities are important in getting our results. Finally we conclude in section 7. All the proofs are collected in the appendices.

2 The Environment

The economic environment consists of two economies, a *small economy* and the *rest of the world economy*. Population is fixed in both the countries. We assume that the population size in the small country is of measure zero relative to the rest of the world. Motivated by this assumption, and for brevity, we refer to the rest of the world economy as simply the *world economy*.

The two economies are assumed to have identical preferences and technologies (up to the stochastic productivity factors), though the nature of uncertainty faced by the economies might be different. In each economy there are two intermediate goods, a and m, and one final good, Y. The intermediate goods are produced using capital and labor in each intermediate good sector. Technology for producing good a is less capital intensive than the technology producing m. The intermediate goods are traded between the economies. The final good is produced by combining the two intermediate goods. It can be either invested or consumed domestically but cannot be traded across economies. Capital and labor are also immobile across borders.

2.1 Preferences

The agents in both the economies are assumed to have identical preferences. Representative agents in each economy supply labor inelastically and derive utility from consumption.

Assumption 1

The utility function, $u : \mathcal{R}_+ \rightarrow \mathcal{R}_+$ *has the following properties:*

1. u *is continuous on* $\mathcal{R}_+$*, bounded below, and (without loss of generality)*

$u(0) = 0$.

2. *u is twice continuously differentiable and strictly concave, i.e.,* $u'(c) > 0$, $u''(c) < 0 \;\; \forall \;\; c \in \mathcal{R}_{++}$

3. $\lim_{c \to 0} u'(c) = \infty$.

2.2 Production

Each economy has access to three technologies: two intermediate good technologies a and m, and one final good technology Y. All the production function are assumed to be standard neoclassical production functions: homogeneous of degree one in all inputs, twice continuously differentiable with positive and diminishing marginal products of each input.

Final good is produced by combining intermediate goods a and m:

$$Y = H(a, m), \tag{2.1}$$

$H(a, m)$ satisfies the following assumptions [1]

Assumption 2

$H(a, m)$ *exhibits constant returns to scale, and for all* $a \geq 0$ *and* $m \geq 0$*,*

1. $H(0, m) = H(a, 0) = 0$

2. $H_1(a, m) > 0$*,* $H_2(a, m) > 0$*,* $H_{11}(a, m) < 0$ *and* $H_{22}(a, m) < 0$

[1] Here we use H_1 to represent the partial derivative of H with respect to its first argument. We do so for all other first and second derivatives.

There are two distinct production functions, which combine capital and labor to produce intermediate goods. The technology for producing intermediate good a is given by,

$$a = \lambda F(K_a, L_a) \tag{2.2}$$

where λ is the productivity factor and is potentially stochastic. K_a and L_a are capital and labor employed in sector a.

The technology for producing intermediate good m is given by,

$$m = \theta G(K_m, L_m) \tag{2.3}$$

where θ is the productivity factor and is also potentially stochastic. Similarly, K_m and L_m are capital and labor employed in sector m.

Assumptions on both production functions F and G are similar to that on H – they are constant returns to scale, and their marginal products of capital and labor are positive and strictly diminishing. In addition the intermediate technologies satisfy the following boundary conditions.

Assumption 3

Inada conditions for intermediate technologies:

1. *For all* $L > 0$*,* $\lim_{K\to 0} F_1(K, L) = \lim_{K\to 0} G_1(K, L) = \infty$
2. *For all* $L > 0$*,* $\lim_{K\to\infty} F_1(K, L) = \lim_{K\to\infty} G_1(K, L) = 0$*.*

We also assume, as standard in Heckscher-Ohlin models, that the good m technology is more capital intensive than the good a technology for all relevant factor price ratios (i.e., there are no factor intensity reversals). More formally, we have the following assumption,

Assumption 4

For all $K > 0$ *and* $L > 0$ $\quad \frac{F_2(K,L)}{F_1(K,L)} > \frac{G_2(K,L)}{G_1(K,L)}$.

2.3 International Trade

As we said above final goods, capital and labor are not tradable across the countries. The only commodities that can be traded between the economies are the two intermediate goods. Thus, the quantities of intermediate goods utilized in a small economy for the production of final goods can be different from the quantities produced in the small economy. We assume, as is standard in deterministic dynamic Heckscher-Ohlin models, that trade is balanced in each period for each economy.

Assumption 5

In all periods t*, and for both countries* $(i = s, w)$

$$p_{at}(a_t^{i,d} - a_t^i) + p_{mt}(m_t^{i,d} - m_t^i) = 0. \tag{2.4}$$

Where variables with superscript d are quantities demanded in the country i, variables without superscript d are quantities produced in the country i, and p_{at}, p_{mt} are the world prices of intermediate goods. This assumption has no implication on the equilibrium outcomes in deterministic Heckscher-Ohlin models when both the economies produce both intermediate goods. In that case balanced trade is an equilibrium outcome. With country specific productivity shocks, however, the period-by-period balanced trade constraint is binding and precludes risk sharing opportunities through borrowing and lending. As we will see later, this constraint plays an important role in determining the equilibrium outcomes. Balanced trade implies that countries cannot borrow or lend. Thus the balanced trade constraint will also be reflected in the budget constraint of a representative household.

2.4 Uncertainty

In this paper, except in section 5, we assume that the world economy faces no uncertainty. Further, productivity factors in the intermediate technologies used in the world economy λ_t^w and θ_t^w are both normalized to be equal to one for all t. The small economy, however, faces uncertainty - λ_t^s and θ_t^s are stochastic. We assume the following distributions for these productivity terms:

Assumption 6

1. *Both λ_t^s and θ_t^s are i.i.d. random variables drawn from their respective time invariant distributions.*

2. *The support of λ_t^s is $\Lambda = [\underline{\lambda}, \overline{\lambda}]$, where $0 < \underline{\lambda} \leq \overline{\lambda} < \infty$, while the support of θ_t^i is $\Theta = [\underline{\theta}, \overline{\theta}]$, where $0 < \underline{\theta} \leq \overline{\theta} < \infty$.*

3. *$E[\lambda] = 1$ and $E[\theta] = 1$.*

The last part of the assumption above ensures that the expected productivity of each sector in the small economy is equal to productivity of the world economy's sectors.

Let η be probability measure for the joint distribution of $z = (\lambda, \theta)$, defined on the Borel subsets of $Z = \Lambda \times \Theta$. The assumption that Λ and Θ are full supports imply that $\eta(A) > 0$ for any non-degenerate rectangle in $Z = \Lambda \times \Theta$ space.

At this point it is useful to state the timing of various events and decision processes in the economy. At the beginning of every period the uncertainty about current productivity levels is resolved. The consumers, final good producers, intermediate goods producers all take their decisions after that. The consumers choose how much to consume and save. The savings decisions determine the next period's capital. The intermediate goods producers decides how to allocate the capital and labor available in the economy between the two sectors. Notice here that the aggregate capital in the economy is decided before the uncertainty for the period is resolved (it is decided a period earlier), but the allocation of capital and labor across sectors takes place after the uncertainty is resolved. Also, the final good producers decide the amount of each intermediate goods to demand, which in turn determines the quantity of exports and imports of each intermediate good. With this timing, the subscript t on a variable signifies that the variable is measurable with respect to the information available up to period t, including period t productivity terms in both sectors.

3 Equilibrium in the World and the Small Economy

In this section we characterize the equilibrium of the world and the small economy. We begin with the world economy.

3.1 Equilibrium in the world economy

Our assumption that the small economy's population is of zero measure compared to the population of the world economy implies that the world economy behaves as a closed economy and the prices of the intermediate goods are determined by the world economy's equilibrium alone.

In the absence of uncertainty the world economy will converge to a unique steady state. In the steady state the prices of the intermediate goods p_a and p_m and the interest rate in the world economy will be constant across time.

In our analysis of the equilibrium for the small economy we will assume that the world is in the steady state. The world economy's equilibrium determines the intermediate good prices, p_a and p_m, prevailing universally in both the world and the small economy. So, in our analysis the prices of intermediate goods are given and constant across time. Also, since we are concentrating on the equilibrium of the small economy only, we will drop the superscript i from all variables. We will distinguish world variables with superscript w whenever necessary.

3.2 Decision Problems in the Small Economy

In the small economy the representative household maximizes her lifetime expected utility subject to the period budget constraint and taking prices of labor, $w_t,$ and capital, r_t, as given. Thus the representative household's decision problem is to choose the consumption c_t, investment x_t and capital k_t to solve:

$$\max_{\{c_t,x_t,k_t\}_{t=1}^{\infty}} \quad E_1[\sum_{t=1}^{\infty}\beta^{t-1}u(c_t)]$$
$$s.t.$$
$$c_t + x_t \leq w_t + r_t k_{t-1} \tag{3.1}$$
$$k_t = (1-\delta)k_{t-1} + x_t \tag{3.2}$$

given initial level of per-capita capital k_0.

Notice that markets are incomplete, there are no contingent assets available to the households to insure themselves against risk. Moreover, the budget constraints above do not allow for borrowing or lending. Capital accumulation is the only available instrument to transfer resources across periods and states of nature. The lack of borrowing or lending is a reflection of the period-by-period balanced trade constraint described earlier.

The above maximization problem results in the following dynamic optimality conditions:

$$u'(c_t) = \beta E_t\left[u'(c_{t+1})(1-\delta+r_{t+1})\right] \tag{3.3}$$
$$c_t + k_t = w_t + r_t k_{t-1} + (1-\delta)k_{t-1} \tag{3.4}$$

These are the equations which determine the dynamics of per-capita capital and per-capita wealth in this model.

On the production side, there are two kinds of firms in the economy, final good firms and intermediate goods firms. We assume each firm operates in a perfectly competitive environment. The representative final good firm takes the prices of the intermediate goods as given and solves the following problem:

$$\begin{aligned} \min_{\{a_t^d, m_t^d\}} & \quad p_a a_t^d + p_m m_t^d \\ s.t. & \\ Y_t & \leq H(a_t^d, m_t^d) \end{aligned} \tag{3.5}$$

As mentioned earlier, variables with superscript d are the quantities demanded in the economy, while variables without the superscripts are the quantities produced in the economy. The first order conditions for the final good firm are,

$$p_a = H_1(a_t^d, m_t^d) \tag{3.6}$$

$$p_m = H_2(a_t^d, m_t^d) \tag{3.7}$$

Given world prices of intermediate goods, these equations determine the relative quantities of intermediate goods demanded in the small economy.

The representative intermediate goods firm in each economy chooses how to allocate the total capital and labor available in that economy across the two sectors. It takes world prices of intermediate goods and domestic factor prices as given and solves,

$$\begin{aligned} \min_{\{K_{at}, L_{at}, K_{mt}, L_{mt}\}} & \quad r_t(K_{at} + K_{mt}) + w_t(L_{at} + L_{mt}) \\ s.t. & \\ p_a a_t + p_m m_t & \leq p_a \lambda_t F(K_{at}, L_{at}) + p_m \theta_t G(K_{mt}, L_{mt}). \end{aligned} \tag{3.8}$$

Let us define the intensive form of the intermediate production functions as :

$$
\begin{aligned}
f(k) &= F(\frac{K}{L}, 1) & (3.9)\\
g(k) &= G(\frac{K}{L}, 1). & (3.10)
\end{aligned}
$$

The following equations give the first order conditions in terms of the intensive production functions,

$$
\begin{aligned}
p_a \lambda_t f'(k_{at}) &\leq r_t & (3.11)\\
p_a \lambda_t \left[f(k_{at}) - f'(k_{at}) k_{at} \right] &\leq w_t & (3.12)\\
p_m \theta_t g'(k^s_{mt}) &\leq r_t & (3.13)\\
p_m \theta_t \left[g(k_{mt}) - g'(k_{mt}) k^i_{mt} \right] &\leq w_t. & (3.14)
\end{aligned}
$$

Inequalities (3.11) and (3.12) hold with equality whenever sector a is operated with positive inputs, while inequalities (3.13) and (3.14) hold with equality whenever sector m is operated with positive inputs.

