

**Economics 8103**  
**Microeconomic Theory**  
**Spring-1 2005**

Lecture 9  
**Perfect Equilibrium  
of the Agent Normal Form  
and Sequential Equilibrium**

I. Introduction

- A. Consider an  $\epsilon$ -perfect equilibrium of the agent normal form.
1. Since all actions are chosen with positive, though possibly small, probability, every node in the tree occurs with positive probability.
    - a. This implies that at every information set we can form a conditional probability distribution over the nodes in the information set.
    - b. We think of this conditional probability distribution as the *belief* of the agent choosing at the information set when the information set occurs.
  2. The expected payoff in the game as a whole for the agent choosing at the information set is the sum of two terms.
    - a. The first term is the probability that the information set will not be reached times the expected payoff conditional on the information set not being reached.
      - i. This term does not depend on behavior at the information set.

- b. The second term is the probability that the information set will be reached times the expected payoff conditional on this event.
    - i. The second term is in turn the sum over nodes in the information set of the probability of the node times the expected payoff conditional on the node.
    - ii. This term depends on the agent's beliefs at the information set, his behavior at the information set, and the behavior at nodes lower down in the tree.
  - c. Thus we see that if all agents play totally mixed strategies, then strategies that maximize the payoff of (the agent who chooses at) an information set in the game as a whole is equivalent to maximizing the expected payoff conditional on the information set being reached.
    - i. This maximization problem depends on the beliefs at the information set and behavior lower down in the game tree.
- B. Now consider the limit of a sequence of  $\epsilon$ -perfect equilibria where the magnitudes of the trembles converge to 0.
1. For each  $\epsilon$ -perfect equilibrium and each information set there is a belief, and by taking convergent subsequences we can insist that the sequences of beliefs also converge.
    - a. The limiting beliefs are related to the strategies in a way that embodies the idea that the beliefs should be conditional probabilities when conditional probabilities are defined, but is somewhat stronger.
    - b. Kreps and Wilson refer to this condition as *consistency*.
  2. In each  $\epsilon$ -perfect equilibrium the strategy at each information set is

$\epsilon$ -optimal relative to the beliefs and the strategies lower down in the tree.

- a. The expected payoffs associated with the various actions available at the information set are continuous functions of these variables.
  - b. This means that the limiting strategy at the information set is optimal relative to the limiting belief and the limit strategies lower down in the game tree.
  - c. Kreps and Wilson refer to this conditional as *sequential rationality*.
3. A *sequential equilibrium* is a specification of strategies and beliefs at each information set that is consistent and sequentially rational.

C. Issues connected with sequential equilibrium.

1. Of course we must develop the ideas above precisely.
2. Since we know that perfect equilibria of the agent normal form exist and give rise to sequential equilibria, we know that every extensive game has at least one sequential equilibrium.
3. We must show that (the strategy component of) every sequential equilibrium is a Nash equilibrium of the agent normal form.
4. In a sequential equilibrium is it necessarily the case that each agent is behaving optimally? That is, if an agent's behavior at each information set is optimal given his behavior at other information sets, is it necessarily the case that he could not improve his expected payoff by changing his behavior at several information sets? The answer is yes for games of perfect recall.

D. The practical and philosophical significance of sequential equilibrium.

1. Sequential equilibrium and its refinements are now the dominant so-

- lution concepts in economic modelling.
2. One important aspect of sequential equilibrium is that it embodies the notion that, conditional on any state of the game, behavior must be rational.
    - a. In this sense it generalizes the notion of subgame perfection.
  3. If we regard the consistency conditional as a reasonable requirement to impose on the relation between strategies and beliefs, then sequential equilibrium is a “minimally restrictive” solution concept.
    - a. Anything that we regard as a reasonable equilibrium must be a sequential equilibrium.
    - b. On the other hand there may be sequential equilibria that are unreasonable, implausible, or violate our intuitions concerning simultaneously rational behavior in games.
      - i. This has led to work on developing refinements of sequential equilibrium.
  4. One important aspect of sequential equilibrium is that for most games it is much easier to compute the set of sequential equilibria than to compute the set of perfect equilibria of the agent normal form.
    - a. Since the sequential rationality condition depends only on beliefs and behavior lower down in the game tree, one can work by backwards induction.
  5. In addition to their computational significance, the beliefs in a sequential equilibrium also have considerable intuitive significance.
    - a. This makes the analysis of the set of sequential equilibria of an economic model more interesting than the analysis of the set of agent normal form perfect equilibria.
    - b. In part this is due to the fact that sequential equilibrium is

