Andrew McLennan **January 26, 1999**

Economics 5113
Introduction to Mathematical Economics
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Lecture 6
Maximization of Univariate Functions

I. Introduction

A. In this lecture we consider the application of the basic results concerning optimization to certain economic models.

1. We begin with a review of the mathematical results concerning optimization of single variable functions.

2. We then consider the following economic problems.

a. Profit maximization for a price taking firm.

b. Profit maximization for a monopolist.

c. Consumer choice in a two good model.

d. The issue of time consistency.

II. Taylor's Theorem and the Theory of Local Maxima

A. The result we wish to review is that a necessary condition for a point x^* to be a local maximum of a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ is that the first derivative of f vanishes at x^*: $f'(x^*) = 0$.

1. First we will prove this directly.

2. Then we will use it to prove Rolle's theorem and Taylor's theorem.

3. Then we will show how this an other conditions of maximization are consequence of Taylor's theorem.

B. The basic result, and Rolle's theorem.

Proposition 1: Let $f : U \to \mathbb{R}$ be a function with open domain $U \subset \mathbb{R}$. If $x* \in \arg\max_{x \in U} f(x)$, then $f'(x^*) = 0$ if $f'(x^*)$ is defined.

Proof: When $f'(x^*)$ is defined we have

$$f'(x^*) = \lim_{x \to x^*} \frac{f(x) - f(x^*)}{x - x^*}.$$

Since x^* is a maximizer, the difference quotient is nonnegative when $x < x^*$ and nonpositive when $x > x^*$, so 0 is the only possible limit. ∎

1. We will use the following in proving Taylor's theorem.

Rolle's Theorem: If $a < b$, $[a, b] \subset U$, and f is differentiable at each point in (a, b), then there is a point $x \in (a, b)$ with

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Proof: If $f(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x - a)$ for all x, then any point in (a, b) will do. If $f(x) > f(a) + \frac{f(b)-f(a)}{b-a}(x - a)$ for some x, we may choose $x*$ to maximize $f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a)$, applying Proposition 1 above to obtain the desired result, and if $f(x) < f(a) + \frac{f(b)-f(a)}{b-a}(x - a)$ for some x, we minimize instead of maximizing, again obtaining the desired point. ∎

C. Taylor's theorem.

1. Fix an open set $U \subset \mathbb{R}$ and a function $f : U \to \mathbb{R}$. We say that f is *continuously differentiable*, or C^1, if f is differentiable at each point in U and f' is a continuous function.
2. For integers $r > 1$ we say that f is *r-times continuously differentiable*, or C^r, if f is C^{r-1} and $f^{(r-1)}$ is C^1. (This is an example of an *inductive definition*.)

Taylor's Theorem: If f is C^r, then for all $x_0 \in \mathbb{R}$ and all $\epsilon > 0$ there exists $\delta > 0$ such that for all x with $x_0 - \delta < x < x_0 + \delta$,

$$\left| f(x) - [f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \ldots + \frac{1}{r!} f^{(r)}(x_0)(x - x_0)^r] \right| \leq \epsilon |x - x_0|^r.$$

Proof:

Simplification: Let

$$g(x) = f(x) - \left[f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \ldots + \frac{1}{r!} f^{(r)}(x_0)(x - x_0)^r \right].$$

If the result is true for g, then it is true for f. But g is C^r, like f, and in addition its first r derivatives vanish. The upshot is that we may assume that

$$f'(x_0) = f''(x_0) = \ldots = f^{(r)} = 0.$$

Case 1 – $r = 1$: In this case the assertion of Taylor's theorem is basically a rewriting of the definition of the assertion that $f'(x_0) = 0$, namely that $|\frac{f(x) - f(x_0)}{x - x_0}| < \epsilon$ for $x \neq x_0$ sufficiently close to x_0.