Thus it is the intermediate firms which decide whether or not to produce both intermediate goods in positive quantities, or in other words, whether or not the country will diversify. Their first order conditions can be used to define the boundaries of the "cone of diversification" k^b_{at} and k^b_{mt}. Whenever, the aggregate capital-labor ratio of a small economy belongs to the interior of this cone, $k_t \in (k^b_{at}, k^b_{mt})$, it is profitable to produce both the intermediate goods in the small economy. The boundaries k^b_{at} and k^b_{mt} are defined as a solution to the following equations

$$
\begin{aligned}
p_a \lambda_t f'(k^b_{at}) &= p_m \theta_t g'(k^b_{mt}) & (3.15)\\
p_a \lambda_t \left[f(k^b_{at}) - f'(k^b_{at}) k^b_{at} \right] &= p_m \theta_t \left[g(k^b_{mt}) - g'(k^b_{mt}) k^b_{mt} \right] & (3.16)
\end{aligned}
$$

The above two equations are the optimality conditions which equate marginal products of capital and labor in two intermediate sectors. They must be satisfied whenever both intermediate sectors are operated, i.e., when economy's aggregate capital-labor ratio is within the cone of diversification. In this case, the optimal capital-labor ratios in intermediate sectors a and m are $k_{at} = k_{at}^b$ and $k_{mt} = k_{mt}^b$ respectively. This allows us to dispense with the superscript b, with k_{at} and k_{mt} signifying both the boundaries of the cone of diversification and the optimal capital-labor ratios in the two sectors of economies within the cone. Observe that the boundaries, k_{at} and k_{mt} are stochastic, they are functions of λ_t and θ_t. In appendix A we show that whenever k_{at} and k_{mt} are positive, $k_{at} < k_{mt}$. Further, k_{at} and k_{mt} are increasing in $\rho = \frac{p_a \lambda}{p_m \theta}$.

A crucial point is that k_{at} and k_{mt} are independent of the domestic capital-labor ratio k_{t-1}. As long as the aggregate capital-labor ratios of two (or more) economies, that face same realizations of both productivity shocks λ_t and θ_t, fall within the cone of diversification $[k_{at}, k_{mt}]$, the economies will have the same capital-labor ratios in both sectors. Further, these economies will have the same factor prices, as is evident from the following equations:

$$r_t = p_a \lambda_t f'(k_{at}) \tag{3.17}$$

$$w_t = p_a \lambda_t [f(k_{at}) - f'(k_{at}) k_{at}] \tag{3.18}$$

This is the essence of factor price equalization effect of international trade in goods. It plays a crucial role in creating multiple steady states and non-convergence in the environment without uncertainty, the focus of the next section.

The allocation of capital and labor between two intermediate sectors, however, does depend on the domestic capital-labor ratio. Countries having a higher capital-labor ratio devote a larger fraction of capital and labor to the capital-intensive sector m.

Finally, in any equilibrium the following market clearing conditions must be satisfied:

$$
\begin{aligned}
a_t &= \lambda_t F(K_{at}, L_{at}) && (3.19)\\
m_t &= \theta_t G(K_{mt}, L_{mt}) && (3.20)\\
C_t + X_t &= H(a_t^d, m_t^d) && (3.21)\\
C_t &= c_t L && (3.22)\\
X_t &= x_t L && (3.23)\\
K_{at} + K_{mt} &= K_{t-1} && (3.24)\\
L_{at} + L_{mt} &= L && (3.25)
\end{aligned}
$$

The market clearing conditions are standard. Observe that in the market clearing condition for capital, equation (3.24), aggregate capital is determined a period earlier than when it is allocated between the two intermediate sectors for production, a consequence of the timing assumptions mentioned before.

3.3 Equilibrium in the Small Economy without Uncertainty

Before we proceed to talk about convergence in a stochastic environment, let us first understand why there are multiple steady state equilibria, non-convergence and specialization in the economies without uncertainty. Suppose a small economy faces no uncertainty and has $\lambda_t = 1 (= \lambda_t^w)$, and $\theta_t = 1 (= \theta_t^w)$ for all t and in all states of nature.

Since λ_t and θ_t are fixed, the boundaries of the diversification cone, k_{at} and k_{mt}, are constant over time. Further, since the technology is identical across the world and the small economies, the boundaries of the diversification cone in the two economies coincide: $k_a = k_a^w$ and $k_m = k_m^w$.

There are two possible scenarios for the small economy, it may start with a capital-labor ratio within the diversification cone, or it may start with a capital-labor ratio outside the diversification cone. First suppose the initial capital in the small economy, k_0, is within the diversification cone, i.e., $k_0 \in [k_a, k_m]$. Then, since $k_a = k_a^w$, we have

$$r_t = p_{at} f'(k_{at}) = r_t^w. \tag{3.26}$$

Similarly, $w_t = w_t^w$ Thus, there is factor price equalization across the economies. The fact that interest rates are equal across countries mean that there is no incentive for cross-economy borrowing and lending, and trade is balanced period-by-period in the equilibrium. Further, identical rates of return in both economies mean that the incentives to accumulate capital is the same in both economies, and since the world economy is in the steady state, the small economy will also be in the steady state at the initial capital-labor ratio. Thus, any capital-labor ratio within the diversification cone can be sustained as a steady state.

Now consider the case where the small economy starts at a capital-labor ratio that is outside the diversification cone. In particular, suppose that the economy starts with a very low capital-labor ratio, $k_0 < k_a$. In this case, as long as $k_{t-1} < k_a$, it is optimal to produce only the less capital intensive good and we will have

$$r_t = p_a f'(k_{t-1}) > r_t^w = p_a f'(k_a).$$

The interest rate in the small economy will be larger than the world interest rate and the small economy will accumulate capital till it reaches (asymptotically) the lower boundary of the diversification cone, i.e., till the point where $k_{t-1} = k_a$.

Once it reaches the boundary, once again there is factor price equalization and the economy will stop accumulating capital any further. Hence, the small economy will be at a steady state at the lower boundary of the diversification cone and will produce only the less capital intensive good. Thus, the country that starts with a very low level of capital will permanently specialize in producing good a.

In the case where the economy starts at $k_0 > k_m$, the economy will de-accumulate capital till it reaches the upper boundary of the diversification cone and it will remain there forever producing only good m.

Hence economies starting at any capital-labor ratio within the diversification cone will remain at that capital-labor ratio and those starting from outside the diversification cone will reach steady state at the boundaries of the diversification cone. This implies that there will be no convergence in per-capita capital stock or income levels if various economies start with different initial capital-labor ratios.

Notice, one crucial difference between a one-sector closed economy and an open economy with two tradeable sectors is that while in the former the interest rate is a function of the aggregate capital in the economy, it is independent of the aggregate capital in the open economy within the diversification cone. As a result, there is a unique capital-labor ratio for a given interest rate in a closed economy. In the case of an open economy, several aggregate capital-labor ratios are sustainable for a given interest rate, all that differs is the share of the two intermediate goods. This is crucial in delivering multiple steady states in a deterministic dynamic Heckscher-Ohlin model.

3.4 Equilibrium in a stochastic small economy

We now turn our attention to the case when the small economy faces uncertainty, i.e., when λ_t and θ_t are stochastic.

We assume that for all values of $z = (\lambda, \theta)$, the corresponding capital-labor ratios $k_a(\frac{p_a \lambda}{p_m \theta})$ and $k_m(\frac{p_a \lambda}{p_m \theta})$ are strictly positive and finite. This assumption is not crucial for our results, but ensures that at very low values of aggregate capital-labor ratio k the small economy will produce only good a, while at very large values of capital, the small economy will produce only good m.

Also, notice that since p_a and p_m are constants and λ_t and θ_t are i.i.d., the variables $p_a\lambda_t$ and $p_m\theta_t$ are also i.i.d. Define the per capita income of the small economy as $y_t = w_t + r_t k_{t-1} + (1-\delta)k_{t-1}$. It is a function of the small economy's capital-labor ratio k_{t-1} and TFP shocks $z = (\lambda, \theta)$: $y_t = y(k_{t-1}, z_t)$. Now the representative household's problem can be restated as

$$\max_{c_t, k_t} E \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \tag{3.27}$$

$$s.t.$$

$$c_t + k_t \leq y(k_{t-1}, z_t)$$

$$y(k_0, z_1) \text{ is given}$$

where the expectation is defined over the Borel sigma-algebra of partial shock histories $z^t = (z_0, z_1, ...z_t) \in Z^t$. This set up of the household's problem makes it clear that from the household's perspective the problem is essentially the same as that faced by an agent in an one-sector stochastic growth model with i.i.d. shocks. Given this setup, the optimal consumption and investment policy functions in any period t will be functions of current income y_t only. For our main result on convergence we need to establish the continuity and monotonicity properties of our policy functions. To do that we first need to understand the continuity and monotonicity properties of the income function, which is achieved in the next lemma.

Lemma 1. ***Properties of the income function,*** y.

- *Income of the small economy* y *is continuous in* k, λ, *and* θ. *It is strictly increasing in* k, *nondecreasing in* λ *and* θ, *and strictly increasing in either* λ, *or* θ, *or both.*

- *For every* $z \in Z$ *the function* $y(\cdot, z) : R_+ \to R_+$ *is concave, and continuously differentiable. For every* $k > 0$ *the derivative* $\frac{\partial y(k,\cdot)}{\partial k}$ *is continuous in* λ, *and* θ.

- *There exists the maximum sustainable level of capital* $\bar{k}$ *such that* $y(k, z) < \bar{k}$ *for all* $k > \bar{k}$ *and for all* $z \in Z$.

Let $X = \left[0, \bar{k}\right]$. Define the value function $v(k_0, z_1)$ as the maximum lifetime expected utility attained in the program (3.27). It is a standard result that the value function is unique, bounded, strictly concave, continuously differentiable in k (for $k > 0$) and solves the following Bellman equation,

$$v(k, z) = \max_{k' \in [0, y(k,z)]} \left[u(y(k, z) - k') + \beta \int v(k', z')\eta(dz') \right] \tag{3.28}$$

Further, for each $z \in Z$, $v(\cdot, z) : X \to R_+$ is strictly increasing and $v(0, z) = 0$.

The investment policy function $h(k, z)$ is defined so that

$$v(k, z) = u(y(k, z) - h(k, z)) + \beta \int v(h(k, z), z')\eta(dz') \tag{3.29}$$

In the following proposition we establish existence, continuity and monotonicity properties of both, the investment policy function $h(k,z)$ and the consumption policy function $c(k,z)$.

