**Economics 8117-8** **Noncooperative Game Theory**

**October 22, 1997** **Lecture 4**

**Andrew McLennan**

# The Maximal Number of Regular Totally Mixed Nash Equilibria

**I.** Introduction

A. The motivating theme of this lecture is computation of equilibrium, and in particular the worst-case complexity of this computation as it is usually measured in computer science.

B. After posing the problem, we look at the specific algebra of computation of totally mixed equilibrium in normal form games with three agents, each of whom has two strategies.

C. The issue of bounding the complexity of the problem of computing totally mixed Nash equilibrium leads to a discussion of the largest number of such equilibria a game can have, which has recently been addressed, for normal form games, by Richard McKelvey and myself.

II. The Maximal Number of Regular Equilibria

A. We fix a finite player, finite strategy normal form determined by a set of players $I = \{1, \ldots, n\}$, and finite nonempty sets of pure strategies $S_1, \ldots, S_n$.

1. As usual, let $S = S_1 \times \ldots \times S_n$.
2. For each $i \in I$, let $\Sigma_i = \mathbb{R}^{S_i}$, let $\Sigma = \Sigma_1 \times \ldots \times \Sigma_n$, and let

$$\Sigma_{-i} = \Sigma_1 \times \ldots \times \Sigma_{i-1} \times \Sigma_{i+1} \times \ldots \times \Sigma_n.$$

B. The system of equations and inequalities for *totally mixed equilibrium* is:

$$(\mathrm{a}_1) \quad 0 = u_i(s_i, \sigma_{-i}) - u_i(t_i, \sigma_{-i});$$

$$(\mathrm{b}_1) \quad 1 = \sum_{s_i \in S_i} \sigma_i(s_i);$$

$$(\mathrm{c}_1) \quad 0 < \sigma_i(s_i).$$

1. Such an equilibrium is *regular* if, for the given utilities, the derivative of the function of $\sigma$ whose components are the right hands sides of $(\mathrm{a}_1)$ and $(\mathrm{b}_1)$ is nonsingular.

   a. This means precisely that the hypotheses of the implicit function theorem are satisfied, so that there is an open neighborhood $U$ of $u$ and a $C^\infty$ function $\hat{\sigma} : U \to \Sigma$ with $\hat{\sigma}(u) = \sigma$ such that the graph of $\hat{\sigma}$ is the intersection of the Nash equilibrium correspondence with a neighborhood $W \subset (\mathbb{R}^Z)^I \times \Sigma$ of $(u, \sigma)$.

   b. In particular, if there are, say, $q$ regular equilibria at a particular utility, then there is a neighborhood of this utility, all elements of which have at least $q$ regular equilibria.

C. The central result is

**Theorem:** Let $s = (s_{10}, \ldots, s_{n0}) \in S$ be any strategy $n$-tuple, and let $T_i = S_i - \{s_{i0}\}$, $i = 1, \ldots, n$. There are utilities $u$ for which there are as many regular totally mixed equilibria as the number of partitions $U_1, \ldots, U_n$ of $\cup_i T_i$ such that, for each $i$, $\#U_i = \#T_i$ and $U_i \cap T_i = \emptyset$. There are no utilities with more than this number of regular totally mixed equilibria.

1. The following table gives the maximal number of equilibria for games in which all agents have the same number of pure strategies.

MAXIMUM GENERIC NUMBER OF TOTALLY MIXED NASH EQUILIBRIA

**$n$-person game, each player has $k$ pure strategies**

| $n$ \ $k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 2 | 10 | 56 | 346 | 2252 | 15184 | 104960 |
| 4 | 9 | 297 | 13833 | 748521 | $4.4\times10^{7}$ | $2.8\times10^{9}$ | $1.8\times10^{10}$ |
| 5 | 44 | 13756 | $6.7\times10^{6}$ | $4.0\times10^{9}$ | $2.7\times10^{12}$ | $1.9\times10^{15}$ | $1.5\times10^{18}$ |
| 6 | 265 | 925705 | $5.7\times10^{9}$ | $4.5\times10^{13}$ | $4.1\times10^{17}$ | $4.2\times10^{21}$ | |
| 7 | 1854 | $8.5\times10^{7}$ | $7.8\times10^{12}$ | $9.6\times10^{17}$ | | | |
| 8 | 14833 | $1.0\times10^{10}$ | $1.6\times10^{16}$ | | | | |
| 9 | 133496 | $1.6\times10^{12}$ | | | | | |
| 10 | $1.3\times10^{6}$ | | | | | | |

