Note on Existence of Minimum Positive Fixed Point in the Stochastic Growth Model

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Abstract

In the context of the classical stochastic growth model we prove existence of a minimum positive fixed point for the optimal capital policy function under assumptions more general than in Brock and Mirman (1972), solving a problem which had been extensively pointed out in the literature. This proof removes one of the technical stumbling blocks in the theory of stochastic growth.

1 Introduction

The existence of a minimum positive fixed point have been a technical stumbling block in classical stochastic growth models and has received much attention. To understand the importance of this issue let us introduce the per-capita production function $f(k, \lambda)$ where $k \geq 0$ is the capital input and λ is the random variable affecting the production function. Commonly made assumptions are that $f(0, \lambda) = 0$ for all λ and that, in any period t the investment can not exceed gross output: $k_t \leq y_t = f(k_{t-1}, \lambda_t)$. Under these assumptions, $k = 0$ is an ergodic set with uninteresting and counterfactual implications. Hence, in the one-sector stochastic growth literature researchers were interested in the conditions which guarantee that the model economy avoids the trivial steady state. A sufficient condition is that the optimal capital policy function $k_t = h(k_{t-1}, \lambda_t)$ has a stable minimum positive fixed point $\underline{k} > 0$ such that for all $k \in (0, \underline{k})$, $h(k, \lambda) > k$.

Brock and Mirman (1972) have been the pioneers to scrutinize this problem. They assume that the random production shocks λ are i.i.d. draws from a compact set $[\underline{\lambda}, \overline{\lambda}]$, where $\underline{\lambda} > 0$ is the worst shock, in the sense that it gives the least amount of output for any given input. They proved that if $\Pr(\lambda = \underline{\lambda}) > 0$ then the unique optimal investment policy function $k_t = h(k_{t-1}, \lambda_t)$ possess such a minimum fixed point, given by the minimum positive fixed point of $h(\cdot, \underline{\lambda})$. In case $\Pr(\lambda = \underline{\lambda}) = 0$, however, they could not rule out the existence of an infinite number of arbitrarily small positive fixed points for $h(k, \underline{\lambda})$. Mirman and Zilcha (1976) revisited this problem and, using a specific example, proved that it is possible for $h(k, \underline{\lambda})$ to have no positive fixed point at all. In both cases, when there are infinite number of arbitrarily small fixed points or no fixed point at all, the stationary distribution of capital may have a support on an interval with zero as the lower endpoint. Thus, under the usual boundary condition on the utility function, $\lim_{c \to 0} u'(c)$, shadow prices of income and capital may diverge to infinity as capital approaches zero. This fact complicates proofs and makes researchers assume the existence of minimum positive fixed point by imposing this property directly on the policy function without establishing the sufficient conditions.[1] Also, in applied work it is often necessary to approximate the capital policy function on a compact set capturing all possible values of capital (from the given initial conditions). It is important for this invariant set not to include too small values of capital where large nonlinearities render overall approximation imprecise. It is therefore convenient to claim that the model at hand has a support of capital-labor ratios (and output) on a set bounded away

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[1] For example, Donaldson and Mehra (1983) and Olson (1989) assumed that the optimal investment policy function has a minimum positive fixed point. For the discussion of possible problems with unbounded shadow prices see Mirman and Zilcha (1976).

from zero.

Hopenhayn and Prescott (1992) suggested a line of proof for the existence of the minimum fixed point in the stochastic growth model with multiplicative TFP shocks and bounded utility function, but they did not work out the details of the proof.[2] In this note we give a complete proof along the lines of Hopenhayn and Prescott (1992), but for a more general Brock and Mirman (1972) production function, $f(k, \lambda)$. Further, we use a weaker version of the Inada conditions than in Brock and Mirman (1972).

In the next section we state the stochastic optimal growth problem studied by Brock and Mirman (1972) and Mirman and Zilcha (1975) and prove the main result.