Proposition 1. ***Existence, continuity and monotonicity of the policy functions***

- *There exist unique consumption and investment policy functions,* $c_t = c(k_{t-1}, z_t)$ $k_t = h(k_{t-1}, z_t)$*. They both are continuous with respect to* k_t, λ_t*, and* θ_t *and measurable with respect to the Borel subsets of* $Z = \Lambda \times \Theta$*.*

- *Functions* $c(k_{t-1}, (\lambda_t, \theta_t))$ *and* $h(k_{t-1}, (\lambda_t, \theta_t))$ *are strictly increasing in* k_{t-1}*, nondecreasing in* λ_t *and* θ_t*, and strictly increasing in either* λ_t*, or* θ_t*, or both. Also,* $c(0,z) = 0$ *and* $h(0,z) = 0$ *for all values of* z*.*

Now, to explore the dynamic properties of the investment policy function we need to be more specific about its shape. Let us start with fixed points of the function. For any realization z, define k_z to be a fixed point for the investment policy function $h(k,z)$, i.e. k_z is such that $k_z = h(k_z, z)$. We can also define the maximum and minimum positive fixed points for any given realization z as follows,

$$k_z^{\max} = \max\{k > 0 | h(k,z) = k\} \tag{3.30}$$

$$k_z^{\min} = \min\{k > 0 | h(k,z) = k\} \tag{3.31}$$

Let us also define the *best*, $\overline{z}$, and the *worst*, $\underline{z}$, shocks in the sense of giving the most and the least amount of income, for any given level of capital available. Notice that since the small economy's income is a nondecreasing function of both λ and θ, these are uniquely defined: $\overline{z} = \left(\overline{\lambda}, \overline{\theta}\right)$ and $\underline{z} = (\underline{\lambda}, \underline{\theta})$.

In the next proposition we show that for each z, minimum and maximum positive fixed points are well defined and such, that the investment policy function possesses certain stability properties.

Proposition 2. ***Fixed points and stability properties of the investment policy function***

- *For all* $z \in Z$, $k_z^{\min} > 0$ *exists and for all* $k < k_z^{\min}$, $h(k,z) > k$.
- *For all* $z \in Z$, $k_z^{\max}$ *exists and for all* $k > k_z^{\max}$, $h(k,z) < k$.
- *The function* $h(k,z)$ *has a stable configuration, i.e.,* $k_{\underline{z}}^{\max} < k_{\overline{z}}^{\min}$.

These stability properties imply that for all positive initial values of capital-labor ratio k_0, the optimal capital-labor ratio sequence $\{k_t\}_{t=1}^{\infty}$ will be bounded away from zero. Thus, without loss of generality, we can restrict the domain of possible capital-labor ratios to a compact set $\left[\underline{k}, \overline{k}\right]$, where $\underline{k} > 0$. Let $X = \left[\underline{k}, \overline{k}\right]$.

Given that we have assumed the shocks to be i.i.d., the policy function $h(k,z)$ defines a Markov process on the set of capital-labor ratios X. Let $\mathcal{B}$ be the Borel sigma field generated by X. For all $B \subset \mathcal{B}$ let $P(k_{t-1}, B) = \Pr(k_t \in B)$ be the transition probability function of the capital-labor ratio process in the small economy. Let $P^t(B) = \Pr(k_t \in B)$ be the probability measure for small economy's capital-labor ratio in period t defined on Borel subsets B of X. It is generated by the transition probability function as

$$P^t(B) = \int_X P(k,B) P^{t-1}(dk)$$

starting from some initial distribution P_0 defined on $(X, \mathcal{B})$. The invariant distribution over X then, is any probability measure μ such that

$$\mu(B) = \int_X P(k,B)\mu(dk)$$

The economy is generally assumed to start from a given value of the capital, that means P^0 is a degenerate distribution concentrated on some positive value of capital-labor ratio. Our objective here is to prove that no matter which positive value of capital we start from, the limit $\lim_{t\to\infty} P^t$ is the unique invariant distribution. More precisely, let δ_{k_0} be a degenerate distribution concentrated on k_0. Let $P^0(k_0, B) = \delta_{k_0}, \quad P^1(k_0, B) = P(k_0, B)$, and $P^t(k_0, B) = \int_X P(k, B)P^{t-1}(k_0, dk)$ for any set $B \subset \mathcal{B}$. We need to show that $\lim_{t\to\infty} P^t(k_0, B) = \mu(B)$ for all positive k_0 and any Borel subset B in $\mathcal{B}$.

Theorem 1. ***Convergence***

There exists the unique invariant probability measure μ *on* $(X, \mathcal{B})$*, such that* $\lim_{t\to\infty} P^t(k_0, B) = \mu(B)$ *for all* $k_0 > 0$*. The full support of* μ *is the unique non-degenerate compact interval on* $\mathcal{R}_{++}$ *given by* $[k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}]$.

The above theorem says that no matter where different small economies start from, their long run average per-capita capital stock will be the same. Thus, in the long run, there will be convergence in the per-capita capital stock and hence, convergence in the per-capita income levels across countries. This result is in stark contrast to the result in the deterministic Heckscher-Ohlin model, where two countries with different initial conditions will end up with different levels of steady state variables.

One key assumption in our model is the balanced trade condition. As already pointed out, in the non-stochastic case the requirement that trade be balanced period-by-period does not constrain equilibrium when both the tradable commodities are produced in the economy. However, in case with uncertainty, there is an incentive for the small economy to borrow and lend from the world economy. Not being able to do that, the small economy will try to smooth consumption by saving more when income is higher than expected and less when the income is lower than expected. In addition the rate of return in the diversification cone is determined by the realization of the shocks (see equation (3.11) and (3.13)). Thus, even within the diversification cone, the incentive to accumulate or de-accumulate capital in the small economy is different from that of the world economy. Thus, the small economy can grow (or have negative growth) inside the diversification cone. This is an important distinction between the stochastic and non-stochastic versions.

Theorem 1 states that support of the invariant distribution is the unique non-degenerate interval given by $[k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}]$. The lower boundary of the interval $k_{\underline{z}}^{\max}$ is the maximum fixed point of the policy function for the *worst* possible shocks, while the upper boundary is the minimum fixed point of the policy function for the *best* possible shocks. As shown in Proposition 2, this interval is non-degenerate, and since $k_{\underline{z}}^{\max} > 0$, its lower boundary is strictly positive.

The fact that the invariant distribution is unique can be illustrated using figure 1. In that figure we have drawn two policy functions, one for the worst shock $\underline{z}$ and the other for the best shock $\overline{z}$. The capital-labor ratio in the shaded region, marked as the invariant set, is the full support for the invariant distribution. Notice, first that any economy that has capital-labor ratio in that region will always remain there - the worst that can happen is that the economy faces the worst shock each period, then its capital-labor ratio will converge to the lower boundary, whereas it goes to the upper boundary in the best possible case when the country faces the

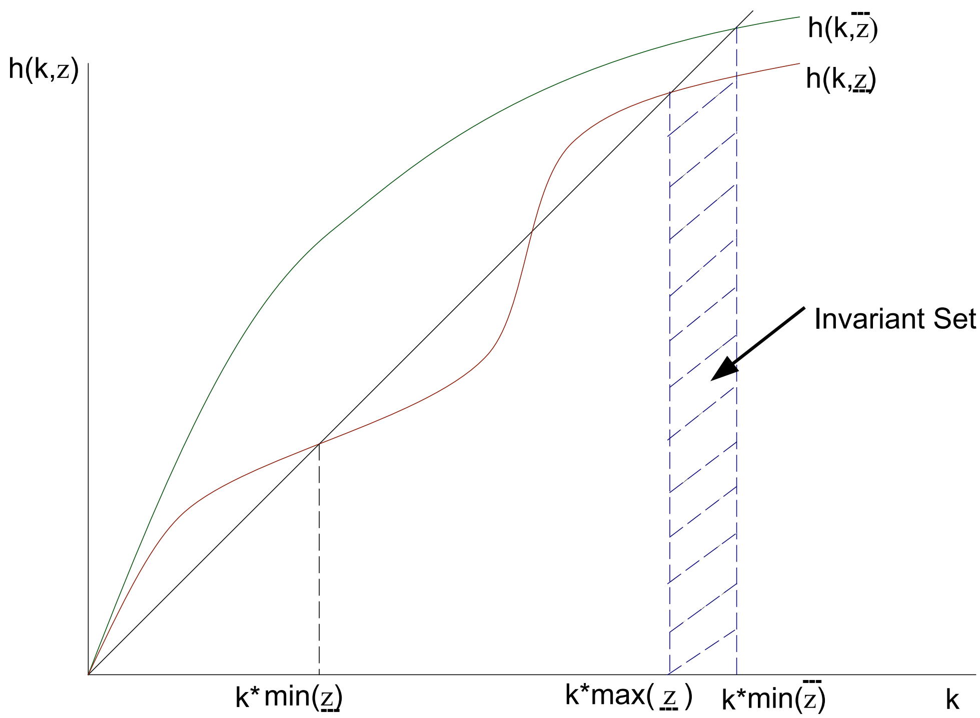

Figure 1: Invariant set

best shock every period. Since, the policy function, $h(k, z)$, is continuous and non-decreasing in z and the shocks come from a full compact support, every non-degenerate interval of capital-labor ratios within the invariant set is attainable with positive probability. Now consider the case when the initial capital-labor ratio is below the minimum point of the interval $[k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}]$. A sequence of good shocks, that happens with positive probability, will eventually bring it inside the interval. The case when the capital-labor ratio is above the interval is symmetric. Thus, $[k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}]$ will be the unique full support for the invariant distribution.

This characterization of the support also helps us to determine whether the economy will diversify. To do that we need to find out whether there is any intersection between the support of the invariant set and the diversification cone. Recall that k_{at} and k_{mt} are the capital labor ratios in sectors a and m in the small economy, whenever it produces positive amounts of both intermediate goods. In the Appendix A we show that they are strictly increasing, continuous functions of $\rho = \frac{p_a \lambda}{p_m \theta}$. So the minimum values of k_a, k_m are the ones corresponding to $z^* = (\underline{\lambda}, \overline{\theta})$. The maximum values of k_a, k_m correspond to $z^{**} = (\overline{\lambda}, \underline{\theta})$.

Theorem 2. ***Diversification***

The fixed points of the optimal policy function satisfy $k_{\overline{z}}^{\min} > k_a(z^*)$ *and* $k_{\underline{z}}^{\max} < k_m(z^{**})$.[2]

The above theorem implies that there is a positive measure of z such that $[k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}] \cap [k_a(z), k_m(z)]$ is a non-degenerate interval. Which in turn implies, in an infinite horizon setting, that the support of the invariant set will intersect with the diversification cone in at least some periods. Thus, the small economy

[2]Note that given constant prices p_a and p_m, there is a unique single valued map from $z = (\lambda, \theta)$ to $\rho = \frac{p_a \lambda}{p_m \theta}$. This allows us to write $k_a(z)$ and $k_m(z)$.

will surely diversify in some periods. This result also differs from the finding in the non-stochastic version where a small economy starting from outside the diversification cone will permanently specialize.

4 Simulation of the Small Economy

So far, we have shown that with uncertainty and borrowing constraint there will be convergence, but our results do not give a sense about how fast that convergence will occur. To find out about the speed of convergence, we simulate our model. In the simulations we assume that there is no uncertainty in the production of the intermediate good m, but uncertainty is present only in the production of good a.[3] We assume that there are two possible states, high and low, with equal probability.[4] We then simulate our model for different magnitudes of shocks. We fix the mean of the shocks in sector a, λ_t, to be 1 and take different symmetric deviations from that, λ^H being the good shock and λ^L being the bad shock. We find that the bigger are the possible shocks in the small economy, the faster will be the convergence. This is illustrated in the figures 2 and 3 where we report two cases: (i) deviation from the mean is 1%, (ii) deviation from the mean is 10%.