2. The Theorem automatically provides lower bounds on the worst-case complexity measures of algorithms that compute equilibria, in any sense that involves doing at least one thing (e.g. printing a line of output) for each equilibrium.
3. The hypothesis that a regular totally mixed equilibrium is a useful representation of emperical data, or of the mental processes of the agents in question, seems implausible if there are 1,334,960 other regular totally mixed equilibria.
   a. My assessment of the situation is that game theory works well for those games for which it works well.

**III.** Three Agents, Each with Two Strategies

A. In the last lecture we discussed a procedure in which we first enumerate the possible supports or bases for equilibrium, then, for each basis, determine the set of equilibria with that basis.

   1. In particular, we found that as long as at most two agents were mixing, the equilibrium problem for any given basis was a matter of linear algebra, and therefore presumably well understood.

B. The simplest example that is not (in the sense just mentioned) trivial occurs when there are three agents, each of whom is mixing over two pure strategies.

1. Let $S_i = \{0,1\}$, $i = 1,2,3$.

2. In the equations expressing each agent's indifference between the two choices, which are homogeneous, it is simplifying (in the sense of reducing the number of variables) to replace the normalization $\sigma_i(0)+\sigma_i(1) = 1$ with the normalization $\sigma_i(0) = 1$, after which the equations have the form

$$w^1_{11}\sigma_2(1)\sigma_3(1) + w^1_{10}\sigma_2(1) + w^1_{01}\sigma_3(1) + w^1_{00} = 0$$
$$w^2_{11}\sigma_1(1)\sigma_3(1) + w^2_{10}\sigma_1(1) + w^2_{01}\sigma_3(1) + w^2_{00} = 0$$
$$w^3_{11}\sigma_1(1)\sigma_2(1) + w^3_{10}\sigma_1(1) + w^3_{01}\sigma_2(1) + w^3_{00} = 0.$$

Here $w^i_{jk}$ is the difference in utility for agent $i$ that results when agent $i$ switches to his/her second strategy, given that the other agents are playing $j$ and $k$.

3. This system is equivalent to

$$\begin{aligned} &&(\sigma_2(1) - a_{12})\quad &(\sigma_3(1) - a_{13}) &= \delta_1 \\ (\sigma_1(1) - a_{21})&& &(\sigma_3(1) - a_{23}) &= \delta_2 \\ (\sigma_1(1) - a_{31})\quad &&(\sigma_2(1) - a_{32})& &= \delta_3 \end{aligned}$$

where

$$\begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} = -\begin{pmatrix} 0 & \frac{w^1_{01}}{w^1_{11}} & \frac{w^1_{10}}{w^1_{11}} \\ \frac{w^2_{01}}{w^2_{11}} & 0 & \frac{w^2_{10}}{w^2_{11}} \\ \frac{w^3_{01}}{w^3_{11}} & \frac{w^3_{10}}{w^3_{11}} & 0 \end{pmatrix} \text{ and}$$

$$\delta_i = \frac{-\begin{vmatrix} w^i_{00} & w^i_{01} \\ w^i_{10} & w^i_{11} \end{vmatrix}}{(w^i_{11})^2} = \frac{w^i_{01}w^i_{10} - w^i_{00}w^i_{11}}{(w^i_{11})^2} \qquad (i = 1,2,3).$$

a. This representation of the equations represents an important idea, as we will see in generality later.

