

Economics 3012
Strategic Behavior
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Lecture 4

Topics

- Problem Set 3
- Final Remarks on Auction Theory
- Mixed Strategy Nash Equilibrium

Problem Set 2

Exercise 49.1

Problem Description:

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$$u_i(y_1, y_2) = \begin{cases} f(y_1, y_2), & y_i > y_j, \\ \frac{1}{2}f(y_1, y_2), & y_i = y_j, \\ 0, & y_i < y_j, \end{cases}$$

where f is continuous, decreasing in both variables, and satisfies $f(1, 1) = 0$.

- Taking the question very literally, Exercise 79.2 is a special case when $v_1 = v_2 = 1$.
 - Set $f(y_1, y_2) = 1 - \min\{y_1, y_2\}$.
 - * *This is a continuous function.*
 - Interpret y_1 and y_2 as the two amounts of time.
- Up to rescaling, the two games are the same whenever $v_1 = v_2$.

Analysis:

The only Nash equilibrium is $(y_1, y_2) = (1, 1)$.

- If $y_1 \neq y_2$, then the player devoting more resources to fighting can increase her payoff by reducing the resources devoted to fighting slightly.
- If $y_1 = y_2 < 1$ then (because f is continuous) either agent can increase her payoff by increasing the resources devoted to fighting slightly, obtaining all of a slightly reduced output.
- If $y_1 = y_2 = 1$, neither agent can increase her payoff by reducing the resources devoted to fighting because the result is to deliver all the output to the other agent.

Exercise 86.1

Problem Description:

- Each agent $i = 1, 2$ has a unique favorite action in a set of alternatives A .
 - Let a_1^* and a_2^* be the favorite actions.
- Each person chooses a bid m_i .
- The high bidder pays the other person's bid and implements their favorite action:

$$U_i(m_1, m_2) = \begin{cases} u_i(a_i^*) - m_j, & m_i > m_j, \\ \frac{(u_i(a_i^*) - m_i) + u_i(a_j^*)}{2}, & m_i = m_j, \\ u_i(a_j^*), & m_i < m_j. \end{cases}$$

- Examples:
 - You and your friend want to go on vacation together, with
$$A = \{ \text{Paris, Rio de Janeiro} \}.$$
 - The Roman Legions auctioned off the Emperorship plus execution of the rival.

Analysis

- Let $m_1^* = u_1(a_1^*) - u_1(a_2^*)$ and $m_2^* = u_2(a_2^*) - u_2(a_1^*)$.
- It is a weakly dominant strategy for person 1 to bid m_1^* :
 - If player 2 bids less than this, then player 1 prefers winning, thereby paying player 2's bid but enjoying a_1^* , to the relevant alternative, which is to lose and suffer a_2^* .
 - If player 2 bids more than m_1^* , then player 1 prefers suffering a_2^* to paying enough to change the outcome.

Exercise 97.1

Problem Description:

- The liability rule is that $X_1 = \infty$ and $X_2 = 0$, so that the injurer is liability regardless of the levels of care:

$$u_1(a_1, a_2) = -a_1 - L(a_1, a_2),$$

$$u_2(a_1, a_2) = -a_2.$$

Analysis:

- For the victim $a_2 = 0$ is a dominant strategy.
- Taking $a_2 = 0$ as given, the injurer will choose a_1 to maximize $-a_1 - L(a_1, a_2)$.
- There is no clear relationship between the injurer's level of care \hat{a}_1 in the socially optimal pair (\hat{a}_1, \hat{a}_2) and a_1^* : either could be greater than the other, depending on whether the two types of care are
 - substitutes;
 - complements.

Multiunit Auctions

Case I: Identical objects: suppose the n bidders are competing for m identical objects. Each bidder submits a list $b_i = (b_i^1, \dots, b_i^k)$ with $b_i^1 \geq \dots \geq b_i^k$. The objects are awarded to the m largest bids.

- There are several payment rules that can be used:
 - *Discriminatory*: each winning bid pays the amount bid.
 - *Uniform Price*: all winning bids pay the highest rejected bid.
 - *Vickrey*: A bidder winning k objects pays the sum of the k highest bids submitted by other bidders.
 - * In the Vickrey auction bidding your true values is weakly dominant: your bid affects whether you win but not how much you pay when you win.
 - * Examples show that the other two rules do not have this property.