We also simulate the economy when there is no uncertainty. The path for the capital is plotted in figure 4. This simulation replicates the results of the Heckscher-Ohlin models without uncertainty - a country that start with capital-labor ratio less than k_a grows till it reaches the lower boundary of the diversification cone and then its capital-labor ratio is fixed at that level. The case with countries that start with a capital-labor ratio greater than k_m is symmetric.

[3]None of our qualitative results change because of this assumption.

[4]Simulations with continuous state space give similar results.

The finding that the speed of convergence is increasing in the magnitude of shocks relates our convergence result with non-convergence in the deterministic version. It suggests that for small degrees of uncertainty it will take extremely long for economies to converge. In the limit, when uncertainty is driven to zero, convergence disappears altogether. Thus, the deterministic Heckscher-Ohlin model is a very special case of the stochastic model.

Our simulation is useful in another dimension as well, it allows us to see the actual shape of the investment policy function. A plot of the policy function in figure 5 reveals the effect of uncertainty and market incompleteness in our model. Recall that in the deterministic case the investment policy function coincides with the 45 degree line everywhere within the diversification cone, every point there is a fixed point and a steady state. With uncertainty the policy functions for good and bad shocks shift apart from each other. It is a consequence of representative agent's desire to self-insure herself by accumulating more (less) capital when income is higher (lower) than expected due to good (bad) productivity shock. This pattern of capital accumulation makes it much less likely for the small economy to find itself in a fixed point of the investment policy function. Without uncertainty or under complete markets self-insurance would not be necessary and multiple steady-states become a possibility.

The above discussion suggests that there is nothing special about the uncertainty being introduced by sector-specific productivity shocks. Other kinds of uncertainty, like endowment shocks or the economy-wide productivity shocks should do. This intuition is correct and in the next section we consider a version of the model with economy-wide productivity shocks. There is however, another interesting property of the model with sector-specific productivity shocks. It is apparent from the figure 5 with the simulated policy function. On that figure we can see that both policy functions - for the good shock and for the bad shock - tilt clockwise relative to the 45 degree line. The tilt shrinks the invariant set (full support of the invariant distribution) and, in this particular simulation, makes it fit completely within the diversification cone. This effect is due to changes in comparative advantage induced by sector-specific shocks. Recall that changes in sectoral shocks affect the boundaries of the diversification cone, thus the small economy's own diversification cone shifts as various values of sectoral shocks are realized. To fully take advantage of uncertain changes in comparative advantage the small economy is induced to have capital-labor ratio closer to the intersection of its own possible diversification cones. This intersection is smaller than the world's diversification cone.

It would be interesting to see what happens in the model where the changes in comparative advantage are absent, and only self-insurance motives are at work. In the next section we consider a model with economy-wide productivity shocks, and show that in that case the entire world economy's diversification cone is a strict subset of the invariant set.

5 A model with economy-wide shocks

In this section we assume that both intermediate sectors are affected by the same productivity shock.[5] That is, we impose a restriction that $\Lambda = \Theta$ and that λ and θ are perfectly correlated. It is immediately apparent then that this model is a special

[5]It is trivial to show that this set up is equivalent to the - perhaps more intuitive - environment with productivity shocks on the final good technology only. We stick with this interpretation for ease of comparison with the previous sections.

case of the one with sector-specific shocks. The convergence result of theorem 1 then immediately applies.

With regard to diversification first observe, that with economy-wide productivity shock, the boundaries of the diversification cone of the small economy are fixed and coincide with the ones of the world economy. It is immediately apparent from the equations (3.15) and (3.16) once you substitute λ for θ. Thus, in this case, for all z we have $k_a(z) = k_a^w$, and $k_m(z) = k_m^w$. We are now ready to prove the following proposition:

Proposition 3. *If both intermediate sectors are affected by the same productivity shock, e.g. λ_t, then the fixed points of the optimal policy function satisfy $k_{\underline{z}}^{\max} < k_a^w$ and $k_{\overline{z}}^{\min} > k_m^w$.*

This proposition proves that with economy-wide shocks the entire diversification cone is a proper subset of the invariant set. So, in the invariant distribution a small economy may visit not only the entire diversification cone, but also some areas outside of it.

We simulated the model with economy-wide shocks to see what kind of dynamics and policy functions will the model generate. Figure 6 shows the optimal policy functions for both high and low productivity shocks. As we can see, without sector-specific shocks the entire diversification cone is in the invariant set. The figure 7 shows the dynamics of capital-labor ratio for two small economies - one initially poor, and another, initially rich - that face the same sequence of realized economy-wide shocks. They do converge, in accordance with our theoretical results. It takes, however, a lot of time. Also in the invariant distribution they visit not only the entire diversification cone, but also areas outside of it, again in accordance with theoretical predictions.

The example with economy-wide shocks highlights again the role of sector-specific shocks: they make the invariant set smaller than the diversification cone, but they are not necessary for convergence. It is self-insurance motive due to income uncertainty and market incompleteness that drives the convergence result.

Are country-specific shocks with market incompleteness necessary for convergence? Do countries have to be different? Could we prove similar convergence results with "global" technological shocks that affect all the countries? May be it is just uncertainty in income that matters, and not the way it is introduced. The next section presents a model in which all countries are affected by same productivity shocks. We show analytically, that in this particular example countries do not converge and may permanently specialize in producing only one tradable good.

6 Analytical example with no convergence

In this example we assume a different structure for uncertainty. We assume that both the world and the small economy face identical shocks, i.e., $\lambda_t^w = \lambda_t^s$, and $\theta_t^w = \theta_t^s$ for all t.

We use specific functional forms for the utility and the production functions. The utility function is logarithmic, $u(c) = \ln(c)$, while all production functions are Cobb-Douglas. The individual production functions are given by,

- Final good technology $H(a, m) = a^{\mu} m^{1-\mu}$
- Intermediate good a technology $\lambda F(K, L) = \lambda K^{\alpha} L^{1-\alpha}$
- Intermediate good m technology $\theta G(K, L) = \theta K^{\gamma} L^{1-\gamma}$,

where $1 > \gamma > \alpha > 0$. Further, we assume full depreciation, i.e., $\delta = 1$. Under these assumptions we can find an analytical solution to the dynamic problems of both the world and the small economies. Since, in this example we provide the dynamics of both the world and the small economy, we will again use superscript w for world and s for small economy to distinguish them.

Given the specific functional forms it is easy to show that in the world economy the allocation of labor across intermediate sectors is fixed.

$$L^w_{at} = \frac{(1-\alpha)\mu}{(1-\alpha)\mu + (1-\gamma)(1-\mu)} \tag{6.1}$$

$$L^w_{mt} = \frac{(1-\gamma)(1-\mu)}{(1-\alpha)\mu + (1-\gamma)(1-\mu)}. \tag{6.2}$$

Further, optimal capital-labor ratios in both intermediate sectors of the world economy are proportional to the aggregate capital labor-ratio:

$$k^w_{at} = \frac{1}{L_a(1-\Omega)+\Omega} k^w_{t-1} \tag{6.3}$$

$$k^w_{mt} = \frac{\Omega}{L_a(1-\Omega)+\Omega} k^w_{t-1} \tag{6.4}$$

where $\Omega = \frac{\gamma(1-\alpha)}{\alpha(1-\gamma)} > 1$. Denote $\phi_a = \frac{1}{L_a(1-\Omega)+\Omega}$, and $\phi_m = \frac{\Omega}{L_a(1-\Omega)+\Omega}$. Thus, capital-labor ratio in each sector is a constant fraction of the aggregate capital-labor ratio. Notice that $\phi_a \in (0,1)$, while $\phi_m > 1$, a consequence of technology m being more capital intensive than technology a.

The optimal capital-labor ratio in the world economy evolves according to the following law of motion:

$$k^w_t = A_t \left(k^w_{t-1}\right)^q \tag{6.5}$$

where, $A_t = Q\lambda_t^{\mu}\theta_t^{1-\mu}$ is an aggregate productivity (Q is a fixed positive number) and $q = \lambda\mu + \gamma(1-\mu)$. Since $q < 1$ the world capital-labor ratio converges to a unique invariant distribution. The above law of motion for world capital-labor ratio determines a markov process for intermediate good prices p_{at}, p_{mt}. Notice that the prices of the intermediate goods are no longer constant across time.

Now suppose, at the beginning of period t, the small economy's capital-labor ratio is $k_{t-1}^s > 0$. Let $\tau_t = \frac{k_{t-1}^s}{k_{t-1}^w}$, be the capital labor ratio in the small economy relative to that in the world economy. There are three possible cases to consider:

- If $\tau_t \leq \phi_a$, then $k_{t-1}^s \leq k_{at}^w = \phi_a k_{t-1}^w$. In this case the small economy will produce only good a in period t. The optimal level of investment in this case will be $k_t^s = \alpha\beta p_{at}\lambda_t \left(k_{t-1}^s\right)^{\alpha} = \alpha\beta\tau_t^{\alpha} p_{at}\lambda_t \left(k_{t-1}^w\right)^{\alpha}$.

- If $\phi_a < \tau_t < \phi_m$, then $k_{t-1}^s \in (k_{at}^w, k_{mt}^w)$ and the small economy will produce both goods, a and m, in period t. The optimal level of investment in this case will be $k_t^s = \alpha\beta\tau_t\phi_a^{\alpha-1}p_{at}\lambda_t \left(k_{t-1}^w\right)^{\alpha}$ or equivalently, $k_t^s = \gamma\beta\tau_t\phi_m^{\gamma-1}p_{mt}\theta_t \left(k_{t-1}^w\right)^{\gamma}$.

- Finally, if $\tau_t \geq \phi_m$, the small economy will produce only good m in period t, and will invest $k_t^s = \gamma\beta\tau_t^{\gamma}p_{mt}\theta_t \left(k_{t-1}^w\right)^{\gamma}$.

This investment rule implies that in the next period, $t+1$, the capital-labor ratio in the small economy relative to that in the world economy will depend on whether or not the small economy is inside the diversification cone. Thus, in period $t+1$ we have,

- If $\tau_t < \phi_a$, then $\frac{k_t^s}{k_t^w} = \frac{\alpha\beta\tau_t^{\alpha}p_{at}\lambda_t\left(k_{t-1}^w\right)^{\alpha}}{\alpha\beta\phi_a^{\alpha-1}p_{at}\lambda_t\left(k_{t-1}^w\right)^{\alpha}} = \tau_t\left(\frac{\tau_t}{\phi_a}\right)^{\alpha-1} > \tau_t$.

- if $\tau_t \in [\phi_a, \phi_m]$, then $\frac{k_t^s}{k_t^w} = \frac{\alpha\beta\tau_t\phi_a^{\alpha-1}p_{at}\lambda_t\left(k_{t-1}^w\right)^{\alpha}}{\alpha\beta\phi_a^{\alpha-1}p_{at}\lambda_t\left(k_{t-1}^w\right)^{\alpha}} = \tau_t$.