4. It is tedious but straightforward to verify that there are two solutions, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $\sigma' = (\sigma_1', \sigma_2', \sigma_3')$, to this system of equations, given by
$$\sigma_i = \frac{d_i + \Delta^{1/2}}{2c_i}, \quad \sigma_i' = \frac{d_i - \Delta^{1/2}}{2c_i},$$
where
$$\omega_i = a_{i+2,i} - a_{i+1,i},$$
$$c_i = \frac{\omega_1\omega_2\omega_3}{\omega_i} + \delta_i, \quad d_i = (a_{i+1,i} + a_{i+2,i})c_i - \omega_{i+1}\delta_{i+1} + \omega_{i+2}\delta_{i+2},$$
$$\Delta = (\omega_1\omega_2\omega_3 + \omega_1\delta_1 + \omega_2\delta_2 + \omega_3\delta_3)^2 + 4\delta_1\delta_2\delta_3.$$
(In the definitions of $\omega_i$ and $d_i$ the indices are to be interpreted as integers modulo 3.)
5. The solutions $\sigma$ and $\sigma'$ are distinct if and only if $\Delta \neq 0$.
6. They are real if and only if $\Delta \geq 0$; this will certainly be the case if any one of the three equations factor, in the sense that $\delta_i = 0$, in which case $\delta_1\delta_2\delta_3 = 0$.
7. We could figure out when all probabilities are positive, but it seems unlikely to be very informative.
8. These calculation are, in my humble opinion, quite pretty, but not very encouraging, in that they suggest that this sort of calculation is hard.

**IV.** Reformulation of the Algebraic System

A. For each $i$ let $d_i = \#S_i - 1$, and let $S_i = \{s_{i0}, \ldots, s_{id_i}\}$.

1. For $k = 1, \ldots, d_i$ let $v_i^k : \Sigma_{-i} \to I\!R$ be given by
$$v_i^k(\sigma_{-i}) = u_i(s_{ik}, \sigma_{-i}) - u_i(s_{i0}, \sigma_{-i}).$$
2. The system of equations and inequalitities for totally mixed equilibrium is now
$$(\mathrm{a}_2) \quad 0 = v_i^k(\sigma_{-i});$$
$$(\mathrm{b}_2) \quad 1 = \sum_{s_i \in S_i} \sigma_i(s_i);$$
$$(\mathrm{c}_2) \quad 0 < \sigma_i(s_i).$$

3. **N.B.** – Each $v_i^k$ is *multilinear*, in the sense of being linear in each $\sigma_j$, $j \neq i$. Conversely, any collection of multilinear functions is derived from some utility.

B. Now consider a system of linear isomorphisms $\ell_i : \Sigma_i \to \Sigma_i$, $i \in I$, with each $\ell_i$ mapping the affine subspace of $\Sigma_i$ given by ($\mathrm{b}_2$) onto itself.

1. If $\sigma$ satisfies both ($\mathrm{a}_2$) and ($\mathrm{b}_2$), then $(\ell_1(\sigma_1), \ldots, \ell_n(\sigma_n))$ is a solution of ($\mathrm{a}_2$) and ($\mathrm{b}_2$) with each $v_i^k$ replaced by $\hat{v}_i^k$, where $\hat{v}_i^k \in V_i$ is the multilinear function given by

$$\hat{v}_i^k(\sigma_{-i}) = v_i^k({\ell_1}^{-1}(\sigma_1), \ldots, {\ell_n}^{-1}(\sigma_n)).$$

2. For any finite set of solutions of ($\mathrm{a}_2$) and ($\mathrm{b}_2$), say $\sigma^1, \ldots, \sigma^r$, it is possible to find $\ell_1, \ldots, \ell_n$ such that ($\mathrm{c}_2$) is satisfied by each $\ell_i(\sigma_i^h)$.