Case II: Nonidentical Objects and Positive Complementarities

- Leaving incentives aside, hard problems abound:
 - Suppose each $i = 1, \dots, n$ has a value v_{ik} for each object $k = 1, \dots, m$.
 - * How hard is it (in a computational sense) to find an allocation of at most one object to each bidder that maximizes the sum of values?
 - When bidders' values of packages of objects may differ from the sum of values of individual objects, the value maximization problem is computationally intractable.
- Spectrum Auctions in the US
 - Simultaneous ascending bid.
 - * The intent was to encourage information flows.
 - It was vulnerable to implicit collusion.
 - Results in practice were variable.

Chopstick Auction

Problem (not in Osborne) Description:

- Each $i = 1, 2$ submits bids (b_i^1, b_i^2, b_i^3) on three identical objects.
- Each object is sold to the high bidder, with the bidder paying her bid.
 - In the event of a tie, the winner is decided by coin flip.
- Each agent loses the money spent on winning bids and wins \$1 if she wins two or more objects.
- *That is, one chopstick is worth nothing, and the value of three chopsticks is the same as the value of two.*

Analysis:

- There is no Nash equilibrium.
 - Say that player i 's strategy is *vulnerable* if $b_i^k + b_i^\ell < 1$ for some distinct $k, \ell = 1, 2, 3$.
 - In any Nash equilibrium one player's strategy must be invulnerable, and the other player must win no objects.
 - If, in equilibrium, player i is playing the invulnerable strategy (b_i^1, b_i^2, b_i^3) , then player i wins all three objects, paying

$$\begin{aligned} & b_i^1 + b_i^2 + b_i^3 \\ &= \frac{1}{2}(b_i^1 + b_i^2) + \frac{1}{2}(b_i^1 + b_i^3) + \frac{1}{2}(b_i^2 + b_i^3) \\ &\geq 3/2 \end{aligned}$$

and could do better by bidding $(0, 0, 0)$.

- A remarkable, and remarkably difficult, paper by Rosenthal and Szentes gives an equilibrium in mixed strategies.

Mixed Strategies

Example: Penalty Kick

- Problem Description:
 - The kicker and the goalie each choose whether to go left or right. The advantage is with the kicker if they choose opposite directions, and with the goalie if they choose the same direction:

$$\begin{array}{cc} & L_2 & R_2 \\ \begin{array}{c} L_1 \\ R_1 \end{array} & \left(\begin{array}{cc} (-1, 1) & (1, -1) \\ (1, -1) & (-1, 1) \end{array} \right) \end{array}$$

- This game is also called *Matching Pennies*.
- Analysis:
 - In general, a Nash equilibrium is *a pair of mutually consistent expectations about the other agent's behavior*.
 - In this case these expectations must be uncertain.

- Let p be the probability that player 1 chooses L_1 .
- Let q be the probability that player 2 chooses L_2 .
- If P and q constitute a Nash equilibrium, then:
 - (a) if $p > 0$, then choosing L_1 is at least as good as choosing R_1 , *given* q ;
 - (b) if $1 - p > 0$, then choosing R_1 is at least as good as choosing L_1 , *given* q .
- and:
 - (c) if $q > 0$, then choosing L_2 is at least as good as choosing R_2 , *given* p ;
 - (d) if $1 - q > 0$, then choosing R_2 is at least as good as choosing L_2 , *given* p .
- Note that $p = 0$ implies that $q = 0$ (by (c)) which implies that $p = 1$ (by (b)) which implies that $q = 1$ (by (d)) which implies that $p = 0$ (by (a)).

- Therefore any Nash equilibrium has $0 < p < 1$ and $0 < q < 1$.
 - Now (a) and (b) imply that L_1 and R_1 must be equally good for player 1:
$$q \cdot -1 + (1 - q) \cdot 1 = q \cdot 1 + (1 - q) \cdot -1.$$
 - Similarly, (c) and (d) imply that L_2 and R_2 must be equally good for player 2:
 - From (c) and (d) we have
$$p \cdot 1 + (1 - p) \cdot -1 = p \cdot -1 + (1 - p) \cdot 1.$$
- Therefore $p = \frac{1}{2}$ and $q = \frac{1}{2}$.

A Conceptual Point

Paradox: Although we refer to the assignment of p to L_1 and $1 - p$ to R_1 as *player 1's mixed strategy*, it is adjusted to make player 2 indifferent between L_2 and R_2 .

- It *is* a solution to the problem of maximizing player 1's expected payoff, but in reality player 1 usually just chooses L_1 or R_1 without first “choosing” p .
- It works better to think of the assignment of p to L_1 and $1 - p$ to R_1 as *player 2's expectations concerning player 1's behavior*.