- if $\tau_t > \phi_m$, then $\frac{k_t^s}{k_t^w} = \frac{\gamma\beta\tau_t^{\gamma} p_{mt}\theta_t \left(k_{t-1}^w\right)^{\gamma}}{\gamma\beta\phi_m^{\gamma-1} p_{mt}\theta_t \left(k_{t-1}^w\right)^{\gamma}} = \tau_t \left(\frac{\tau_t}{\phi_m}\right)^{\gamma-1} < \tau_t$.

Thus, whenever the small economy has an aggregate capital-labor ratio outside the diversification cone, $[k_{at}, k_{mt}]$, the optimal investment policy will push it closer to the diversification cone. If, on the other hand, the small economy starts within the diversification cone, it will maintain a constant ratio between the domestic aggregate capital-labor ratio k_t^s, and the world aggregate capital-labor ratio k_t^w. Thus, if two small economies started within the diversification cone, but with different capital-labor ratios relative to that of the world economy, they will maintain those relative positions. Hence, there is no convergence in capital or income. Also, if any small economy starts with capital labor ratios outside the diversification cone, they will always specialize in the production of only one commodity.

Thus we get the same results as in the non-stochastic version of the dynamic Heckscher-Ohlin model:

- Multiplicity of invariant distributions of capital,
- No income convergence, and
- Permanent specialization in production.

The only difference between this example and the stochastic version considered earlier is that here both the small and the world economies face identical shocks, i.e. $z_t^w = z_t^s$ for all t. As a result of global shocks, there are no risk sharing opportunities. Inside the diversification cone, the small economy and the world economy have same return to capital. There is no incentive for borrowing or lending between the economies and the trade balance constraint does not bind.

Balanced trade is an equilibrium outcome in this case, as in the non-stochastic version.

This shows that the fact that the trade balance constraint binds, as countries realize different productivity shocks, is crucial for our convergence and diversification results. It is not just uncertainty that is important. Here we have uncertainty and yet the results are very similar to what we see in deterministic models.

On a more technical level, this example also shows the importance of i.i.d. shocks. It is the only assumption that is violated here. As a result of the world economy being disturbed by shocks, intermediate good prices p_{at} and p_{mt} follow a markov process induced by the optimal capital accumulation of the world economy. It follows that the small economy now faces autocorrelated productivity shocks $\lambda_t p_{at}$ and $\theta_t p_{mt}$, rather than i.i.d. shocks as in the previous sections. So, the example shows that with suitably correlated shocks there could be multiplicity of invariant distributions of capital.

7 Conclusion

This paper shows that in an uncertain world, when markets are not complete, different economies will have the same average long-run income irrespective of where they start from. This reverses the predictions of the deterministic dynamic Heckscher-Ohlin model. Thus, our results extend the predictions of income convergence, standard in one sector neoclassical growth models, to the dynamic multi-country Heckscher-Ohlin environment. The results of our model differ from the deterministic dynamic Heckscher-Ohlin model in another front. We find that there will surely be some periods in which a small open economy diversifies, even if it starts with a very low capital stock. This is in contrast to the deterministic version, where countries may permanently specialize in producing a subset of tradable goods. We find that the restriction on risk sharing opportunities imposed by the period-by-period balanced trade requirement is crucial for our results. However, as noted in the paper, our results will remain unchanged if we allowed for limited borrowing and lending.

The results of the deterministic version and the stochastic version seems to be in two different extremes, but our simulation results give a sense of continuity between the two cases. It shows that the smaller the shocks are, the slower is the convergence and in the limit when there is no uncertainty there is no convergence. Thus, how fast will countries catch up with each other depends on the extent of uncertainty prevailing in the world.

Further, we show that the assumption that shocks are sector-specific is not necessary to prove convergence, but play a role in contracting the set of possible capital-labor ratios observed in the long-run.

Finally, we constructed an example that shows the importance of country-specific shocks and of the balanced trade constraint being binding. We show that with global shocks affecting all countries the income convergence may disappear.

References

[1] A. Atkeson and P. Kehoe. Paths of development for early- and late-bloomers in a dynamic heckscher-ohlin model. *Federal Reserve Bank of Minneapolis, Research Department Staff Report*, (256), 2000.

[2] S. Rao Aiyagari. Uninsured idiosyncratic risk and aggregate saving. *Quarterly Journal of Economics*, 109(3):659–684, August 1994.

[3] W.A. Brock and L.J. Mirman. Optimal economic growth and uncertainty: The discounted case. *Journal of Economic Theory*, 4:479–513, 1972.

[4] G. Chamberlain and C.A. Wilson. Optimal intertemporal consumption under uncertainty. *Review of Economic Dynamics*, 3:365–395, 2000.

[5] Partha Chatterjee and Malik Shukayev. Note on existence of minimum positive fixed point in the stochastic growth model. Mimeo,University of Minnesota, 2004.

[6] Z. Chen. Long-run equilibria in a dynamic heckscher-ohlin model. *Canadian Journal of Economics*, XXV(4), 1992.

[7] R.H. Clarida. Consumption, liquidity constraints and asset accumulation in the presence of random income fluctuations. *International Economic Review*, 28:339–351, 1987.

[8] H.A. Hopenhayn and E.C. Prescott. Stochastic monotonicity and stationary distributions for dynamic economies. *Econometrica*, 60(6):1387–1406, 1992.

[9] M. Maffezzoli. Neoclassical growth and commodity trade. Working Paper, 2003.

[10] L.J. Mirman and I. Zilcha. On optimal growth under uncertainty. *Journal of Economic Theory*, 11:329–339, 1975.

[11] L.J. Mirman and I. Zilcha. Unbounded shadow prices for optimal stochastic growth models. *International Economic Review*, 17(1):121–132, 1976.

[12] N. Stokey, R. Lucas, and E. Prescott. *Recursive Methods in Economic Dynamics*. Harvard University Press, 1989.

[13] J. Ventura. Growth and interdependence. *Quarterly Journal of Economics*, 112:57–84, February 1997.