2 Formulation of the Problem and Results

Since the purpose of this note is to prove the existence of the minimum positive fixed point in a well known problem, we will be brief in outlining the model, and will give only the proof of the main result.

Let $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the nonnegative and positive reals respectively. Given any metric space X, $\mathfrak{B}(X)$ is the Borel set of X. The capital accumulation problem evolves as follows. At the start of period t the representative agent receives income $y_t = f(k_{t-1}, \lambda_t) \in \mathbb{R}_+$. Given this income a level of consumption $c_t \in (0, y_t]$ is chosen, yielding current utility $u(c_t)$. The remainder is invested into capital $k_t \in \mathbb{R}_+$ to be used in production of the following period output $y_{t+1} = f(k_t, \lambda_{t+1})$. Next period the process is repeated. The objective of the agent is to maximize the discounted expected utility

$$E \sum_{t=0}^{\infty} \delta^t u(c_t) \tag{1}$$

where $0 < \delta \leq 1$, subject to the feasibility constraint

$$0 \leq c_t \leq f(k_{t-1}, \lambda_t), \tag{2}$$

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[2] See Hopenhayn and Prescott (1992), footnote 18 on page 1402.

and the initial condition given by $k_{-1} \in \mathbb{R}_{++}$.

Production shocks, the production and the utility functions are assumed to satisfy the following assumptions.

(A.1) The random variable λ_t is i.i.d. drawn from a compact set $\Lambda = [\underline{\lambda}, \overline{\lambda}] \in \mathbb{R}_{++}$. Let $v(B) = \Pr(\lambda \in B)$ be the probability measure defined for all $B \in \mathfrak{B}(\Lambda)$.

(A.2) The production function $f(\cdot, \lambda) : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, strictly concave, continuously differentiable function with $f(0, \lambda) = 0$ for all $\lambda \in \Lambda$.

(A.3) For all $k \in \mathbb{R}_{++}$, both $f(k, \cdot) : \Lambda \to \mathbb{R}_+$ and $\frac{\partial f(k,\cdot)}{\partial k} : \Lambda \to \mathbb{R}_+$ are continuous functions of λ.

(A.4) For all $k \in \mathbb{R}_{++}$, $f(k, \lambda) \geq f(k, \underline{\lambda}) > 0$ with strict inequality for a positive measure of λ.

(A.5) For all $\lambda \in \Lambda$, the following boundary conditions are satisfied
1) $\lim_{k \to 0} \frac{\partial f(k,\lambda)}{\partial k} > \frac{1}{\delta}$ and
2) $\lim_{k \to \infty} \frac{\partial f(k,\lambda)}{\partial k} = 0$.

(A.6) The utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing, strictly concave and continuously differentiable function satisfying the boundary condition $\lim_{c \to 0} u'(c) = \infty$.

(A.7) The utility function is bounded below. Hence, without loss of generality, we can normalize $u(0) = 0$.

Let $\mathbf{k} = (k_0, k_1, ...)$, $\mathbf{c} = (c_0, c_1, ...)$.

Definition*:* The pair $(\mathbf{k}, \mathbf{c})$ is said to be a *feasible program* from the initial output $f(k_{-1}, \lambda_0)$ if for all $t = 0, 1, ...$ it satisfies condition (2).

Let

$$V(k_{-1}, \lambda_0) = \max \sum \delta^t E\left[u(c_t) | f(k_{-1}, \lambda_0)\right] \tag{3}$$

where the maximum is taken over all $\mathbf{c}$ for which $(\mathbf{k}, \mathbf{c})$ is a feasible program from $f(k_{-1}, \lambda_0)$.