psychologically natural. When we actually play games our tendency is to form beliefs about the relative probability of states of the game and optimize relative to these beliefs. From an evolutionary point of view it seems reasonable to conjecture that the fact that we think in this way is due to the computational efficiency of the sequential equilibrium concept.

## II. Consistency of Strategies and Beliefs

- A. Let an extensive form game  $G = ((T, \prec), H, (A, \alpha), (I, \iota), \rho, u)$  be given.
- B. Given a behavior strategy  $\pi = \prod_{h \in H} \Delta(A_h)$  we can define the probability of  $t$  conditional on a predecessor node  $x$  as well as the absolute probabilities of each node and information set:

$$\begin{aligned} \mathbf{P}^\pi(t|x) &= \prod_{\ell=0, \dots, L-1} \pi(\alpha(p_\ell(t))) \quad (x = p_\ell(t)); \\ \mathbf{P}^\pi(t) &= \rho(p_{\ell(t)}(t)) \mathbf{P}^\pi(t|p_{\ell(t)}(t)); \\ \mathbf{P}^\pi(h) &= \sum_{x \in h} \mathbf{P}^\pi(x). \end{aligned}$$

1. We adopt the convention that  $\mathbf{P}^\pi(t|x) = 0$  whenever  $x$  is not a predecessor of  $t$ .
- C. The space of *beliefs* is  $\prod_{h \in H} \Delta(h)$ .
1. A belief describes agents' subjective probability distributions over the nodes in each information set in the game.
  2. At the very least we should like the beliefs and the strategies to satisfy Bayesian updating when that is well defined.
  3. We will take the closure of the set of belief-strategy pairs for which this requirement determines a unique system of beliefs.
- D. Recall that for any finite set  $X$ , the set of interior probability measures is  $\Delta^0(X) = \{\mu: X \rightarrow (0, 1] \mid \sum_X \mu(x) = 1\}$ .

1. The set of *interior strategies* is  $\Pi^0 = \prod_{h \in H} \Delta^0(A_h)$ .

2. If  $\pi \in \Pi^0$  then  $\mathbf{P}^\pi(t) > 0$  for all  $t$ , and for any  $x \in X$  we can define  $\mu^\pi(x) = \mathbf{P}^\pi(x)/\mathbf{P}^\pi(H(x))$ .

E. **Definition:** The space of *interior consistent assessments* is

$$\Psi^0 = \{(\mu^\pi, \pi) | \pi \in \Pi^0\}.$$

The space of *consistent assessments* is  $\Psi$ , the closure of  $\Psi^0$  in the topology of  $(\prod_{h \in H} \Delta(h)) \times \Pi$ .

F. Remarks

1. Consistency is maximally restrictive: any smaller set of assessments containing the interior ones is not compact. This troubled Kreps and Wilson, who could not find any intuitive justification for the full strength of this assumption. They describe several weaker notions.
2. For two reasons I find Kreps and Wilson's concern unnecessary.
  - a. Consistency seems to be the precise expression of the statistical independence of the stents' strategies.
  - b. It is extremely attractive mathematically insofar as the space of consistent assessments is very well behaved.

#### IV. Expected Payoffs, Rationality, and Sequential Equilibrium

A. We define an agent's expected payoff at a strategic node, relative to a behavior strategy  $\pi$ , and at information set, relative to a behavior strategy-belief pair  $(\mu, \pi)$ , in the obvious way:

$$\mathbf{E}^\pi(u_i|x) = \sum_{z \in Z(x)} \mathbf{P}^\pi(z|x)u_i(z);$$

$$\mathbf{E}^{\mu, \pi}(u_{\iota(h)}|h) = \sum_{x \in h} \mu(x)\mathbf{E}^\pi(u_{\iota(h)}|x);$$

**Lemma:**  $\mathbf{E}^{(\mu, \pi)}(u_{\iota(h)}|h)$  is a continuous (actually polynomial) function of  $\mu$  and  $\pi$ .

