Economics 8117-8 **Noncooperative Game Theory**

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Elimination of Dominated Strategies

I. Introduction

A. The simplest and least controversial consequence of rationality is that an agent will not choose an action that is worse than another one no matter what else happens.

B. There are two forms of dominance.

1. Strict dominance occurs when the dominating action has a strictly higher payoff in every case.
2. Weak dominance replaces the strict inequality with a weak one, except that at least one inequality must be strict.

C. Suppose the rationality of all agents is common knowledge.

1. No agent plays a dominated strategy, and everybody can count on this.
2. It is as if the dominated strategy was not a possibility in the game in the first place.
3. Elimination of dominated strategies can be iterated: after a dominated strategy has been eliminated new strategies may be dominated.

D. Following Moulin, we will study this technique in two contexts.

1. Games in normal form
2. Extensive games of perfect information

II. Normal Form Games

A. The set of *agents* is $I = \{1, \ldots, n\}$– this will be fixed throughout the course.

B. An n–*person normal form game* is a $2n$–tuple

$N = \langle S_1, \ldots, S_n; u_1, \ldots, u_n \rangle$

1. S_i is a nonempty finite set of *strategies* for agent i.

2. Let $S = \Pi_{i \in I} S_i$ be the set of *strategy vectors.*

3. $u_i : S \to I\!R$ is agent i's *utility* or *payoff function.*

C. We adopt the following notation for "everyone but i"

1. $S_{-i} = S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n$

2. If $s_i \in S_i$ and $s_{-i} \in S_{-i}$, then $\langle s_i, s_{-i} \rangle \in S$ is the obvious strategy vector.

D. We will be especially interested in truncations by elimination. If $R_1 \subset S_1, \ldots, R_n \subset S_n$ are nonempty set of strategies, let

$N(R_1, \ldots, R_n) = \langle R_1, \ldots, R_n; u_1|_R, \ldots, u_n|_R \rangle$, $R = \Pi_{i \in I} R_i$.

III. Strict Dominance

A. **Definition:** We say that s_i *strictly dominates* t_i, and that t_i is *strictly dominated by* s_i, if

$$u_i(s_i, r_{-i}) > u_i(t_i, r_{-i}) \text{ for all } r_{-i} \in S_{-i}.$$

B. **Definition:** We say that s_i is *strictly dominant* if it strictly dominates all other strategies in S_i.

1. The strongest possible solution concept is that everyone plays a strictly dominant strategy.

2. This is unlikely, but not unheard of in modelling praxis.

a. Perhaps the best known example is the Groves–Ledyard mechanism for allocating a public good – see Groves, T., and J. Ledyard, "Optimal Allocation of Public Goods: a Solution to the 'Free Rider' Problem," *Econometrica,* **45**, (1977), 783-809.

C. **Definition:** A truncation $N(R_1, \ldots, R_n)$ is a *strict dominance truncation* if, for all i, all elements of $S_i \setminus R_i$ are strictly dominated in N. Let $N^0, \ldots, N^J$ be a sequence of strict dominance truncations, that is, N^j is a strict dominance truncation of N^{j-1}, $j = 1, \ldots, J$. We say that the sequence $N^0, \ldots, N^J$ is *complete* if N^J has no further nontrivial strict dominance truncation.

D. Example – first eliminate R, then eliminate U

$$\begin{array}{ccc} 1\backslash 2 & L & R \\ U & (0,3) & (2,2) \\ D & (3,1) & (1,0) \end{array}$$

Lemma 1: If t_i strictly dominates s_i and s_i strictly dominates r_i, then t_i strictly dominates r_i.

Proof: Obvious. ∎

Lemma 2: If $N(R_1, \ldots, R_n)$ is a strict dominance truncation of N, r_i, $s_i \in R_i$, and s_i strictly dominates r_i, in N, then s_i strictly dominates r_i in $N(R_1, \ldots, R_n)$.

Proof: Obvious. ∎

Lemma 3: Let $N = N^0, \ldots, N^J$ be a sequence of strict dominance truncations with $N^j = N(R_1^j, \ldots, R_n^j)$. Suppose $R_i^J \subset \bar{R}_i \subset S_i$ for all i, and for some particular i there is a strategy $r_i \in R_i^J$ that is strictly dominated in $N(\bar{R}_1, \ldots, \bar{R}_n)$. Then r_i is strictly dominated in N^J.