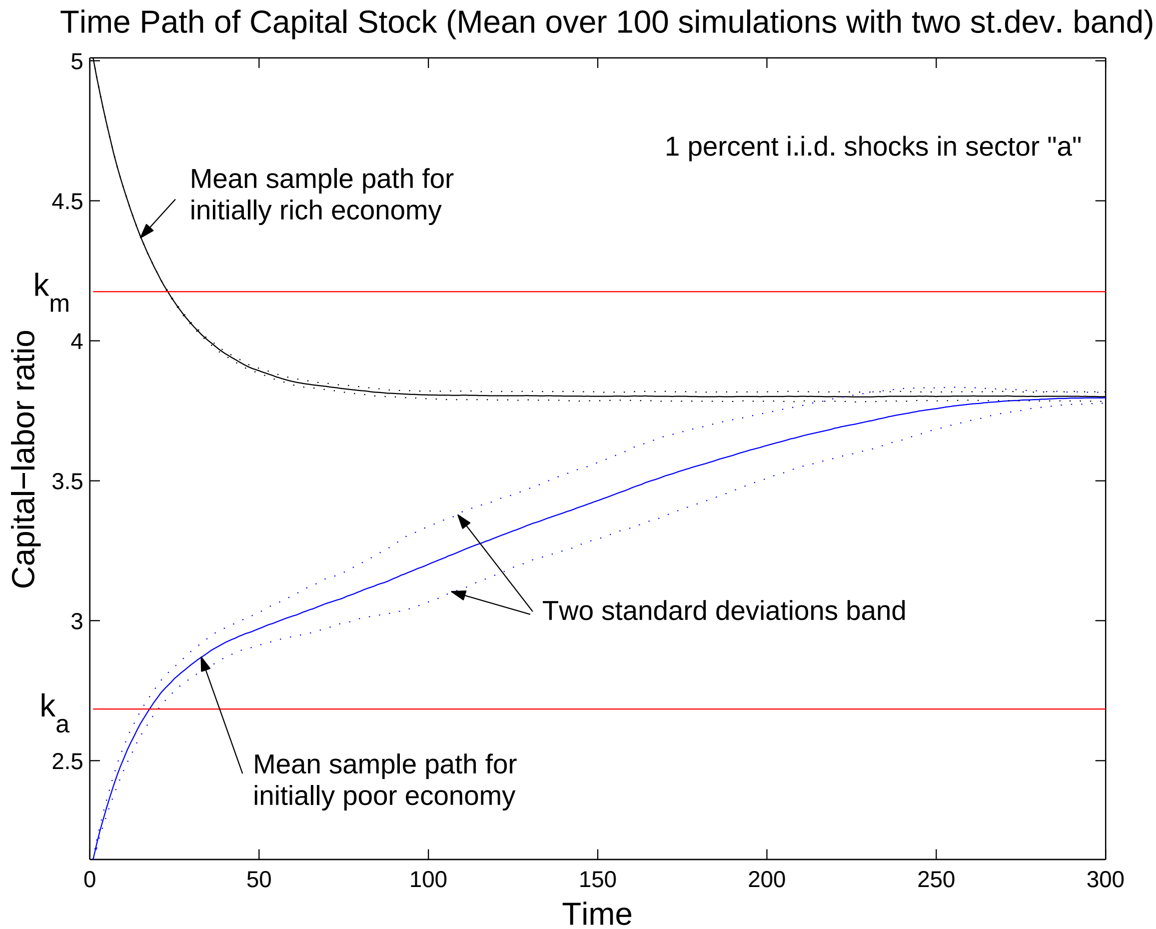

Figure 2: Path of Capital: 1% Shock

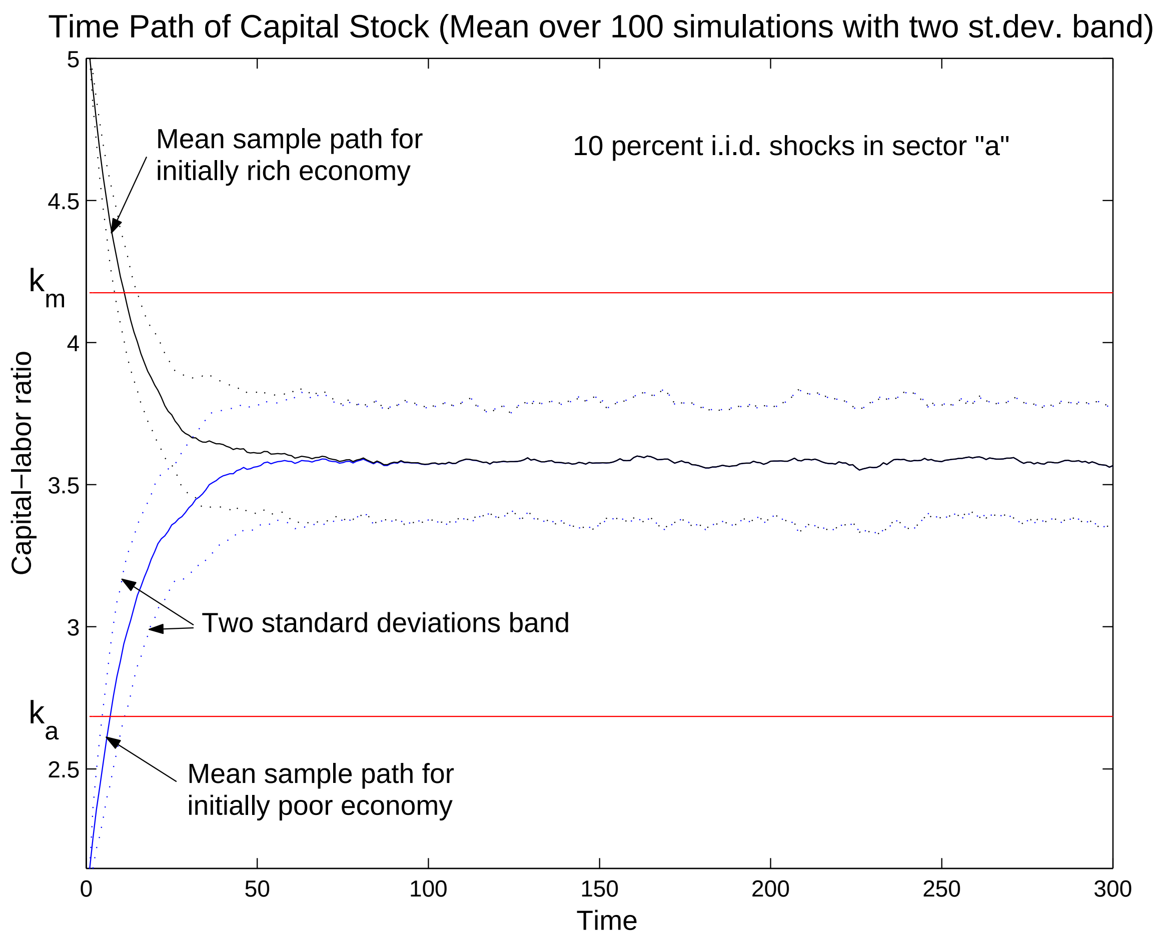

Figure 3: Path of Capital: 10% Shock

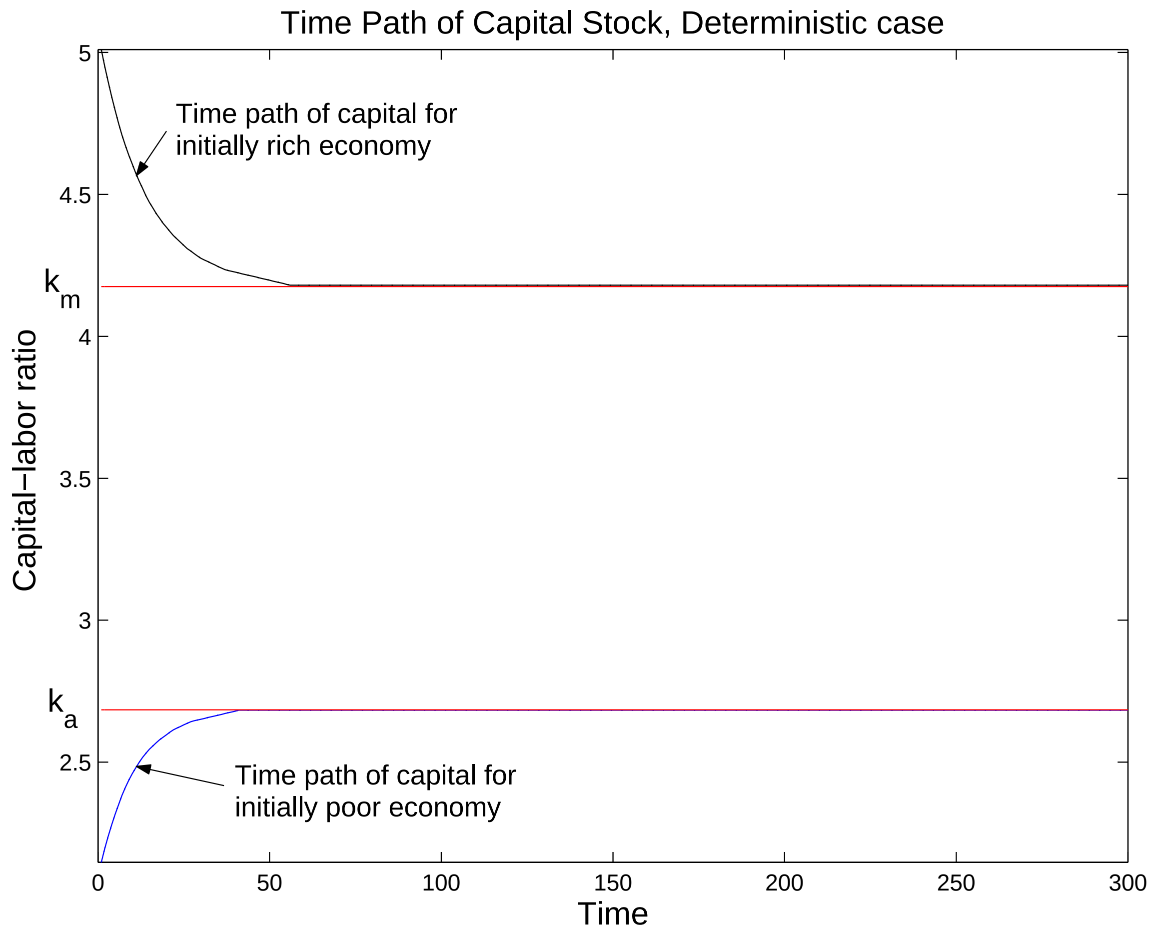

Figure 4: Path of Capital: No Uncertainty

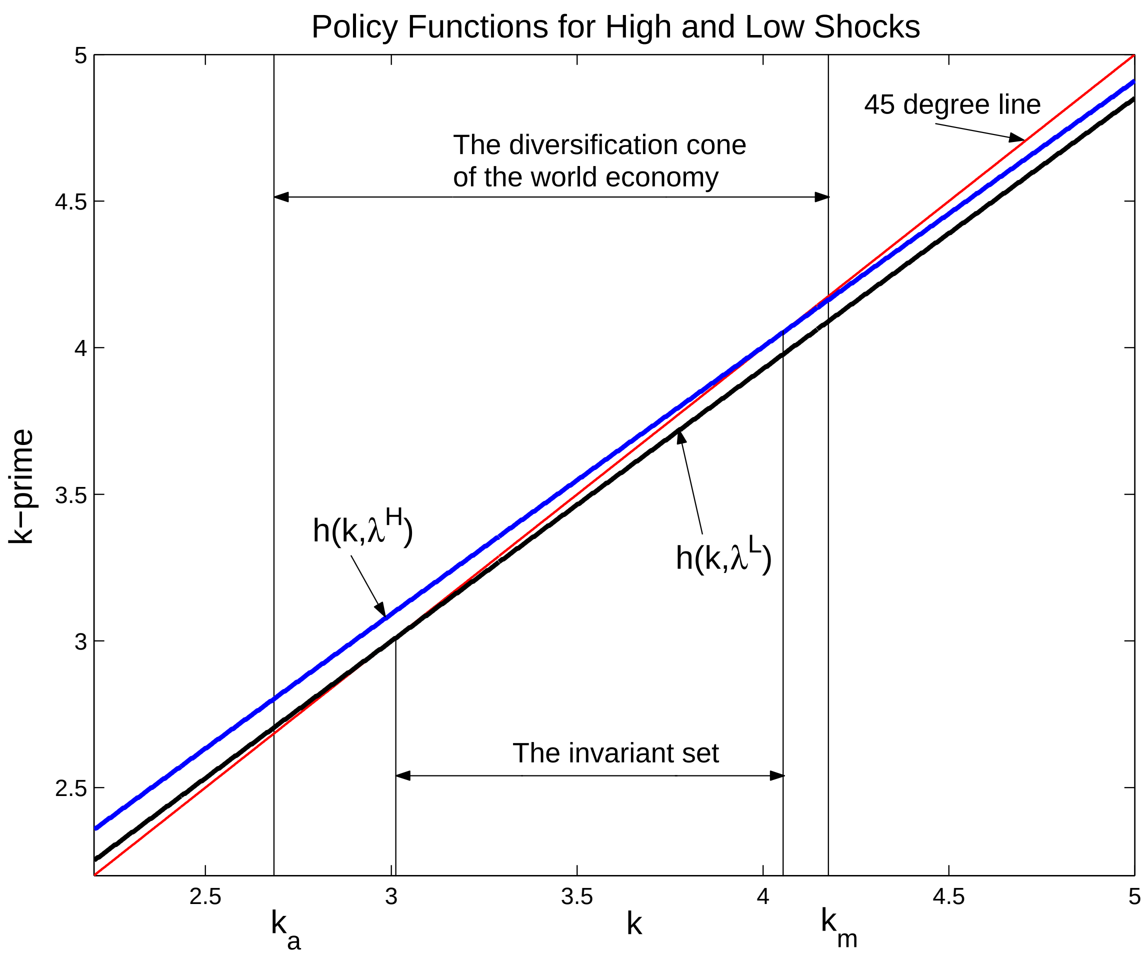

Figure 5: Policy Functions for "High" and "Low" shocks

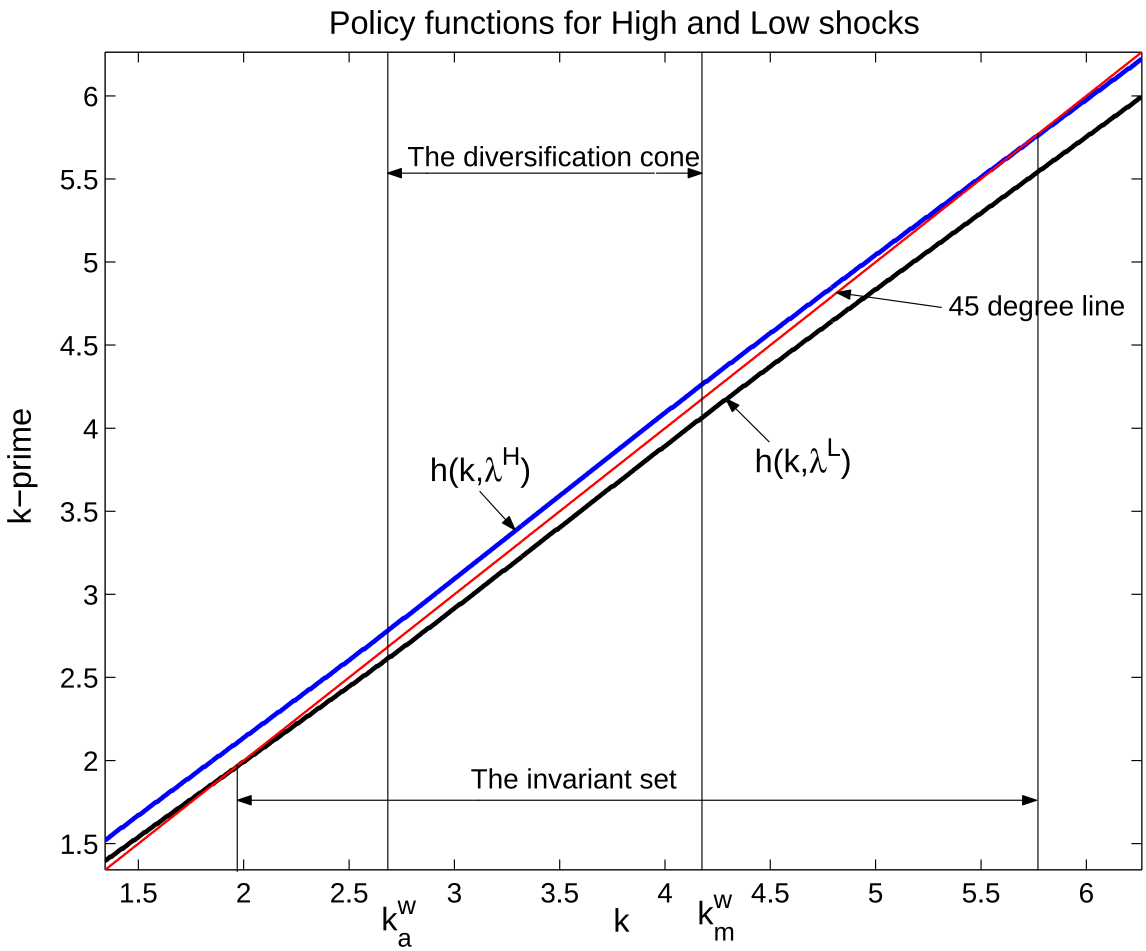

Figure 6: Policy Functions for “High” and “Low” economy-wide shocks

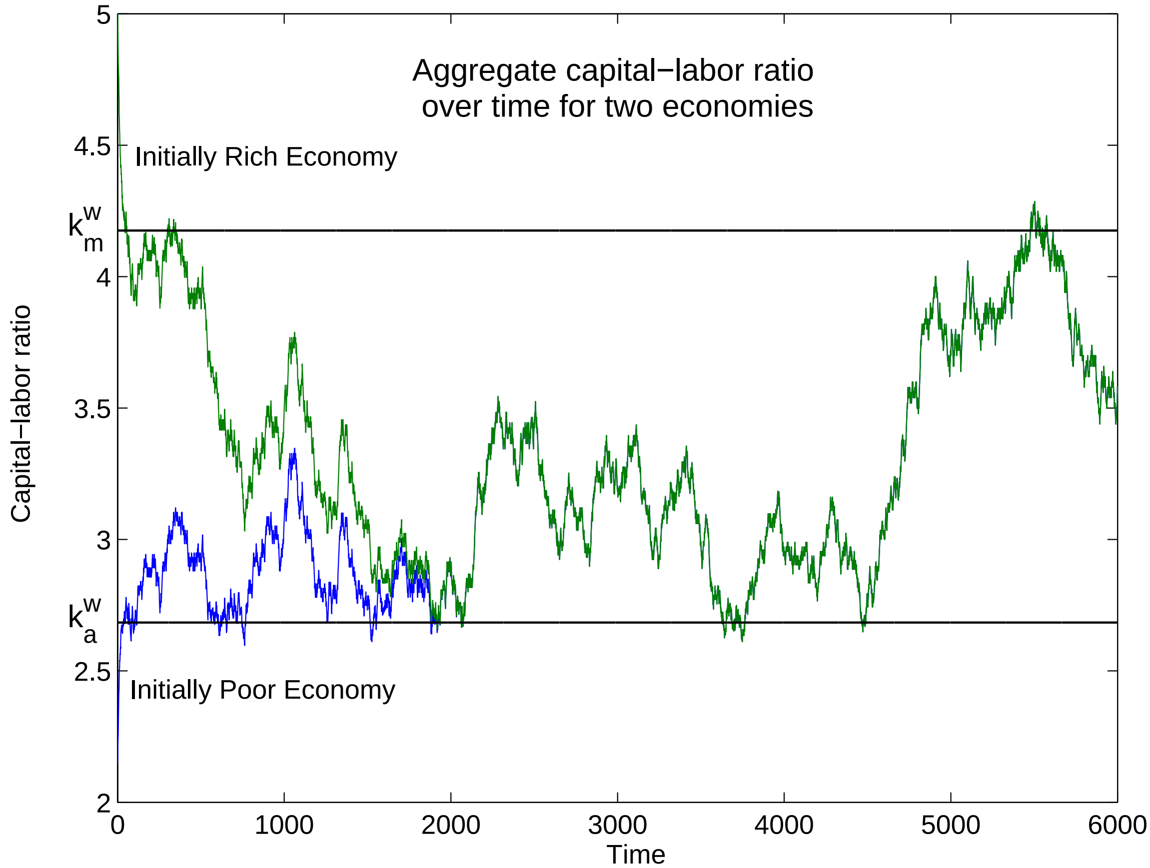

Figure 7: Capital dynamics with economy-wide shocks

Appendix

A Properties of the boundaries of the diversification cone

In this appendix we will ignore time subscripts and the country superscripts since, if the small economy produces both intermediate goods, the analysis applies to both economies.

To simplify notation let's denote:

$$
\begin{aligned}
x &= k_a \\
y &= k_m \\
\sigma &= p_a \lambda \\
\chi &= p_m \theta \\
\rho &= \frac{\sigma}{\chi}
\end{aligned}
$$

We established that when both intermediate sectors produce the following equalities must hold:

$$r = \sigma f'(x) = \chi g'(y) \tag{A.1}$$

$$w = \sigma \left[f(x) - f'(x)x \right] = \chi \left[g(y) - g'(y)y \right] \tag{A.2}$$

Dividing the second set of equalities by the first we obtain

$$\frac{f(x) - f'(x)x}{f'(x)} = \frac{g(y) - g'(y)y}{g'(y)}$$

This is an implicitly defined function $y = \Omega(x)$, the derivative of which can be found using the implicit function theorem. For that define

$$\Phi(x,y) = \frac{f(x)}{f'(x)} - x - \frac{g(y)}{g'(y)} + y$$

Then we have:

$$\begin{aligned}\Omega'(x) &= -\frac{\Phi_1(x,y)}{\Phi_2(x,y)} = -\frac{\frac{[f'(x)]^2 - f''(x)f(x)}{[f'(x)]^2} - 1}{-\frac{[g'(y)]^2 - g''(y)g(y)}{[g'(y)]^2} + 1} \\ &= \frac{f''(x)f(x)}{[f'(x)]^2}\frac{[g'(y)]^2}{g''(y)g(y)} > 0\end{aligned}$$

Thus k_m is a strictly increasing and continuous function of k_a. Also from the above equations it is clear that whenever $k_a = 0$ so is $k_m = \Omega(k_a)$.

From (A.1) and (A.2)we can derive another expression that we will need later

$$\rho f(x) - g(y) = \rho f'(x)x - g'(y)y = \rho f'(x)(x-y)$$

Or alternatively

$$\rho f'(x) = \frac{g(y) - \rho f(x)}{y - x} \tag{A.3}$$

Once we have $\Omega'(x)$ we can use the equality $\sigma f'(x) = \chi g'(\Omega(x))$ to find x as a function of sectoral shocks. It is clear that x depends only on the ratio of sectoral shocks $\rho = \frac{\sigma}{\chi}$. Define a function of x and ρ :

$$\widetilde{\Phi}(x,\rho) = \rho f'(x) - g'(\Omega(x))$$

By the implicit function theorem we again can find the derivative of $x = \pi(\rho)$

function:

$$\begin{aligned}
\pi'(\rho) &= -\frac{\widetilde{\Phi}_2(x,\rho)}{\widetilde{\Phi}_1(x,\rho)} = -\frac{f'(x)}{\rho f''(x) - g''(\Omega(x))\Omega'(x)} \\
&= -\frac{f'(x)}{\rho f''(x) - g''(y)\frac{f''(x)f(x)}{[f'(x)]^2}\frac{[g'(y)]^2}{g''(y)g(y)}} \\
&= -\frac{f'(x)}{\rho f''(x) - \frac{f''(x)f(x)}{[f'(x)]^2}\frac{[g'(y)]^2}{g(y)}} \\
&= -\frac{f'(x)}{\rho f''(x)(1-\frac{\rho f(x)}{g(y)})} \\