3. *If the set of vectors* $\mathbf{v}$ *of multilinear functions having a certain number of solutions of ($a_2$) and ($b_2$) has a nonempty interior, then so does the set of such* $\mathbf{v}$ *having that number of solutions of ($a_2$), ($b_2$), and ($c_2$).*

   a. Sometimes it turns out to be mathematically convenient to retain the condition that strategic probabilities not vanish.

   b. According to circumstances our goal will be characterization of the maximal generic number of solutions either the system consisting of ($\mathrm{a}_2$) and ($\mathrm{b}_2$) or the system

$$(\mathrm{a}_3) \quad 0 = v_i^k(\sigma_{-i});$$

$$(\mathrm{b}_3) \quad 1 = \sum_{s_i \in S_i} \sigma_i(s_i);$$

$$(\mathrm{c}_3) \quad 0 \neq \sigma_i(s_i).$$

4. We generalize the system above by dropping the requirement that the number $d_i$ of equations expressing agent $i$'s indifference between his/her various pure strategies coincides with

$$c_i = \dim \{ \sigma_i \in I\!R^{\Sigma_i} : 1 = \sum_{s_i \in S_i} \sigma_i(s_i) \}.$$

a. We continue to assume that $c_1 + \ldots + c_n = d_1 + \ldots + d_n$.

b. For vectors $c = (c_1, \ldots, c_n)$ and $d = (d_1, \ldots, d_n)$ satsfying this condition, consider sets $C_1, \ldots, C_n$ where each $C_i$ has $c_i$ elements. Let $\mathcal{L}_n(c, d)$ be the number of partitions of $\bigcup_i C_i$ into sets $D_1, \ldots, D_n$ where, for each $i$, $C_i \cap D_i = \emptyset$ and $C_i \cap D_i = \emptyset$.

5. Since ($a_3$) and ($c_3$) are homogeneous — that is, their truth values are unaffected by replacing any $\sigma_i$ with $\lambda_i \sigma_i$ for any nonzero scalar $\lambda_i$ – they define a subset of the associated product of projective spaces.

   a. For any real vector space $W$, let $\mathbb{P}(W)$ be the associated projective space consisting of the 1-dimensional linear subspaces of $W$.

   b. For any $w \in W - \{0\}$ we let $[w] \in \mathbb{P}(W)$ be the span of $w$. In the case of $\sigma \in \Sigma$ we will abuse notation, letting $[\sigma]$ denote

$$([\sigma_1], \ldots, [\sigma_n]) \in \mathbb{P}(\Sigma_1) \times \ldots \times \mathbb{P}(\Sigma_n).$$

   c. We may consider the problem of characterizing the largest integer $K$ such that, for an open set of vectors $\mathbf{v}$, there are $K$ points $[\sigma] \in \mathbb{P}(\Sigma_1) \times \ldots \times \mathbb{P}(\Sigma_n)$ such that

$$(*) \qquad v_i^k(\sigma_{-i}) = 0. \quad (i = 1, \ldots, n)(k = 1, \ldots d_i)$$

## V. A Utility that Attains the Maximal Number

A. Consider the special case of system $(*)$ in which, for each $i$ and $k = 1, \ldots, d_i$,

$$v_i^k(\sigma_{-i}) = \prod_{j \neq i} \ell_{ij}^k(\sigma_j),$$

where the $\ell_{ij}^k : \Sigma_j \to \mathbb{R}$ are nonzero linear functionals.

1. For each $j = 1, \ldots, n$, let

$$L_j = \{ \ell_{ij}^k : i \neq j, k = 1, \ldots, d_i \}.$$

**Proposition:** *If the functionals $\ell_{ij}^k$ are in general position in the sense that, for any $j$ and any subset $L \subset L_j$ of cardinality $c \leq c_j$, the intersection of the kernels of the elements of $L$ has dimension $c_j + 1 - c$, then the system $(*)$ has $\mathcal{L}_n(c,d)$ solutions, each of which is regular.*

