University of Sydney

Department of Economics

Mathematical Methods of Economic Analysis

Multivariable Functions Questions

PART I

From Simon and Blume, do the following:

- Chapter 13: 13.11–12, 13.23.
- Chapter 20: 20.1, 20.8 (use Theorem 20.6), 20.12 (by “equivalent”, S & B mean “have the same level sets”, i.e., the same indifference curves or whatever).

PART II

Q1. Let $f\colon \mathbf{R} \to \mathbf{R}$, where

$$f(x) = \min\{\, x^2, 5 \,\},$$

i.e., for each value of x, $f(x)$ is the lesser of x^2 and 5 (if $x^2 = 5$, then $f(x)$ equals both).

(i) Roughly graph this function.

(ii) What are $f(-4)$, $f(-1)$ and $f(6)$ and what is the range of f?

(iii) What is $f(y)$?

(iv) For $x \geq 0$, what is $f(\sqrt{x})$? Roughly graph $f(\sqrt{x})$ against x for $x \geq 0$.

(v) Now define a new function g, where $g(x) = f(\sqrt{x})$. What must be true of the domain of g? Graph g for the largest possible domain.

(vi) Now graph $g(\sqrt{x})$ against x for $x \geq 0$.

Q2. Define the difference quotient:[1] $g\colon \mathbf{R}\backslash\{0\} \to \mathbf{R}$ where

$$g(h) = \frac{(x+h)^2 - x^2}{h}$$

and $x \in \mathbf{R}$ (there is a different g function for each $x \in \mathbf{R}$). Prove that

$$\lim_{h \to 0} g(h) = 2x.$$

Do you know what the meaning of this limit is?

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[1] $\mathbf{R}\backslash\{0\}$ is the set difference of $\mathbf{R}$ and $\{0\}$, i.e., it consists of all real numbers except zero.

Q3. (i) Assume that a consumer has a utility function, $U = U(I, L)$, where I is income and L is labour. Further assume that utility is a strictly *increasing* function of income and a strictly *decreasing* function of labour and that the utility function is *continuous.* Suppose that we start with $(I, L) = (I_0, L_0)$. Show that there must be *some* combination of more income and more labour that will yield more utility than does (I_0, L_0).

Hint: Consider the change in two steps. First increase I and then increase L. Show, from the assumptions stated earlier and the definition of continuity, that it must be possible to do this in such a way that utility increases (you may take it for granted that the utility function's domain permits the required increases in I and L).

(ii) Suppose that there are two consumers A and B with *continuous, strictly increasing* utility functions, $U_A = U_A(M_A)$ and $U_B = U_B(M_B)$, where M_A is the amount of money received by A and M_B is the amount of money received by B. There is a fixed sum of money M to be distributed between the two consumers, so that $M_A + M_B = M$.

Assume that there is an initial distribution of the money, (M_A^0, M_B^0), with the property that $U_A(M_A^0) = 100$ and $U_B(M_B^0) = 80$. If money is taken from A and given to B, then A becomes worse off than before and B becomes better off than before. Show, using the fact that the utility functions are continuous and strictly increasing, that it is possible to do this in such a way that A is still better off than B, i.e.,

$$80 < U_B(M_B) < U_A(M_A) < 100.$$

Q4. Prove that if A and B are convex, then $A + B$ is convex, i.e., let $C \equiv A + B$ and show that if $x_1, x_2 \in C$, then

$$\lambda x_1 + (1 - \lambda)x_2 \in C \quad \text{for all } 0 < \lambda < 1.$$

Similarly, show that $A \cap B$ and $A \times B$ are convex.

Q5. Let $f\colon \mathbf{R}_+^n \to \mathbf{R}_+$ be a concave production function, with $y = f(\mathbf{x})$ satisfying $f(\mathbf{0}) = 0$. Prove that f *cannot* exhibit increasing returns to scale.

Q6. Assume that a consumer maximises a utility function $u\colon \mathbf{R}_+^n \to \mathbf{R}$, where $u = u(\mathbf{x})$, subject to a budget set: $\beta = \{\, \mathbf{x} \in \mathbf{R}_+^n \mid \mathbf{p} \cdot \mathbf{x} \leq M \,\}$, where $M > 0$ is a fixed level of income and $\mathbf{p}$ is a fixed price vector. Show that if $u(\mathbf{x})$ is a strictly quasi-concave function, then a utility maximising consumption vector, if it exists, is unique.

Q7. One definition of a quasi-concave function $y = f(\mathbf{x})$ is that, for any $\mathbf{x}^a$, $\mathbf{x}^b$ in the domain of the function,

$$f(\lambda \mathbf{x}^a + (1 - \lambda)\mathbf{x}^b) \geq \min\{f(\mathbf{x}^a), f(\mathbf{x}^b)\} \qquad \text{for all } 0 < \lambda < 1.$$

Suppose that a consumer maximises a utility function $u = u(\mathbf{x})$ subject to a budget set: $\beta = \{\, \mathbf{x} \in \mathbf{R}_+^n \mid \mathbf{p} \cdot \mathbf{x} \leq M \,\}$, where $M > 0$ is a fixed level of income and $\mathbf{p}$ is a fixed price vector. It may be the case that more than one $\mathbf{x}$ vector maximises utility subject to the budget set. Prove, *using the above definition of quasi-concavity*, that if $u = u(\mathbf{x})$ is a quasi-concave function, then the set of all such utility maximising vectors is a convex set.