B. **Definition:** An assessment  $(\mu, \pi)$  is *sequentially rational* if for all  $h \in H$  and all  $\pi' \in \Pi$  with  $\pi'^j = \pi^j$  for all  $j \neq \iota(h)$ ,  $\mathbf{E}^{\mu, \pi}(u_{\iota(h)}|h) \geq \mathbf{E}^{\mu, \pi'}(u_{\iota(h)}|h)$ . A *sequential equilibrium* is a consistent sequentially rational assessment.

C. **Definition:** An assessment  $(\mu, \pi)$  is *myopically rational* if, for all  $h \in H$  and  $\pi' \in \Pi$  with  $\pi'_{h'} = \pi_{h'}$  for all  $h' \neq h$ ,  $\mathbf{E}^{\mu, \pi}(u_{\iota(h)}|h) \geq \mathbf{E}^{\mu, \pi'}(u_{\iota(h)}|h)$ .

**Proposition:** If  $(\mu, \pi)$  is consistent and myopically rational, then it is sequentially rational.

D. The proof of the Proposition.

1. **Definition:** The *personal decision arborescence* of  $i \in I$  is the set  $T^i = H^i \cup A^i \cup Z$  ordered by the personal precedence relation  $\prec^i$  derived from  $\prec$ . (Exercise: define  $\prec^i$  precisely.)

a. The set of nonterminal nodes in  $T^i$  is  $X^i = H^i \cup A^i$ , and the set of immediate successors of  $x \in X^i$  is  $F^i(x)$ .

i.  $F^i(h) = A(h)$ ,  $h \in H^i$ .

ii.  $F^i(a) \subset H^i \cup Z$ ,  $a \in A^i$ .

b. The assumption of Perfect Recall is used here to guarantee that  $(T^i, \prec^i)$  is an arborescence.

2. Some definitions.

a. **Definition:** For  $\pi \in \Pi$ ,  $h \in H$ , and  $a \in A(h)$ , let  $\pi|a \in \Pi$  be the strategy with  $\pi_h$  replaced by playing  $a$  with probability 1.

b. **Definitions:**

i.  $\mathbf{P}^{\mu, \pi}(h'|h) = \sum_{x' \in h', x \in P(x') \cap h} \mu(x) \mathbf{P}^{\pi}(x'|x)$ ,  $h \prec^i h'$ .

ii.  $\mathbf{P}^{\mu, \pi}(z|h) = \mu(x) \mathbf{P}^{\pi}(z|x)|_{x \in P(z) \cap h}$ ,  $z \prec^i h$ .

iii.  $\mathbf{P}^{\mu, \pi}(h'|a) = \mathbf{P}^{\mu, \pi|a}(h'|h)$ ,  $h \prec^i h'$  and  $a \in A(h)$ .

iv.  $\mathbf{P}^{\mu, \pi}(z|a) = \mathbf{P}^{\mu, \pi|a}(z|h)$ ,  $a \prec^i z$ ,  $a \in A(h)$ .

3. Claims.

a. If  $h \prec^i h'$  and  $a$  is the action that must be chosen at  $h$  to reach  $h'$ , then by Perfect Recall we have

$$\mathbf{P}^{\mu, \pi}(h'|h) = \pi(a) \mathbf{P}^{\mu, \pi}(h'|a).$$

b.  $\mathbf{E}^{\mu, \pi}(u_{\iota(h)}|h) = \sum_{a \in A(h)} \pi(a) \mathbf{E}^{\mu, \pi}(u_{\iota(h)}|a)$ .

c. If  $(\mu, \pi)$  is consistent and  $a \in A^i$ , then

$$\begin{aligned} \mathbf{E}^{\mu, \pi}(u_i|a) &= \Sigma_{F^i(a) \cap Z} \mathbf{P}^{\mu, \pi}(z|a) u_i(z) \\ &\quad + \Sigma_{F^i(a) \cap H^i} \mathbf{P}^{\mu, \pi}(h|a) \mathbf{E}^{\mu, \pi}(u_i|h). \end{aligned}$$