&= -\frac{g(y)}{\rho^2 f''(x)(y-x)}
\end{aligned}$$

Where in the last two formulas we used the fact that $\frac{[g'(y)]^2}{[f'(x)]^2} = \rho^2$, and $g(y) - \rho f(x) = \rho f'(x)(y-x)$. The assumption of g technology being more capital intensive than f technology implies that $(y-x) > 0$. So we obtain $\pi'(\rho) > 0$. Hence, k_a and k_m are strictly increasing functions of ρ whenever both sectors are operated with positive inputs.

B Proofs

B.1 Proof of Lemma 1

Fix some $z = (\lambda, \theta)$ and positive k.

If $k \in (0, k_a(z))$ then $y = p_a\lambda f(k) + (1-\delta)k$. Hence, $\frac{\partial y}{\partial k} = p_a\lambda f'(k) + 1 - \delta > 0$. Also $\frac{\partial y}{\partial \lambda} = p_a f(k) > 0$ and $\frac{\partial y}{\partial \theta} = 0$. Thus in $k < k_a(z)$ case monotonicity and continuity of y is established. For concavity of $y(\cdot, z)$ and continuity of $\frac{\partial y}{\partial k}$ observe that its $\frac{\partial^2 y}{\partial k^2} = p_a\lambda f''(k) < 0$, $\frac{\partial^2 y}{\partial k \partial \lambda} = p_a f'(k)$, $\frac{\partial^2 y}{\partial k \partial \theta} = 0$.

If $k > k_m(z)$ then $y = p_m \theta g(k) + (1-\delta)k$. Hence $\frac{\partial y}{\partial k} = p_m \theta g'(k) + 1 - \delta > 0$. Also $\frac{\partial y}{\partial \lambda} = 0$ and $\frac{\partial y}{\partial \theta} = p_m g(k) > 0$. Thus in $k > k_m(z)$ case monotonicity and continuity of y is established. For concavity of $y(\cdot, z)$ and continuity of $\frac{\partial y}{\partial k}$ observe that $\frac{\partial^2 y}{\partial k^2} = p_m \theta g''(k) < 0$, $\frac{\partial^2 y}{\partial k \partial \lambda} = 0$, and $\frac{\partial^2 y}{\partial k \partial \theta} = p_m g'(k)$.

If $k \in (k_a(z), k_m(z))$ then $y = p_a \lambda f(k_a(z)) L_a + p_m \theta g(k_m(z)) L_m + (1 - \delta)k$. Where $L_a + L_m = 1$ and $k_a(z) L_a + k_m(z) L_m = k$. By the assumption of perfect competition, we know that optimal k_a, k_m, L_a, L_m maximize y. Therefore by the envelope theorem we have $\frac{\partial y}{\partial \lambda} = p_a f(k_a(z)) L_a > 0$. Similarly, $\frac{\partial y}{\partial \theta} = p_m g(k_m) L_m > 0$. It is also easy to see that $\frac{\partial y}{\partial k} = p_a \lambda f'(k_a(z)) + 1 - \delta = p_m \theta g'(k_m(z)) + 1 - \delta > 0$. This establishes monotonicity and continuity results for y. From the appendix A it follows that for all $k_a(z) < k < k_m(z)$, $\frac{\partial r}{\partial (p_a \lambda)}$ and $\frac{\partial r}{\partial (p_m \theta)}$ are well defined finite numbers. This along with the fact that $r = p_a \lambda f'(k_a(z))$, implies that the derivative $\frac{\partial y}{\partial k}$ is continuous in λ and θ. To establish concavity of $y(\cdot, z)$ and continuity of $\frac{\partial y}{\partial k}$ in k observe that $\frac{\partial^2 y}{\partial k^2} = \frac{\partial r}{\partial k} = 0$ by the fact that interest rate r is independent of k in the diversification cone.