Proof: Choose $s_i^0 \in \bar{R}_i$ that strictly dominates r_i in $N(\bar{R}_1, \ldots, \bar{R}_n)$, and construct a sequence $s_i^1, \ldots, s_i^J$ inductively by letting $s_i^j \in R_i^j$ be either s_i^{j-1} or some strategy in R_i^j that strictly dominates s_i^{j-1} in N^{j-1}. Then Lemmas 1 and 2 imply that s_i^J dominates r_i in N^J. ∎

Proposition: All complete sequences of strict dominance truncations have the same final game.

Proof: Let $N = N^0, \ldots, N^J$ with $N^j = N(R_1^j, \ldots, R_n^j)$ and $N = \bar{N}^0, \ldots, \bar{N}^K$ with $\bar{N}^k = N(\bar{R}_1^k, \ldots, \bar{R}_n^k)$ be two such sequences. Applying Lemma 3 shows that any

$r_i \in (S_i \setminus R_i^1) \cap \bar{R}_i^K$ is dominated in $\bar{N}^K$, so the completeness of the second sequence implies that $\bar{R}_i^K \subset R_i^1$ for all i. This inclusion allows us to apply Lemma 3 to show that any $r_i \in (R_i^1 \setminus R_i^2) \cap \bar{R}_i^K$ is dominated in $\bar{N}^K$, so the completeness of the second sequence implies that $\bar{R}_i^K \subset R_i^2$ for all i. Proceeding inductively, we eventually arrive at $\bar{R}_i^K \subset R_i^J$ for all i, and since the situation is symmetric we also have the reverse inclusion. ∎

E. **Definition:** The *strict dominance core* of a normal form is the game obtained by iterative elimination of strictly dominated strategies until no further such eliminations are possible.

F. **Definition:** A normal form game is *individually flat* if, for all i, agent i's payoff does not depend on his own strategy:

$u_i(r_i, t_{-i}) = u_i(s_i, t_{-i})$ for all $r_i, s_i \in S_i$ and all $t_{-i} \in S_{-i}$.

1. If the strict dominance core is individually flat then we say that the game is *strictly dominance solvable.*
2. In this case the strict dominance core is certainly a reasonable notion of the "solution" of the game.

IV. Weak Dominance

A. **Definition:** We say that s_i *weakly dominates* t_i, and that t_i is *weakly dominated by* s_i, if

$u_i(s_i, r_{-i}) \geq u_i(t_i, r_{-i})$ for all $r_{-i} \in S_{-i}$ with strict inequality for some r_{-i}.

1. To play a weakly dominated strategy is not so unambiguously irrational.
2. Later we will see topological reasons for allowing weakly dominated strategies to be used.
3. Still, some game theorists argue that iterative elimination of weakly dominated strategies should be a fundamental principle of game theory.

B. **Definition:** We say that s_i is *weakly dominant* if it weakly dominates all other strategies in S_i.

1. One of the leading results in social choice theory, the Gibbard-Satterthwaite theorem, asserts the nonexistence of nondictatorial social choice functions with the property that each agent has a weakly dominant strategy. See Gibbard, A., "Manipulation of schemes that mix voting and chance," *Econometrica*, **45**, (1977), 665-681, and earlier work cited there.

C. **Definition:** A truncation $N(R_1, \ldots, R_n)$ is a *weak dominance* truncation if, for all i, all elements of $S_i \setminus R_i$ are weakly dominated in N. Let $N^0, \ldots, N^J$ be a sequence of weak dominance truncations: N^j is a weak dominance truncation of N^{j-1}, $j = 1, \ldots, J$. We say that the sequence $N^0, \ldots, N^J$ is *complete* if N^J has no further nontrivial weak dominance truncation.

D. Different complete sequences of weak dominance truncations may have different final games.

1. *Example 1* –

$$\begin{array}{cc} 1\backslash 2 & \begin{array}{ccc} \ell & m & r \end{array} \\ \begin{array}{c} U \\ D \end{array} & \begin{pmatrix} (1,2) & (0,0) & (1,0) \\ (1,3) & (1,0) & (0,0) \end{pmatrix} \end{array}$$

2. *Example 2* – Two agents each pick an integer between 1 and 10. They both win 1 util if their answers differ by no more than 1 (so that choosing 10 is weakly dominated by choosing 9) and otherwise each gets 0 utils.