**Proof:** At a solution $\tau$, for each $j$ at most $c_j$ elements of $L_j$ vanish. Since there are $\sum_i d_i = \sum_i c_i$ functions $w_i^k$, in order for each to vanish, for each $j$ exactly $c_j$ elements of $L_j$ must vanish at $\tau_j$, and for each $i$ and $k$ there must be exactly one $j \neq i$ such that $a_{ij}^k$ vanishes. Thus each solution $\tau$ induces a partition of $\{ w_i^k : i = 1, \ldots, n, k = 1, \ldots, d_i \}$ into sets $U_1, \ldots, U_n$ where $w_i^k \in U_j$ if $a_{ij}^k(\tau_j) = 0$. Note that $\#(U_j) = c_j$ for each $j$ and $v_i^k \in U_j$ implies that $j \neq i$. Conversely, given a partition with these properties, there is exactly one corresponding solution.

By varying $\tau_j$ in the intersection of the kernels of the other $a_{i'j}^{k'} \in L_j$ with $a_{i'j}^{k'}(\tau_j) = 0$, one can change the value of $w_i^k$ (since no other $a_{ij'}^k$ vanishes at $\tau$) without affecting any other $w_{i'}^{k'}$. Thus the image of the derivative of $\mathbf{w}$ includes each standard basis vector of $\mathbb{R}^{d_1+\ldots+d_n}$, whence $\tau$ is a regular solution. ∎

**VI.** Bernstein's Theorem

A. The *punctured complex plane* is $\mathbb{C}^* = \mathbb{C} - \{0\}$.

1. For an *exponent vector* $a = (a_1, \ldots, a_m) \in \mathbb{Z}^m$ let $x^a$ denote the monomial ${x_1}^{a_1} \cdot \ldots \cdot {x_m}^{a_m}$.

B. We consider the system of *Laurent polynomials*

$$\begin{aligned} 0 = f_1(x) &= c_{11}x^{a_{11}} + \ldots + c_{1\ell_1}x^{a_{1\ell_1}}, \\ &\vdots \\ 0 = f_m(x) &= c_{m1}x^{a_{m1}} + \ldots + c_{m\ell_m}x^{a_{m\ell_m}}. \end{aligned}$$

1. Let $\mathcal{A}_j = \{ a_{j1}, \ldots, a_{j\ell_j} \}$, $j = 1, \ldots, m$.
2. The *Newton polytope* of $f_j$ is $Q_j = con(\mathcal{A}_j)$.

3. The *mixed volume* of $Q_1, \ldots, Q_m$ is the coefficient of $\lambda_1 \cdot \ldots \cdot \lambda_m$ in the expansion of

$$\mathrm{vol}(\lambda_1 \mathbf{Q}_1 + \ldots + \lambda_m \mathbf{Q}_m)$$

as a polynomial in $\lambda_1, \ldots, \lambda_m$. Denote this quantity by $MV(Q_1, \ldots, Q_m)$.

**Theorem:** (Bernstein (1975), Kushnirenko (1975)) There is an open dense set of coefficient vectors $c \in \mathbb{C}^{\mathcal{A}_1} \times \ldots \times \mathbb{C}^{\mathcal{A}_m}$ for which the system above has exactly $MV(Q_1, \ldots, Q_m)$ distinct roots in $(\mathbb{C}^*)^m$.

4. Bernstein's paper is about three pages long, and "elementary" in the sense of not appealing to many advanced results. Nonetheless the proof is quite hard, and the argument quite remarkable. The central idea is to choose some $a \in \mathcal{A}_1$ and consider the homotopy, as $\tau$ varies between 0 and $\infty$, of the system with the equation $f_1(x) = 0$ replaced by $f_1(x) + \tau \cdot x^a = 0$. General principles of solutions of algebraic systems implies that the number of roots of this system is the same for almost all $\tau$. At $\tau = 0$ we have the original system, while in the limit as $\tau \to \infty$ we approach roots of simpler systems with $f_1$ replaced by the condition that some variable vanishes.

5. The argument showing that the mixed volume for the system of interest is, in fact, $\mathcal{L}_n(c, d)$, is a lengthy, and somewhat tricky, calculation that will not be discussed here.