

University of Sydney

Department of Economics

Mathematical Methods of Economic Analysis

LIMITS AND SETS QUESTIONS

PART I

From Simon and Blume, do the following:

- Chapter 12: 12.3, 12.6, 12.11, 12.14, 12.20, 12.29.

Note. In some of these exercises, the question will arise as to what theorems you are allowed to assume for the purposes of the proof. You learn most by assuming as little as possible, but you may use any theorems that Simon and Blume present prior to the exercise in question.

PART II

Q1. Prove that if

$$x_k = \frac{k+100}{3k} \quad \text{then} \quad x_k \rightarrow \frac{1}{3} \quad \text{as} \quad k \rightarrow \infty,$$

i.e., find K (which will vary with ϵ) such that for any $\epsilon > 0$

$$\left\| \frac{k+100}{3k} - \frac{1}{3} \right\| < \epsilon \quad \text{for all } k > K.$$

Q2. Prove that if $r \in (0, 1)$ and $x_k = r^k$, then $x_k \rightarrow 0$ as $k \rightarrow \infty$.

Hint. Let

$$s = \frac{1-r}{r} > 0.$$

This re-arranges to

$$r = \frac{1}{1+s}.$$

Use the result from the first set of practice questions that

$$(1+s)^k > 1+sk \quad \text{for all } k \geq 2.$$

Q3. Four definitions of closed sets are contained in the notes using: (i) boundary points, (ii) accumulation points, (iii) the complement of an open set, (iv) limits of sequences. Prove the equivalence of all four definitions.

Hint. The long way to do this is to show for each definition pair that the two definitions are equivalent (i.e., if a set is closed in terms of one definition, then it is closed in terms of the other definition and *vice versa*). The short way to do it is to create a “closed loop of implication”. For example, suppose you could show

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i), \quad (1)$$

where $(a) \Rightarrow (b)$ means that if a set is closed in terms of definition (a), then it is closed in terms of definition (b). The chain in (1) would prove that all four definitions are equivalent because the chain finishes with the same definition as that with which it starts, so the chain is in fact a loop or circle. Thus, by proceeding around the loop, we see that the satisfaction of any one definition implies the satisfaction of the other three.

The order in which the four definitions appear in the chain is not important; all that is important is that *all four* do appear and that the first definition is also the last (so you have a closed loop).

Q4. A consumer has a fixed income $M > 0$ and faces a *strictly positive* price vector \mathbf{p} . The consumer’s budget set X is therefore:

$$X = \{ \mathbf{x} \in \mathbf{R}_+^n \mid \mathbf{p} \cdot \mathbf{x} \leq M \}$$

- (i) Prove that the budget set is compact.
- (ii) Would X still be compact if some prices p_i were allowed to be zero? Explain briefly.