E. Let $D(N)$ be the game obtained from N by eliminating all weakly dominated strategies. Define the sequence $N^0 = N, N^1 = D(N^0), \ldots, N^J = D(N^{J-1})$, where no strategies are weakly dominated in N^J, i.e. $D(N^J) = N^J$. Moulin says that N is *dominance solvable* if N^J is individually flat.

1. This particular sequence of dominance truncations is to some extent arbitrary.

V. Nice Weak Dominance and Very Weak Nice Dominance

A. Marx and Swinkels have pointed out that the ambiguities arising out of different possible orders of elimination are not troublesome, under certain conditions, if one also allows elimination under the following weaker condition.

1. **Definition:** We say that s_i *very weakly dominates* t_i, and that t_i is *very weakly dominated by* s_i, if $u_i(s_i, r_{-i}) \geq u_i(t_i, r_{-i})$ for all $r_{-i} \in S_{-i}$.
2. In words, s_i very weakly dominates t_i if agent i either weakly prefers s_i, in the sense of weak dominance, or is always indifferent.

B. Elimination of a pure strategies for agent i that are equivalent, from his/her point of view, will have no effect on the strategic nature of the game, and in particular on the possibilities for further eliminations, if all other agents are also indifferent.

1. Requiring that whenever one agent is indifferent between two pure strategy vectors, then all other agents are also indifferent, is equivalent to the existence of a space of *outcomes* Ω and maps $\hat{\omega} : S \to \Omega$ and $v_i : \Omega \to I\!R$ ($i \in I$) such that $u_i(s) = v_i(\hat{\omega}(s))$ ($i \in I$, $s \in S$) and $v_i(\omega) \neq v_i(\omega')$ ($i \in I$, $\omega, \omega' \in \Omega$, $\omega \neq \omega'$).
2. For the following argument, the condition that if one agent is indifferent between two strategy vectors, then so are all other agents, is only required to hold in certain circumstances.
3. **Definition:** We say that s_i *nicely (very) weakly dominates* (N(V)WDs) t_i, and that t_i is *nicely (very) weakly dominated* (N(V)WDed) *by* s_i, if s_i *(very) weakly dominates* t_i, and, for all $r_{-i} \in S_{-i}$ such that $u_i(s_i, r_{-i}) = u_i(t_i, r_{-i})$, it is the case that $u_j(s_i, r_{-i}) = u_j(t_i, r_{-i})$ for all $j \in I$.
4. **Definition:** The game $N = \langle S_1, \ldots, S_n; u_1, \ldots, u_n \rangle$ satisfies *transference of decision maker indifference* if, for all $i, j \in I$, $s_i, t_i \in S_i$, and $r_{-i} \in S_{-i}$, $u_i(s_i, r_{-i}) = u_i(t_i, r_{-i})$ implies $u_j(s_i, r_{-i}) = u_j(t_i, r_{-i})$.
   a. When N satisfies transference of decision maker indifference, all weak and very weak dominations are nice.

C. **Definition:** The game $N^* = \langle S_1^*, \ldots, S_n^*; u_1^*, \ldots, u_n^* \rangle$ is *equivalent to a subset* of the game $N = \langle S_1, \ldots, S_n; u_1, \ldots, u_n \rangle$ if there are maps $m_i : S_i^* \to S_i$ $(i \in I)$ such that, for all agents i and all $s^* \in S^*$, $u_i^*(s^*) = u_i(m_1(s_1^*), \ldots, m_n(s_n^*))$. We say that N^* is *almost equivalent to* N if, in addition, m_i is surjective (onto) for all but at most one agent, and if j is this agent, then $S_j \setminus m_j(S_j^*) = \{s_j\}$ where s_j is NVWDed in N.

1. The relation "is equivalent to a subset of" is obviously transitive.
2. If each of two games is equivalent to a subset of the other, then the two games are evidently equivalent in the intuitive sense that, up to relabelling, the second can be obtained from the first by addition and deletion of redundant strategies.
3. The following lemmas are the key steps in the reasoning. The first (in the special case $\sum_i \#(S_i \setminus T_i) = 1$) shows that any sequence of nice very weak dominance trunctations can be realized as a sequence of what Marx and Swinkels call one-at-a-time truncations.

Lemma A: Suppose $R_1 \subset T_1 \subset S_1, \ldots, R_n \subset T_n \subset S_n$, $i \in I$, and $s_i \in T_i \setminus R_i$. If $N'' = N(R_1, \ldots, R_n)$ is obtained from the given game N by eliminating some collection of strategies that are NVWDed, in N, by strategies in $R_1, \ldots, R_n$, then s_i is NVWDed in $N' = N(T_1, \ldots, T_n)$.