Example: Penalty Kick with a Strong Side

- Problem Description:
 - The kicker and the goalie each choose whether to go left or right, but now there is a goal for sure if they choose opposite directions, a goal with probability one third if they both choose right, and a goal with probability two thirds if they both choose left:

$$\begin{array}{cc} & L_2 & R_2 \\ \begin{array}{c} L_1 \\ R_1 \end{array} & \left(\begin{array}{cc} (\frac{2}{3}, -\frac{2}{3}) & (1, -1) \\ (1, -1) & (\frac{1}{3}, -\frac{1}{3}) \end{array} \right) \end{array}$$

- There is still no pure Nash equilibrium.
 - What we have called “Nash equilibrium” up to now is (in this more general setting) called a *Nash equilibrium in pure strategies* or a *pure Nash equilibrium*.

Analysis:

- In any Nash equilibrium with $0 < p < 1$ and $0 < q < 1$:

- L_1 and R_1 must be equally good for player 1:

$$q \cdot \frac{2}{3} + (1 - q) \cdot 1 = q \cdot 1 + (1 - q) \cdot \frac{1}{3},$$

$$1 - \frac{1}{3}q = \frac{1}{3} + \frac{2}{3}q, \quad q = \frac{2}{3}.$$

- L_2 and R_2 must be equally good for player 2:

- From (c) and (d) we have

$$p \cdot -\frac{2}{3} + (1 - p) \cdot -1 = p \cdot -1 + (1 - p) \cdot -\frac{1}{3},$$

$$-1 + \frac{1}{3}p = -\frac{2}{3}p - \frac{1}{3}, \quad p = \frac{2}{3}.$$

- Therefore $p = \frac{2}{3}$ and $q = \frac{2}{3}$.

Example: Battle of the Sexes

- Problem Description:
 - There are two different ways to coordinate behavior, e.g., we can all drive on the left or all drive on the right, and the agents differ as to which is better:

$$\begin{array}{cc} & L_2 & R_2 \\ \begin{array}{c} L_1 \\ R_1 \end{array} & \left(\begin{array}{cc} (10, 9) & (0, 0) \\ (0, 0) & (9, 10) \end{array} \right) \end{array}$$

- Now (L_1, L_2) and (R_1, R_2) are pure Nash equilibria, but there is also a mixed equilibrium.

Analysis:

- In a Nash equilibrium with $0 < p < 1$ and $0 < q < 1$:

- L_1 and R_1 must be equally good for player 1:

$$q \cdot 10 + (1 - q) \cdot 0 = q \cdot 0 + (1 - q) \cdot 9.$$

- L_2 and R_2 must be equally good for player 2:

$$p \cdot 9 + (1 - p) \cdot 0 = p \cdot 0 + (1 - p) \cdot 10.$$

- Therefore $p = \frac{10}{19}$ and $q = \frac{9}{19}$.

Conceptual Point

This equilibrium seems quite unstable and implausible if there is some mechanism (e.g., observing how other people play the game) for coordinating expectations. If there is no such mechanism uncoordinated outcomes may occur, but one would not expect them to be “equilibrated.”

The General 2×2 Game

- Problem Description:
 - The 2×2 game with general payoffs is:

$$\begin{array}{cc} & L & R \\ U & (a_1, a_2) & (b_1, c_2) \\ D & (c_1, b_2) & (d_1, d_2) \end{array}$$

- Assume that:
 - $a_1 \neq c_1, b_1 \neq d_1, a_2 \neq c_2, b_2 \neq d_2$;
 - neither player has a dominant strategy.
- Then there will be a Nash equilibrium with $0 < p < 1$ and $0 < q < 1$.

Analysis:

- If any Nash equilibrium with $0 < p < 1$ and $0 < q < 1$:

- U and D must be equally good for player 1:

$$q \cdot a_1 + (1 - q) \cdot b_1 = q \cdot c_1 + (1 - q) \cdot d_1.$$

- L and R must be equally good for player 2:

$$p \cdot a_2 + (1 - p) \cdot b_2 = p \cdot c_2 + (1 - p) \cdot d_2.$$

- Therefore

$$p = \frac{b_2 - d_2}{(b_2 - d_2) - (a_2 - c_2)} \text{ and } q = \frac{b_1 - d_1}{(b_1 - d_1) - (a_1 - c_1)}.$$

Conceptual Point

The equilibrium mixing probabilities depend only on the *differences* in utility induced by changes in pure strategy. That is, only the numbers $a_1 - c_1$, $b_1 - d_1$, $a_2 - c_2$, and $b_2 - d_2$ affect the equilibrium strategies. Also, multiplying all these numbers by a positive constant does not change the outcome.