The following results are standard and will be given without proofs.[3]

Lemma: *Under the assumptions (A.1-A.7)*

(R.1) *The value function* $V : \mathbb{R}_+ \times \Lambda \to \mathbb{R}_+$ *is finite and satisfies the Bellman equation*

$$V(k, \lambda) = \max_{0 \le k' \le f(k,\lambda)} \left\{ u(f(k, \lambda) - k') + \delta \int V(k', \lambda')v(d\lambda') \right\} \tag{4}$$

(R.2) *There exist a unique optimal policy function* h *such that*

$$V(k, \lambda) = u(f(k, \lambda) - h(k, \lambda)) + \delta \int V(h(k, \lambda), \lambda')v(d\lambda') \tag{5}$$

(R.3) *For every* λ *the value function* $V(\cdot, \lambda) : \mathbb{R}_+ \to \mathbb{R}_+$ is *strictly increasing, strictly concave and differentiable in* k *with the continuous derivative given by*[4]

$$V'(k, \lambda) = u'(f(k, \lambda) - h(k, \lambda))f'(k, \lambda) \tag{6}$$

(R.4) *For every* λ *both the capital policy function* $h(\cdot, \lambda) : \mathbb{R}_+ \to \mathbb{R}_+$, *and the consumption policy function* $[f(\cdot, \lambda) - h(\cdot, \lambda)] : \mathbb{R}_+ \to \mathbb{R}_+$ *are strictly increasing and continuous.*

(R.5) *For all* $k \in \mathbb{R}_+$ *both the capital policy function* $h(k, \cdot) : \Lambda \to \mathbb{R}_+$, *and the consumption policy function* $[f(k, \cdot) - h(k, \cdot)] : \Lambda \to \mathbb{R}_+$ *are continuous.*

(R.6) *For every* $k \in \mathbb{R}_{++}$ *the optimal policy function satisfies* $h(k, \lambda) \geq h(k, \underline{\lambda})$ *with strict inequality for a positive measure of* λ.

(R.7) *The optimal policy function* h *satisfies*

$$u'(f(k, \lambda) - h(k, \lambda)) = \delta \int V'(h(k, \lambda), \lambda')v(d\lambda') \tag{7}$$

(R.8) *For all* λ, $V(0, \lambda) = 0$ *and* $h(0, \lambda) = 0$.

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[3] For details and proofs see Stokey, Lucas and Prescott (1989), Mirman and Zilcha (1975), and Brock and Mirman (1972).

[4] Everywhere below we denote $V'(k, \lambda) = \frac{\partial V(k,\lambda)}{\partial k}$, $f'(k, \lambda) = \frac{\partial f(k,\lambda)}{\partial k}$.

Now we are ready to prove the main result.

Theorem: *Under the assumptions (A.1-A.7) there exists* $\varepsilon > 0$*, such that for all* $k \in (0, \varepsilon)$ *and for all* $\lambda \in \Lambda$, $h(k, \lambda) > k$.

Proof: To prove the result we will show that there exists $\varepsilon > 0$ such that for all $k \in (0, \varepsilon)$ $\; h(k, \underline{\lambda}) > k$. Once this is established the result follows from the property R.6 in the previous lemma.

Consider a deterministic version of the same problem with λ fixed at $\underline{\lambda}$ with probability one. Let $\underline{V}(k)$ and $\underline{h}(k)$ be the value function and the optimal capital policy functions for this deterministic problem respectively. Using a deterministic counterpart of the previous lemma $\underline{V}(k)$ is strictly increasing, strictly concave and continuously differentiable with $\underline{V}(0) = 0$. Similarly, capital policy function $\underline{h}(k)$ is strictly increasing and continuous with $\underline{h}(0) = 0$. Also, $\underline{V}$ and $\underline{h}$ satisfy

$$u'(f(k, \underline{\lambda}) - \underline{h}(k)) = \delta \underline{V}'(\underline{h}(k)) \tag{8}$$

The assumption A.5 implies that there exists a unique $\underline{k} > 0$ that solves the steady state condition $1 = \delta f'(\underline{k}, \underline{\lambda})$. The strict concavity of u and $f(\cdot, \underline{\lambda}) : \mathbb{R}_+ \to \mathbb{R}_+$, together with R.4 imply that $\forall k \in (0, \underline{k})$, $\underline{h}(k) \in (k, \underline{k})$.