Proof: If $r_i \in R_i$ NVWDs s_i in N, then it NVWDs s_i in any truncation of N that contains both r_i and s_i. ∎

Lemma B: Suppose $s_k \in S_k$ is NVWDed in N, and let

$$N^* = \langle S_1^*, \ldots, S_n^*; u_1^*, \ldots, u_n^* \rangle$$

be a game that is almost equivalent to N. Then there is $N^{*\prime}$, either equal to N^* or obtained from N^* by eliminating NVWDed strategies, that is almost equivalent to $N \setminus s_k$:

$$\begin{array}{ccc} N & \stackrel{\text{elim}}{\longrightarrow} & N \setminus s_k \\ \uparrow \text{ae} & & \uparrow \text{ae} \\ N^* & \stackrel{\text{elim}}{\longrightarrow} & N^{*\prime} \end{array}$$

Proof: Suppose that $m_i : S_i^* \to S_i$ ($i \in I$) are the maps that make N^* almost equivalent to N. If all the maps m_i are surjective we may let

$$N^{*\prime} = N^*(S_1^*, \ldots, S_k^* \setminus m_k^{-1}(s_k), \ldots, S_n^*)$$

and obtain the "almost equivalence" by restricting the maps m_i. So suppose that j is the agent for which this is not the case, and let s_j be the unique element of $S_j \setminus m_j(S_j^*)$. Eliminating all elements of ${m_k}^{-1}(s_k)$ will be possible, and sufficient to establish the claim, unless these elements are not NVWDed in N^*, and the only way that this can happen is that $j = k$ and s_j is the only element of S_k that NVWDs s_k. Since s_j is NVWDed, and this relation is transitive, it must also be the case that s_k is the only element of S_k that NVWDs s_j. Since both of these very weak dominations are nice, s_j and s_k must be redundant. But then we let $N^{*\prime} = N^*$, setting $m_i'(t_i) = m_i(t_i)$ for all $(i, t_i) \notin \{k\} \times {m_k}^{-1}(s_k)$ and $m_k'(t_k) = s_j$ for all $t_k \in {m_k}^{-1}(s_k)$. ∎

Proposition: If N' and N'' are two games obtained from N by iterative elimination of NVWDed strategies, then there is N^*, obtained from N' by iterative elimination of NVWDed strategies, that is equivalent to a subset of N''.

Proof: Suppose that N' is obtained from N by eliminating $s^1, \ldots, s^p$. By Lemma A we may assume that these are ordered so that, for $g = 0, \ldots, p-1$, s^{g+1} is NVWDed in the game obtained by eliminating $s^1, \ldots, s^g$, which we denote by N^{g0}. Similarly, let $t^1, \ldots, t^q$ be the strategies in N but not in N'', ordered so that for each $h = 0, \ldots, q-1$, t^{h+1} is NVWDed in the game $N^{0h} = N \setminus t^1, \ldots, t^h$ obtained from N by eliminating $t^1, \ldots t^h$. We claim that it is possible to find $N^{gh} = N(S_1^{gh}, \ldots, S_n^{gh})$ for $g = 1, \ldots, p$ and $h = 1, \ldots, q$ such that for each $g, h \geq 1$, N^{gh} is either equal to $N^{g-1,h}$ or obtained from it by eliminating a NVWDed strategy, and N^{gh} is almost equivalent to $N^{g,h-1}$:

$$\begin{array}{ccccccccc}
N^{00} & \xrightarrow{\text{elim}} & N^{01} & \xrightarrow{\text{elim}} & \cdots & \xrightarrow{\text{elim}} & N^{0,q-1} & \xrightarrow{\text{elim}} & N^{0q} \\
\uparrow \text{ae} & & \uparrow \text{ae} & & & & \uparrow \text{ae} & & \uparrow \text{ae} \\
N^{10} & \xrightarrow{\text{elim}} & N^{11} & \xrightarrow{\text{elim}} & \cdots & \xrightarrow{\text{elim}} & N^{1,q-1} & \xrightarrow{\text{elim}} & N^{1q} \\
\uparrow \text{ae} & & \uparrow \text{ae} & & & & \uparrow \text{ae} & & \uparrow \text{ae} \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
\uparrow \text{ae} & & \uparrow \text{ae} & & & & \uparrow \text{ae} & & \uparrow \text{ae} \\
N^{p-1,0} & \xrightarrow{\text{elim}} & N^{p-1,1} & \xrightarrow{\text{elim}} & \cdots & \xrightarrow{\text{elim}} & N^{p-1,q-1} & \xrightarrow{\text{elim}} & N^{p-1,q} \\
\uparrow \text{ae} & & \uparrow \text{ae} & & & & \uparrow \text{ae} & & \uparrow \text{ae} \\
N^{p0} & \xrightarrow{\text{elim}} & N^{p1} & \xrightarrow{\text{elim}} & \cdots & \xrightarrow{\text{elim}} & N^{p,q-1} & \xrightarrow{\text{elim}} & N^{pq}
\end{array}$$