Case 2 – $r > 1$: To finish the proof we apply induction, assuming that the claim has been established for $r - 1$, so that it holds for f' in particular. Fixing an arbitrary $\epsilon > 0$, choose $\delta > 0$ small enough that if $|x - x_0| < \delta$, then $|f'(x)| \leq \epsilon |x - x_0|^{(r-1)}$. If $|x - x_0| < \delta$, then Rolle's theorem implies the existence of a number z between x_0 and x such that $\frac{f(x) - f(x_0)}{x - x_0} = f'(z)$, so that we have

$$|f(x) - f(x_0)| = |f'(z)| \cdot |x - x_0| \leq \epsilon |z - x_0|^{r-1} |x - x_0| \leq \epsilon |x - x_0|^r.$$

This completes the proof.

D. Necessary and Sufficient Conditions for a Local Maximum

1. We say that x^* is a *local maximum* for f if there is some $\delta > 0$ such that $f(x) \leq f(x^*)$ for all x with $x^* - \delta < x < x^* + \delta$.

Proposition 2: If f is C^2 and x^* is a local maximum for f, then $f''(x^*) \leq 0$.

Proof: In order to obtain a contradiction we suppose that $f''(x^*) > 0$. Proposition 1 implies that $f'(x^*) = 0$, so Taylor's theorem allows us to choose $\delta > 0$ small enough that

$$\left| f(x) - f(x^*) - \frac{1}{2} f''(x^*)(x - x^*)^2] \right| \leq \frac{1}{3}(f''(x^*))(x - x^*)^2.$$

Consequently, if $|x - x^*| < \delta$ and $x \neq x^*$, then

$$\begin{aligned} f(x) &\geq f(x^*) + \frac{1}{2} f''(x^*)(x - x^*)^2) - \frac{1}{3} f''(x^*)(x - x^*)^2 \\ &= f(x^*) + \frac{1}{6} f''(x^*)(x - x^*)^2 > f(x^*), \end{aligned}$$

contradicting the assumption that x^* is a local maximum. ∎

Proposition 3: If f is C^2, $f'(x^*) = 0$, and $f''(x^*) < 0$, then x^* is a local maximum for f.

Proof: By Taylor's theorem we may choose $\delta > 0$ small enough that if $|x - x^*| < \delta$, then

$$\begin{aligned} &\left| f(x) - [f(x^*) + f'(x^*)(x - x^*) + \frac{1}{2} f''(x^*)(x - x^*)^2] \right| \\ &\qquad = \left| f(x) - f(x^*) - \frac{1}{2} f''(x^*)(x - x^*)^2] \right| \\ &\qquad \leq \frac{1}{3}(-f''(x^*))(x - x^*)^2. \end{aligned}$$

Consequently, if $|x - x^*| < \delta$, then

$$\begin{aligned} f(x) &= f(x^*) + (f(x) - f(x^*) - \frac{1}{2} f''(x^*)(x - x^*)^2) + \frac{1}{2} f''(x^*)(x - x^*)^2 \\ &\leq f(x^*) + |f(x) - f(x^*) - \frac{1}{2} f''(x^*)(x - x^*)^2| + \frac{1}{2} f''(x^*)(x - x^*)^2 \\ &\leq f(x^*) + \frac{1}{6} f''(x^*)(x - x^*)^2 \leq f(x^*). \end{aligned}$$

III. Economic Applications

A. Profit Maximization

1. Let $\mathrm{TC}(y)$ be the total cost of producing y units of output. First assume that the output market is competitive, so that total revenue is py, where p is the price of output. Then profits are $\pi(y) = py - \mathrm{TC}(y)$.

a. The first order necessary condition for optimization is $\pi'(y^*) = 0$, which is equivalent to $\text{MC}(y^*) = \frac{d\text{TC}}{dy}(y^*) = p$.

b. The second order sufficient condition for maximization is that both $\pi'(y^*) = 0$ and $\pi''(y^*) = \frac{d\text{MC}}{dy}(y) < 0$.

c. Let $p = 21$, and let $\text{TC}(y) = \frac{y^3}{12} - \frac{5}{2}y^2 + 30y + 100$. This can be solved in two ways:

   i. Set $p = \text{MC}(y)$.

   ii. Maximize π directly.