By A.4 and strict monotonicity of u, for any $k > 0$ and any $\lambda \in \Lambda$, the value function for the stochastic problem $V(k, \lambda)$ is strictly larger than $\underline{V}(k)$. But for all λ, we have $V(0, \lambda) = \underline{V}(0) = 0$. We also know from the result R.3 that for every λ, $V'(k, \lambda)$ is continuous in k. For every $\lambda \in \Lambda$ define

$$\tau(\lambda) = \min \left\{ \underline{k}, \min \left\{ k > 0 \mid V'(k, \lambda) \leq \underline{V}'(k) \right\} \right\}.$$

To show that $\tau(\lambda)$ is well defined for all $\lambda \in \Lambda$ suppose, on the contrary, that for some $\widehat{\lambda}$

$$\inf \left\{ k > 0 \mid V'(k, \widehat{\lambda}) \leq \underline{V}'(k) \right\} = 0.$$

In this case there must exist an infinite sequence $\{k^n\} \in \mathbb{R}_{++}$ such that $\lim_{n \to \infty} k^n = 0$ and for all $n = 1, 2, ...$ we have $V'(k^n, \widehat{\lambda}) - \underline{V}'(k^n) \leq 0$. It follows that

$$\lim_{n \to \infty} \left[V'(k^n, \widehat{\lambda}) - \underline{V}'(k^n) \right] \leq 0. \tag{9}$$

At the same time observe that for all k^n, we have $\int_0^{k^n}(V'(k,\widehat{\lambda}) - \underline{V}'(k))dk = V(k^n,\widehat{\lambda}) - \underline{V}(k^n) > 0$. Hence there must exist an infinite sequence $\{y^n\} \in \mathbb{R}_{++}$ such that for all n, $y^n \in (0,\, k^n)$ and $\left[V'(y^n,\widehat{\lambda}) - \underline{V}'(y^n)\right] > 0$. It follows that

$$\lim_{n\to\infty}\left[V'(y^n,\widehat{\lambda}) - \underline{V}'(y^n)\right] \geq 0. \tag{10}$$

From (9), (10) and continuity of both $V'(\cdot,\widehat{\lambda})$ and $\underline{V}'(\cdot)$ it follows that

$$\lim_{k\to 0}\left[V'(k,\widehat{\lambda}) - \underline{V}'(k)\right] = 0.$$

This implies that there must exist some small neighborhood $(0,\eta)$ of zero, such that $\left[V'(\cdot,\widehat{\lambda}) - \underline{V}'(\cdot)\right]$ is strictly increasing and hence strictly positive everywhere on it. Which contradicts $V'(k^n,\widehat{\lambda}) - \underline{V}'(k^n) \leq 0$ for all n. The contradiction proves that $\tau(\lambda)$ is well defined for all $\lambda \in \Lambda$, and that $\forall\ k \in (0,\tau(\lambda))$, $V'(k,\lambda) > \underline{V}'(k)$.

We need to show next that $\min_{\lambda\in\Lambda}\{\tau(\lambda)\}$ exists (and hence is strictly positive). Define function

$$\psi(k,\lambda) = V'(k,\lambda) - \underline{V}'(k). \tag{11}$$

It is continuous in k, and for each λ, it must be above zero for all $k \in (0,\tau(\lambda))$. Suppose $\xi = \inf_{\lambda\in\Lambda}\{\tau(\lambda)\}$ is equal to zero. Then there must exist a sequence $\{\lambda^n\} \in \Lambda$, such that $\lim_{n\to\infty}\tau(\lambda^n) = 0$. Since Λ is compact, there must exist $\widetilde{\lambda} \in \Lambda$ such that $\lim_{n\to\infty}\lambda^n = \widetilde{\lambda}$. By definition $\tau(\widetilde{\lambda}) > 0$, and $\psi(k,\widetilde{\lambda}) > 0$ for all $k \in (0,\tau(\widetilde{\lambda}))$.