von Neumann-Morgenstern Utility

- The set of pure equilibria depends only on the *orderings* of the strategy profiles induced by the utilities.
 - For a two player game given by pure strategy sets S_1 and S_2 and payoff functions $u_1, u_2 : S \rightarrow \mathbf{R}$ (where $S = S_1 \times S_2$) suppose that $f_1, f_2 : \mathbf{R} \rightarrow \mathbf{R}$ are increasing functions and $\tilde{u}_i(s_1, s_2) = f_i(u_i(s_1, s_2))$ for both i and all $(s_1, s_2) \in S$.
 - * Then (s_1^*, s_2^*) is a (pure) Nash equilibrium of the game

$$(S_1, S_2; u_1, u_2)$$

if and only if it is a (pure) Nash equilibrium of the game

$$(S_1, S_2; \tilde{u}_1, \tilde{u}_2).$$

- * That is, a way to increase u_i is a way to increase \tilde{u}_i and vice versa.

- In a mixed Nash equilibrium each pure strategy gives rise to a *lottery* over the pure outcomes.
- We assume that utility of a lottery for agent i is its *expected utility*: if $u_i(A) = 5$ and $u_i(B) = 20$, then

$$u_i\left(\frac{1}{5}A + \frac{4}{5}B\right) = \frac{1}{5} \cdot 5 + \frac{4}{5} \cdot 20 = 17.$$

- This assumption imposes strong restrictions on the structure of preferences over lotteries.
- Example:
 - (a) Choose between:
 - (i) $0.90 \times \$0 + 0.10 \times \$50M$;
 - (ii) $0.89 \times \$0 + 0.11 \times \$10M$.
 - (b) Choose between:
 - (iii) $\$10M$ for sure;
 - (iv) $0.89 \times \$10M + 0.10 \times \$50M + 0.01 \times \$0$.

– Many people choose (i) over (ii) and (iii) over (iv).

- Rephrasing:
 - (a) Choose between:
 - (i) Getting \$0 89% of the time, and the other 11% of the time getting $\frac{10}{11}\$50M + \frac{1}{11}\0 .
 - (ii) Getting \$0 89% of the time, and the other 11% of the time getting \$10M.
 - (b) Choose between:
 - (iii) Getting \$10M 89% of the time, and the other 11% of the time getting \$10M.
 - (iv) Getting \$10M 89% of the time, and the other 11% of the time getting $\frac{10}{11}\$50M + \frac{1}{11}\0 .
- Preferences over lotteries generated by expected utility satisfy a property called *Independence*, which in this example implies that in each case your preferences are determined by your preference between \$10M for sure and $\frac{10}{11}\$50M + \frac{1}{11}\0 .

Strategic Games with Randomization

Notation: For any finite set X , $\Delta(X)$ is the space of probability distributions on X . A *probability distribution on X* is a function

$$\mu : X \rightarrow [0, 1] \text{ such that } \sum_{x \in X} \mu(x) = 1.$$

Depending on context, probability distributions on X will sometimes be called *lotteries* on X .

Remark: Sometimes we write μ_x rather than $\mu(x)$. Also, we often adopt special notation if it is convenient, such as $(p, 1 - p)$ or (p_1, q_1, r_1) .

A *strategic game with randomization* consists of:

- a set of *players* I ;
- for each player i , a set S_i of *action*;
- for each $i \in I$, preferences over lotteries on $S = \prod_{i \in I} S_i$ that are generated by taking expected utility with respect to some

$$u_i : S \rightarrow \mathbb{R}.$$

Mixed Strategies, Mixed Strategy Profiles

- A *mixed strategy* for player i is a probability distribution on S_i .
- A *mixed strategy profile* is a specification

$$\alpha \in \prod_{i \in I} \Delta(S_i)$$

of a mixed strategy α_i for each agent.

- For each player i a *complementary strategy profile* is a specification

$$\alpha_{-i} \in \prod_{j \in I, j \neq i} \Delta(S_j)$$

of a mixed strategy α_j for each agent other than i .

- Given a mixed strategy profile α and each agent i , α_{-i} denotes the complementary strategy profile obtained by deleting α_i .
- Given a complementary strategy profile α_{-i} and a mixed strategy β_i for i , (β_i, α_{-i}) is the mixed strategy profile obtained by combining β_i with the components of α_{-i} .

Expected Payoffs

- A mixed strategy profile α induces a probability distribution on S in which the probability of a pure strategy profile s is

$$\prod_{i \in S} \alpha_i(s_i).$$

(Sometimes \prod_i denotes a cartesian product of sets, but here it denotes numerical multiplication.)