To see this consider a partial specification of the games N^{gh}, with these properties, in which $N^{g'h'}$ is specified whenever N^{gh} has been specified and $g' \leq g, h' \leq h$. Lemma B implies that this partial specification can be extended by the addition of N^{gh} whenever $N^{g-1,h}$ and $N^{g,h-1}$ have already been specified. It is easy to see that the only partial specifications that cannot be further extended are those that are complete.

For given N^{gh}, N^{pq} is obtained from $N^{p0} = N'$ by iterative elimination of NVWDed strategies. If

$$m_i^g : S_i^{gq} \to S_i^{g-1,q} \quad (g = 1, \ldots, p,\ i = 1, \ldots, n)$$

are the maps that make each N^{gq} almost equivalent to $N^{g-1,q}$, then N^{pq} is equivalent to the subset of $N^{0q} = N''$ given by the sets

$$m_1^1(m_1^2(\cdots m_1^q(S_1^{pq}) \cdots)), \ldots, m_n^1(m_n^2(\cdots m_n^q(S_n^{pq}) \cdots)). \quad \blacksquare$$

Theorem: If N' and N'' are two games obtained from N by iterative elimination of NVWDed strategies, and neither N' nor N'' have any NVWDed strategies, then they are equivalent.

Proof: By the Proposition each must be equivalent to a subset of the other, and since neither has any redundant strategies, both of the equivalences must be realized by injective (one-to-one) maps of the strategy sets. Therefore for each agent the strategy sets in the two games must have the same number of elements, so these maps must be bijections realizing the claimed equivalence. ∎

VI. Games of Perfect Information

A. Draw an extensive form game of perfect information

1. Solve it intuitively.
2. Take the normal form and iteratively eliminate weakly dominated strategies.

B. We say that a normal form game is *flat* if $u_i(s) = u_i(t)$ for all i and all $s,\, t \in S$. The value of a flat game is the payoff vector $(u_1(s), \ldots, u_n(s))$, $s \in S$.

C. **Definition:** A game is *properly dominance solvable* if there is a vector $v \in \mathbb{R}^n$ such that in all complete sequences of dominance truncations $N^0, N^1, \ldots, N^K$, the final game N^K is flat with value v.

Proposition: Consider an extensive game of perfect information with the property that if one agent's payoffs are the same at two terminal nodes, then all agent's payoffs are the same at those two nodes (explain). Then the normal form of this extensive form is properly dominance solvable.

Remark: Moulin states a slightly weaker result. His proof is incorrect, as was pointed out by Gretlein, *Econometrica,* **50**, (1982), 527-528, who supplied a correct proof in another paper. A correct proof of the Proposition is also given in McLennan, "Iterative Elimination of Dominated Strategies", but the argument uses techniques that will be introduced later in the course.

Proof: The hypotheses on the payoffs implies that the derived normal form satisfies the "outcomes" assumption described in V.B.1, so we may apply the Proposition of the last section, concluding that all complete sequences of weak dominance truncations have the same final game, up to renaming of strategies and addition and deletion of redundant strategies. Fixing a particular final game, the terminal nodes that are reached when its strategy vectors are played constitute a "subtree" of the original tree, and if the subtree realizes more than one equivalence class of terminal nodes, there must be a lowest point in the subtree at which some agent chooses between inequivalent terminal nodes. Since that agent cannot be indifferent, a strategy that allows that node to be reached and chooses the inferior branch is weakly dominated by one that makes the same choices everywhere else, and makes the better choice at that point. Now this particular strategy may not be in the final game, but something that very weakly dominates it is, so we obtain a contradiction of the assumption that the sequence of truncations is complete. ∎