

Economics 8117-8**Noncooperative Game Theory****October 29, 1997****Lecture 5****Professor Andrew McLennan****An Example with Continuous Strategy Spaces****I. Introduction**

A. There are no algorithms for the determination of the solution set of games with continua of pure strategies for some agents.

1. Mechanical calculation cannot be substituted for strategic understanding.
2. There is no all inclusive set of analytic techniques. Thus examples are the only way to develop skill in this.

II. We analyze a game of pursuit and evasion.

A. There are two players, the *discount house* and the *boutique*, agents 1 and 2 respectively. The discount house would like to locate near the boutique to take its customers, and the boutique would like to avoid this.

B. The two agents simultaneously choose locations $x \in [-1, 1]$ (for the discount house) and $y \in [-1, 1]$ (for the boutique).

C. The game is zero sum, and the payoffs for the discount house and the boutique are

$$u_1(x, y) = (1 + a(x - y)^2)^{-1} \quad \text{and}$$

$$u_2(x, y) = -(1 + a(x - y)^2)^{-1}.$$

D. Given mixed strategies $\mu \in \Delta([-1, 1])$ for the discount house and $\nu \in \Delta([-1, 1])$ for the boutique, the expected payoffs are

$$u_1(\mu, \nu) = \int_{-1}^1 \int_{-1}^1 (1 + a(x - y)^2)^{-1} d\mu d\nu \quad \text{and}$$

$$u_2(\mu, \nu) = -u_1(\mu, \nu).$$

III. The Analysis

A. We will prove that when $a \leq \frac{1}{3}$, the only equilibrium has agent 1 playing 0 all the time while agent 2 mixes equally between -1 and 1 .

B. Symmetry

1. For $\lambda \in \Delta([-1, 1])$ let $f(\lambda) \in \Delta([-1, 1])$ be the “flip” of λ across the origin, i.e. $f(\lambda)(E) = \lambda(-E)$ for all measurable $E \subset [-1, 1]$.
2. For all $\mu, \nu \in \Delta([-1, 1])$ it is easy to see that $u_1(\mu, f(\nu)) = u_1(f(\mu), \nu)$.
3. Let (μ^*, ν^*) be a Nash equilibrium. The equilibrium conditions give

$$\begin{aligned} u_1(\mu^*, \nu^*) &\geq u_1(f(\mu^*), \nu^*) = u_1(\mu^*, f(\nu^*)) \\ &= -u_2(\mu^*, f(\nu^*)) \geq -u_2(\mu^*, \nu^*) = u_1(\mu^*, \nu^*). \end{aligned}$$

- a. Thus both μ^* and $f(\mu^*)$ are best responses to both ν^* and $f(\nu^*)$, and vice versa.
- b. In general, if a mixed strategy is a best response to two of the opponents mixed strategies, it is a best response to any convex combination of them.
- c. Thus

$$\left(\frac{\mu^*}{2} + \frac{f(\mu^*)}{2}, \frac{\nu^*}{2} + \frac{f(\nu^*)}{2} \right)$$

is a Nash equilibrium.

4. Our analysis now has two phases.
 - a. Find all equilibria that are symmetric: $\mu^* = f(\mu^*)$ and $\nu^* = f(\nu^*)$.
 - b. For each symmetric equilibrium find all equilibria that have the given equilibrium as the symmetric average.

C. Equilibria in which agent 1 plays a pure strategy.

1. Let (μ^*, ν^*) be a symmetric equilibrium in which agent 1 plays a pure strategy.
2. Clearly μ^* must assign all probability to 0.
3. Agent 2’s pure best responses are -1 and 1 , and symmetry requires agent 2 to mix equally between them.

4. A long and careful algebraic computation (see “IV. The Inequality”) shows that agent 1’s equilibrium condition is satisfied if and only if $a \leq \frac{1}{3}$.

D. Uniqueness

1. Let (μ^*, μ^*) be a symmetric equilibrium, $a \leq \frac{1}{3}$.
2. Consider the following thought experiment in which we give agent 1 more information before he chooses. Suppose that agent 1, before choosing x , is told the absolute value of y , so that he will choose the optimal x for the belief $(\frac{1}{2})y + (\frac{1}{2})(-y)$. The expected payoff is

$$\left(\frac{1}{2}\right) \left(1 + ay^2 \left(\frac{x}{y} - 1\right)^2\right)^{-1} + \left(\frac{1}{2}\right) \left(1 + ay^2 \left(\frac{x}{y} + 1\right)^2\right)^{-1}.$$

The verification of the equilibrium condition showed that $x = 0$ is the optimal choice when $ay^2 \leq \frac{1}{3}$, and of course this follows from $a \leq \frac{1}{3}$. (See IV.A.7. below.)

3. Thus $(0, (\frac{1}{2})(-1) + (\frac{1}{2})1)$ is the unique symmetric equilibrium.
4. It is easy to see that there is no other equilibrium that has $(0, (\frac{1}{2})(-1) + (\frac{1}{2})1)$ as its symmetric average.

IV. The Inequality

- A. We now verify the claim in III.C.4 that 0 is a best response for agent 1 to $(\frac{1}{2})1 + (\frac{1}{2})(-1)$ if and only if $a \leq \frac{1}{3}$.

1. In order for 0 to be a best response to $(\frac{1}{2})1 + (\frac{1}{2})(-1)$ it must be the case for all $x \in [-1, 1]$ that

$$(1 + a)^{-1} \geq \frac{[(1 + a(1 - x)^2)^{-1} + (1 + a(1 + x)^2)^{-1}]}{2}.$$

2. We begin by simplifying as much as possible, keeping the expressions $(1 - x)^2$ and $(1 + x)^2$ intact:

$$2(1 + a(1 - x)^2)(1 + a(1 + x)^2) \geq (1 + a)(2 + a(1 - x)^2 + a(1 + x)^2).$$

3. Multiplying out and cancelling terms gives

$$(1-x)^2 + (1+x)^2 + 2a(1-x)^2(1+x)^2 \geq 2 + a(1-x)^2 + a(1+x)^2.$$

4. Solving for a yields the inequality:

$$\frac{[(1-x)^2 + (1+x)^2 - 2]}{[(1-x)^2 + (1+x)^2 - 2(1-x)^2(1+x)^2]} \geq a.$$

5. Although this looks quite complicated, straightforward algebra shows that it reduces to

$$a \leq (3 - x^2)^{-1}.$$

6. Summarizing, we have shown that 0 is a best response to $(\frac{1}{2})1 + (\frac{1}{2})(-1)$ if and only if $a \leq (3 - x^2)^{-1}$ for all $x \in [-1, 1]$, and clearly this is the case if and only if $a \leq \frac{1}{3}$.

7. In fact we have shown that if $a \leq \frac{1}{3}$, then $x = 0$ yields a higher value for the expression for expected payoff than any other point in \mathbb{R} . This is important in that, in the analysis in III.D., we applied this result after the substitutions $x \mapsto \frac{x}{y}$ and $a \mapsto ay^2$.