- This way of computing the probability of s embeds an assumption that the players are acting *independently* in a statistical sense: the action I choose does not influence the probability that you will choose a particular action.
 - This assumption is justified if *the game itself already incorporates all methods the agents have for communicating or coordinating their behavior.*

Expected Payoffs and Mixed Nash Equilibrium

- The expected payoff of agent i at a mixed strategy profile α is the sum over pure strategy profiles of the probability of that profile times agent i 's utility when it occurs:

$$u_i(\alpha) = \sum_{s \in S} \left(\prod_i \alpha_i(s_i) \right) u_i(s).$$

- A mixed strategy profile α^* is a (mixed strategy) *Nash equilibrium* if, for every player i there is no way to increase expected utility:

$$u_i(\alpha_i, \alpha_{-i}^*) \leq u_i(\alpha^*)$$

for every $\alpha_i \in \Delta(S_i)$.

The Best Response Correspondence

In the same way as before we can use best response correspondences to find all the Nash equilibria. Consider Penalty Kick with a Strong Side:

$$\begin{array}{cc} & L_2 & R_2 \\ \begin{array}{c} L_1 \\ R_1 \end{array} & \left(\begin{array}{cc} (\frac{2}{3}, -\frac{2}{3}) & (1, -1) \\ (1, -1) & (\frac{1}{3}, -\frac{1}{3}) \end{array} \right) \end{array}$$

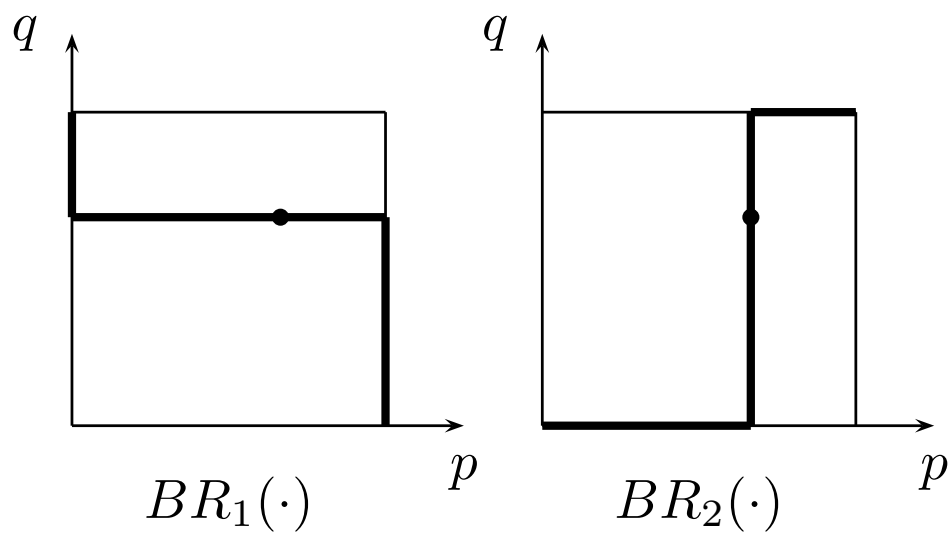
$$BR_1(q) = \begin{cases} 1, & q < \frac{2}{3}, \\ [0, 1], & q = \frac{2}{3}, \\ 0, & q > \frac{2}{3}. \end{cases}$$

$$BR_2(p) = \begin{cases} 0, & p < \frac{2}{3}, \\ [0, 1], & p = \frac{2}{3}, \\ 1, & p > \frac{2}{3}. \end{cases}$$

The graphs of the two best response correspondences intersect at

$$(p, q) = \left(\frac{2}{3}, \frac{2}{3}\right),$$

which is the unique Nash equilibrium.



Best Responses for Battle of the Sexes

The same method can be used to find all Nash equilibria of games like Battle of the Sexes.

Consider Penalty Kick with a Strong Side:

$$\begin{array}{cc} & L_2 & R_2 \\ \begin{array}{c} L_1 \\ R_1 \end{array} & \left(\begin{array}{cc} (10, 9) & (0, 0) \\ (0, 0) & (9, 10) \end{array} \right) \end{array}$$

$$BR_1(q) = \begin{cases} 0, & q < \frac{10}{19}, \\ [0, 1], & q = \frac{10}{19}, \\ 1, & q > \frac{10}{19}. \end{cases}$$

$$BR_2(p) = \begin{cases} 0, & p < \frac{9}{19}, \\ [0, 1], & p = \frac{9}{19}, \\ 1, & p > \frac{9}{19}. \end{cases}$$

The graphs of the two best response correspondences intersect at the Nash equilibria

$$(0, 0), \left(\frac{9}{19}, \frac{10}{19}\right), (1, 1).$$