Now, consider the $\lim_{k\to 0}\psi(k,\widetilde{\lambda})$. It is either equal to zero, or is strictly positive. Suppose first, it is equal to zero. Then for k close enough to zero, say in $(0,\frac{\tau(\widetilde{\lambda})}{100}]$, $\psi(\cdot,\widetilde{\lambda})$ must be strictly increasing, and it cannot be arbitrary close to zero on the compact interval $[\frac{\tau(\widetilde{\lambda})}{100},\frac{\tau(\widetilde{\lambda})}{2}]$ since otherwise it must be equal to zero on some point in it. By R.3

$$\psi(k,\lambda) = u'(f(k,\lambda) - h(k,\lambda))f'(k,z) - \underline{V}'(k) \tag{12}$$

Hence by R.5, A.3, and A.6, $\psi(k,\cdot)$ is continuous in λ. Then for $n > N_1$ large enough, and for k close enough to zero, functions $\psi(\cdot,\lambda^n)$ must also be strictly increasing. Moreover, for $n > N_2$ large enough, and for all $k \in (0, \frac{\tau(\widetilde{\lambda})}{2}]$, $\psi(k, \lambda^n) > 0$. It follows that for $n > \max\{N_1, N_2\}$ we must have $\tau(\lambda^n) > \frac{\tau(\widetilde{\lambda})}{2}$. But this contradicts $\lim_{n\to\infty}\{\tau(\lambda^n)\} = 0$.

Next, if $\lim_{k\to 0}\psi(k, \widetilde{\lambda}) > 0$ then $\psi(\cdot, \widetilde{\lambda})$ cannot be arbitrary close to zero anywhere on $(0, \frac{\tau(\widetilde{\lambda})}{2}]$. By continuity of $\psi(k, \cdot)$ we again arrive at the above contradiction. This contradiction proves $\xi = \min_{\lambda\in\Lambda}\{\tau(\lambda)\} > 0$ is well defined, and $\xi \leq \underline{k}$. Hence for all $k' \in (0, \xi)$, $\int_\Lambda V'(k', \lambda)v(d\lambda) > \underline{V}'(k')$. By continuity and strict monotonicity of $\underline{h}$, the inverse image of $(0, \xi)$ is an open (nondegenerate) interval $(0, \underline{h}^{-1}(\xi)) \subset (0, \xi) \subset (0, \underline{k})$. Let $\varepsilon = \underline{h}^{-1}(\xi)$, and $k \in (0, \varepsilon)$. Suppose $h(k, \underline{\lambda}) \leq \underline{h}(k)$. Then $\int_\Lambda V'(h(k, \underline{\lambda}), \lambda)v(d\lambda) > \underline{V}'(\underline{h}(k))$ and we can write the following Euler equations

$$\begin{aligned} u'\left[f(k, \underline{\lambda}) - h(k, \underline{\lambda})\right] &= \delta \int_Z V'(h(k, \underline{\lambda}), \lambda)v(d\lambda) \\ &> \delta \underline{V}'(\underline{h}(k)) = u'\left[f(k, \underline{\lambda}) - \underline{h}(k)\right]. \end{aligned} \tag{13}$$

But this contradicts the concavity of u. This contradiction proves $h(k, \underline{\lambda}) > \underline{h}(k) > k$ for all $k \in (0, \varepsilon)$. Q.E.D.

Once the previous theorem is established the existence of the minimum positive fixed point of $h(\cdot, \underline{\lambda})$ is assured by the second part of the assumption A.5.

Note that unlike Brock and Mirman (1972), we do not require that the marginal product of capital at zero capital is infinity. In their paper that assumption was crucial for proving existence of positive fixed points for any λ. In some applications researchers use CES production functions which do not satisfy the above condition, so our result is useful in such cases.

Finally, we should comment that the assumption A.7. is crucial for our results. It was the only assumption violated in Mirman and Zilcha (1976) example of a stochastic growth model with no positive fixed points for $h(k, \underline{\lambda})$.

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