Finally, we need to check that when $k = k_a(z)$ or $k = k_m(z)$ the left and right limits of y and its three partial derivatives are equal. But this is true since $L_a = 1$ when $k = k_a(z)$, and $L_m = 1$ when $k = k_m(z)$.

Since k and z were arbitrary we established monotonicity and continuity of both y and $\frac{\partial y}{\partial k}$ everywhere on the domain.

Finally the existence of the upper bound $\overline{k}$ on the set of sustainable capital-labor ratios is implied by the Inada conditions: $\lim_{k \to \infty} p_a \overline{\lambda} f'(k) = \lim_{k \to \infty} p_m \overline{\theta} g'(k) = 0$.

B.2 Proof of Proposition 1

The assumptions on the utility function $u(c)$ place this problem into the domain of "Bounded Return Problems", as defined in section 9.2 of Stokey, Lucas and Prescott (1989). It is straightforward to verify that their assumptions 9.4-9.12 are satisfied by our model. The results of the first part then follow from theorems 9.6, 9.7, 9.8 and 9.10 in Stokey, Lucas and Prescott (1989).

$c(0, z) = 0$ and $h(0, z) = 0$ is obvious. It is easy to show that both policy functions are strictly increasing, continuous functions of y.[6] Therefore, these policy functions inherit all the continuity and monotonicity property of y.

B.3 Proof of Proposition 2

The proof of the main theorem in Chatterjee and Shukayev (2004) can be applied to show that the minimum positive fixed point for the worst possible shock $k_{\underline{z}}^{\min}$ (where $\underline{z} = (\underline{\lambda}, \underline{\theta})$) is well defined and stable. Once this is established, the first two results of the proposition follow trivially from monotonicity and boundness of the investment policy function.

To prove the last result we will show that $k_{\underline{z}} < k_{\overline{z}}$ for any fixed points of $h(k, \underline{z})$ and $h(k, \overline{z})$ correspondingly. To show that we first will prove the following two claims:

Claim 1: For any fixed point $k_{\overline{z}}$ of $h(k, \overline{z})$ we have $1 > \beta \int_Z y'(k_{\overline{z}}, z)\eta(dz)$

Proof: From the Euler equation we have

$$u'(c(k_{\overline{z}}, \overline{z})) = \beta \int_Z u'(c(k_{\overline{z}}, z))y'(k_{\overline{z}}, z)\eta(dz)$$

[6] For example, see proofs of lemmas 1.1 and 1.2 in Brock and Mirman (1972).

Since $u'(c(k_{\overline{z}}, z)) \geq u'(c(k_{\overline{z}}, \overline{z}))$, with strict inequality for some $z \in Z$,

$$
\begin{aligned}
u'(c(k_{\overline{z}}, \overline{z})) &> \beta u'(c(k_{\overline{z}}, \overline{z})) \int_Z y'(k_{\overline{z}}, z)\eta(dz) \\
1 &> \beta \int_Z y'(k_{\overline{z}}, z)\eta(dz)
\end{aligned}
$$

Claim 2: For any fixed point $k_{\underline{z}}$ of $h(k, \underline{z})$ we have $1 < \beta \int_Z y'(k_{\underline{z}}, z)\eta(dz)$

Proof: From the Euler equation we have

$$
u'(c(k_{\underline{z}}, \underline{z})) = \beta \int_Z u'(c(k_{\underline{z}}, z))y'(k_{\underline{z}}, z)\eta(dz)
$$

Since $u'(c(k_{\underline{z}}, z)) \leq u'(c(k_{\underline{z}}, \underline{z}))$, with strict inequality for some $z \in Z$,

$$
\begin{aligned}
u'(c(k_{\underline{z}}, \underline{z})) &< \beta u'(c(k_{\underline{z}}, \underline{z})) \int_Z y'(k_{\underline{z}}, z)\eta(dz) \\
1 &< \beta \int_Z y'(k_{\underline{z}}, z)\eta(dz)
\end{aligned}
$$

The above two claims, along with the fact that $y'(k, z)$ is decreasing in k for every value of z, establish $k_{\underline{z}} < k_{\overline{z}}$.

B.4 Proof of Theorem 1

To prove convergence we check that the assumptions of Theorem 2 of Hopenhayn and

Prescott (1992) are satisfied and then apply that theorem. For brevity let us define $y' = \frac{\partial y}{\partial k}$.

First assumption that must be satisfied is that X contains its lower and upper bounds. Since, $X = \left[\underline{k}, \overline{k}\right]$ is a compact set it satisfies this condition. Next we need to show that the transition probability $P(k, B)$ is increasing in k in the sense of first-order stochastic dominance. Since $h(k, z)$ is increasing in k for every z, $P(k, B)$ is indeed increasing.

The final condition that must be satisfied is the "Monotone Mixing Condition", i.e., it remains to show that there exist some $\tilde{k} \in X$ and an integer M such that $P^M(\overline{k}, [\underline{k}, \tilde{k}]) > 0$, and $P^M(\underline{k}, [\tilde{k}, \overline{k}]) > 0$.

Consider the following set $\tilde{K} = \{k \in X | \, \beta \int_Z y'(k, z)\eta(dz) = 1\}$. Continuity and monotonicity of $y'(\cdot, z)$ for every z guarantee that $\tilde{K}$ is nonempty, although in general it may contain more than one point. Let $\tilde{k}$ be any point in $\tilde{K}$. Let the sequence $\{k_n\}_{n=0}^{\infty}$ be generated as $k_n = h(k_{n-1}, \underline{z})$ with $k_0 = \overline{k}$. By the monotonicity of optimal policy rule $\{k_n\}$ is decreasing and we know from the proposition 2 that $k_n \rightarrow k_{\underline{z}}^{\max}$. For any $\varepsilon > 0$, the rectangle $[(\underline{\lambda}, \underline{\theta}), (\underline{\lambda} + \varepsilon, \underline{\theta} + \varepsilon)]$ has a positive measure under η. This together with continuity of $h(k, \cdot)$ imply that the probability of entering into any neighborhood of $k_{\underline{z}}^{\max}$ in finite number of steps is positive.

From the Claim 2 in the proof of the proposition 2 we have $1 < \beta \int_Z y'(k_{\underline{z}}^{\max}, z)\eta(dz)$. Hence $k_{\underline{z}}^{\max} < \tilde{k}$. Exactly symmetric line of argument establishes that $k_{\overline{z}}^{\min} > \tilde{k}$ and the sequence $\{k_n\}_{n=0}^{\infty}$ started from $k_0 = \underline{k}$ enters with positive probability any neighborhood of $k_{\overline{z}}^{\min}$ in a finite number of steps. The above results prove that there exist some integer M such that $P^M(\overline{k}, [\underline{k}, \tilde{k}]) > 0$, and $P^M(\underline{k}, [\tilde{k}, \overline{k}]) > 0$. Thus, all the assumptions of Theorem 2 in Hopenhayn and Prescott (1992) are satisfied, and it establishes the desired convergence result.

The full support for this invariant distribution is $[k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}]$. To see it observe that the sequence $\{k_n\}_{n=0}^{\infty}$ generated as $k_n = h(k_{n-1}, z)$ started from any $k_0 > k_{\underline{z}}^{\max}$, enters with positive probability any neighborhood of $k_{\underline{z}}^{\max}$. Similarly, $\{k_n\}_{n=0}^{\infty}$ generated as $k_n = h(k_{n-1}, z)$ started from any $k_0 < k_{\overline{z}}^{\min}$, enters with positive probability any neighborhood of $k_{\overline{z}}^{\min}$. It is also clear that once in $[k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}]$ the Markov process $P^t(k_0, \cdot)$ cannot leave this set. Thus $k_{\underline{z}}^{\max}$, and

$k_{\overline{z}}^{\min}$ must be the boundaries of the ergodic set. To show that the whole interval $[k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}]$ is an ergodic set choose any open interval $(k^1, k^2) \in [k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}]$ of a certain length $l > 0$, and any point $k_0 \in [k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}]$. Without loss of generality assume $k_0 < k^1$. Observe that for any $k \in (k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min})$ the image $h(k, Z)$ is a non-degenerate interval $[h(k, \underline{z}), h(k, \overline{z})]$ such that k belongs to the interior of this interval. Then we can construct an increasing sequence $k_n = h(k_{n-1}, z_{n-1})$ such that $0 < \frac{\varepsilon}{2} < |k_n - k_{n-1}| < \varepsilon < \frac{l}{2}$. Clearly, this sequence will enter (k^1, k^2) in finite number of steps, say in N steps. By continuity of $h(\cdot, \cdot)$ this sequence can be constructed with a positive measure of shock histories $z^N = (z_0, z_1, ..., z_N) \in Z \times Z \times ... \times Z$ (N times). Obviously, for $k_0 > k^2$ we can construct a decreasing sequence. So we proved that $P^N(k_0, (k^1, k^2)) > 0$ for some finite N. This establishes irreducibility, and hence ergodicity of $[k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}]$.

B.5 Proof of Theorem 2

We will prove the following two claims, which together with the claims 1 and 2 in the proof of the proposition 2 establish $k_{\overline{z}}^{\min} > k_a(z^*)$ and $k_{\underline{z}}^{\max} < k_m(z^{**})$.

Claim 3: If $k \leq k_a(z^*)$ then $1 \leq \beta \int_Z y'(k, z)\eta(dz)$

Proof: For all $z \in Z$ we have $k \leq k_a(z^*) \leq k_a(z)$ and $k \leq k_a(z^*) \leq k_a^w$. Hence

$$\begin{aligned} \beta \int_Z y'(k, z)\eta(dz) &= \beta \int_Z [p_a \lambda(z) f'(k) + 1 - \delta] \, \eta(dz) \\ &= \beta [p_a f'(k) + 1 - \delta] \geq \beta [p_a f'(k_a^w) + 1 - \delta] = 1 \end{aligned}$$

Claim 4: If $k \geq k_m(z^{**})$ then $1 \geq \beta \int_Z y'(k, z)\eta(dz)$

Proof: For all $z \in Z$ we have $k \geq k_m(z^{**}) \geq k_m(z)$ and $k \geq k_m(z^{**}) \geq k_m^w$.

Hence

$$\beta \int_Z y'(k,z)\eta(dz) = \beta \int_Z \left[p_m \theta(z) g'(k) + 1 - \delta\right] \eta(dz)$$
$$= \beta \left[p_m g'(k) + 1 - \delta\right] \leq \beta \left[p_m g'(k_m^w) + 1 - \delta\right] = 1$$

Now, claim 2 of the proposition 2 and claim 4 here establish $k_{\underline{z}}^{\max} < k_m(z^{**})$, while claim 1 of the proposition 2 and claim 3 here prove $k_{\overline{z}}^{\min} > k_a(z^*)$.

B.6 Proof of Proposition 3

We will prove that for all $k \in [k_a^w, k_m^w]$ the following must be true: $\beta \int_Z y'(k,z)\eta(dz) = 1$. Once that is established the results of the proposition follow from claims 1 and 2 in the proof of the proposition 2.

Fix any $k \in [k_a^w, k_m^w]$. Then we have

$$\beta \int_Z y'(k,z)\eta(dz) = \beta \int_Z \left[p_a \lambda(z) f'(k_a^w) + 1 - \delta\right] \eta(dz)$$
$$= \beta \left[p_a f'(k_a^w) + 1 - \delta\right] = 1.$$