**Proof of the Proposition:** Suppose  $(\mu, \pi)$  is consistent and myopically rational, but not sequentially rational. Let  $h$  be such that there exists  $\pi'$  such the  $\pi'_j = \pi_j$  for  $j \neq i(h)$  and

$$\mathbf{E}^{\mu, \pi'}(u_{i(h)}|h) > \mathbf{E}^{\mu, \pi}(u_{i(h)}|h).$$

Then Claim b above and myopic rationality implies the existence of  $a \in A(h)$  with

$$\mathbf{E}^{\mu, \pi'}(u_{i(h)}|a) > \mathbf{E}^{\mu, \pi}(u_{i(h)}|a),$$

and Claim c implies the existence of  $h' \in F^{i(h)}(a)$  with

$$\mathbf{E}^{\mu, \pi'}(u_{i(h)}|h') > \mathbf{E}^{\mu, \pi}(u_{i(h)}|h').$$

But this shows that one can construct an infinite sequence of information sets with this property, an impossibility. ■

#### E. Existence

**Theorem:** Let  $\Phi$  be the set of sequential equilibria. Then  $\Phi \neq \emptyset$ .

**Proof:** Let  $\{\epsilon_r(\cdot)\}_{r=1,2,\dots}$  be a sequence of behavioral (agent normal form) trembles with  $\lim_{r \rightarrow \infty} \max_{a \in A} \epsilon_r(a) = 0$ , for each  $r$  let  $\pi_r$  be an  $\epsilon_r$ -perfect equilibrium, and let  $\mu_r = \mu^{\pi_r}$ . Choose a subsequence with  $(\mu_r, \pi_r) \rightarrow (\mu^*, \pi^*) \in \Psi$ . Then  $(\mu^*, \pi^*)$  is consistent, obviously, and the continuity of the expected payoff operator implies that it is myopically rational. Thus the Proposition implies that it is sequentially rational. ■

### IV. Forward Induction

A. The general idea of forward induction is to restrict beliefs to be consistent with the idea that other agents are rational, whenever possible.

1. For example, suppose that, at an information set for agent 2, there are two nodes, one of which happens only when agent 1 chooses a dominated action, while the other results from an action that would lead to a better payoff for agent 1, provided that agent 2 believes that that node has occurred. Are equilibria based on the supposition that agent 2 believes that the first node has occurred reasonable?
2. In a model where mistakes are possible, such beliefs do seem reasonable if the dominated action is, in some psychological sense, a typical error, while few people are likely to accidentally choose the other action.
3. Although various definitions have been proposed, none has been accepted as the correct description of this intuition.
4. Iterating the logic of forward induction can lead to funny results.

B. Burning money.

1. A famous example due to van Damme and Ben-Porath and Dekel considers the following version of the battle of the sexes:

$$\begin{array}{cc}
 & L & R \\
 U & (4, 1) & (0, 0) \\
 D & (0, 0) & (1, 4)
 \end{array}$$

2. We consider a variant in which, prior to the beginning of play, agent 1 has the opportunity to publicly throw away 2 utils, which we think of as burning a \$2 bill, leading to the payoff matrix:

$$\begin{array}{cc}
 & L & R \\
 U & (2, 1) & (-2, 0) \\
 D & (-2, 0) & (-1, 4)
 \end{array}$$

3. If agent 2 believes in forward induction, after seeing the \$2 burned she should reason that agent 1 is intending to play  $U$ , since burning the \$2

bill followed by  $D$  is dominated by not burning. Therefore it should be expected that agent 2 will play  $L$  after burning.

4. If you accept all this, you should also believe that agent 1 must be intending to play  $U$  after *not* burning the \$2 bill, since not burning followed by  $D$  yields at most 1, which is less than the 2 that agent 1 can obtain by burning.
5. Thus the unique equilibrium consistent with this form of forward induction has agent 1 not burning and playing  $U$ .
6. Game theorists regard this reasoning with intense suspicion.