2. For a monopolist we may let $\text{TR}(y)$ be the total revenue from marketing y units, so that now profits are $\pi(y) = \text{TR}(y) - \text{TC}(y)$.

   a. The first order necessary condition for optimization, $\pi'(y^*) = 0$, is now equivalent to $\text{MC}(y^*) = \frac{d\text{TR}}{dy}(y^*) = \text{MR}(y^*)$.

   b. The second order sufficient condition for maximization is that both $\pi'(y^*) = 0$ and $\pi''(y^*) = \frac{d\text{MR}}{dy}(y) - \frac{d\text{MC}}{dy}(y) < 0$.

B. Utility Maximization for a Consumer

1. Consider a consumer who consumes two goods, quantities of which are denoted by x and y.

   a. Let the consumer's utility function be $U(x, y) = xy$.

   b. Let the consumer's budget constraint be $p_x x + p_y y \leq M$.

2. Since utility is increasing in both goods, and there are only two goods, we can treat this as a single variable optimization problem by treating the budget constraint as an equality, solving for y in terms of the other quantities, and substituting into the utility function.

   a. Thus $y = \frac{M - p_x x}{p_y}$, and the consumer's problem is to maximize $U(x, \frac{M - p_x x}{p_y}) = x\frac{M - p_x x}{p_y}$.

b. The first order necessary conditions for optimization are

$$0 = \frac{\partial U}{\partial x} - \frac{p_x}{p_y}\frac{\partial U}{\partial y} = \frac{M - p_x x}{p_y} - \frac{p_x}{p_y}x.$$

c. We obtain the intuitively sensible conclusion that, at the optimum, marginal utilities are proportional to prices:

$$\frac{\frac{\partial U}{\partial x}}{p_x} = \frac{\frac{\partial U}{\partial y}}{p_y}.$$

d. For the particular utility function we obtain:

$$x = \frac{M}{2p_x} \quad \text{and} \quad y = \frac{M}{2p_y}.$$

C. Time Inconsistency

1. Suppose that a governmental authority can increase output by increasing inflation, for instance by increasing the money supply. Specifically, suppose that

$$y - y^* = \alpha(\pi - \pi^e),$$

where y is the level of output, $y*$ is the level of output corresponding to no "unanticipated" inflation, and π is the actual rate of inflation, and π^e is that rate of inflation is that the consumers and producers in the economy expect the government to choose.

2. Let the authority's utility function be $U(y, \pi) = y - \beta\pi^2$.

3. The temporal sequence of events is as follows:

   a. The private agents in the economy guess, as best they can based on available information, what the actual inflation rate π chosen by the authority will be.

   b. Once the guess π^e is known, the authority chooses π in order to maximize the utility function above.

4. Substituting $y = y^* + \alpha(\pi - \pi^e)$, the authority's problem is to maximize $y^* + \alpha(\pi - \pi^e) - \beta\pi^2$, which has the unique maximand $\pi^* = \frac{\alpha}{2\beta}$.
5. It is reasonable to assume that the private agents understand that, whatever the government might announce, in the end it will choose π^*, so that we should have $\pi^e = \frac{\alpha}{2\beta}$.
6. With $\pi = \pi^e = \frac{\alpha}{2\beta}$ as above, the government's utility is $U(y^*, \frac{\alpha}{2\beta}) = y^* - \frac{\alpha^2}{4\beta}$. *If the government could credibly commit itself, for instance by a constitutional provision, to set $\pi = 0$, then it could achieve $U(y^*, 0) = y^*$.* Initially it may seem paradoxical that the government's outcome can be improved by taking away strategic flexibility (the freedom to set inflation according to circumstances) but in fact this is a common phenomenon in the analysis of games.