

Economics 8117-8**Noncooperative Game Theory****October 29, 1997****Lecture 6****Professor Andrew McLennan****Perfection****I. Introduction**

A. The outline of the presentation.

1. We present examples of unreasonable Nash equilibria.
2. We then present a general concept to deal with them.
3. We then consider more examples to see how it works.

B. The incredible threat

1. Give the extensive form.
2. The normal form of this game is

$$\begin{array}{c}
 1 \setminus 2 \quad L \quad R \\
 U \quad \left(\begin{array}{cc} (1, 2) & (1, 2) \end{array} \right) \\
 D \quad \left(\begin{array}{cc} (2, 1) & (0, 0) \end{array} \right)
 \end{array}$$

3. (D, L) and (U, R) are both equilibria, and in fact $(U, \alpha L + (1 - \alpha)R)$ is an equilibrium for all $\alpha \leq \frac{1}{2}$. Only (D, L) is “reasonable”.
4. Although this phenomenon is referred to as an “incredible threat”, it does not really depend on (D, L) being less desirable for agent 2 than U .

C. Behavior inconsistent with any beliefs.

1. Present extensive form.
2. The corresponding normal form is

$$\begin{array}{c}
 1 \setminus 2 \quad \ell \quad m \quad r \\
 U \quad \left(\begin{array}{ccc} (2, 3) & (0, 1) & (2, 0) \end{array} \right) \\
 C \quad \left(\begin{array}{ccc} (2, 0) & (0, 1) & (2, 3) \end{array} \right) \\
 D \quad \left(\begin{array}{ccc} (1, 4) & (1, 4) & (1, 4) \end{array} \right)
 \end{array}$$

3. There are two types of Nash equilibria.
 - a. $(\alpha U + (1 - \alpha)C, \beta \ell + (1 - \beta)r)$ where $\beta = 0$ if $\alpha < \frac{1}{2}$, β can be any probability if $\alpha = \frac{1}{2}$, and $\beta = 1$ if $\alpha > \frac{1}{2}$.
 - b. $(D, \gamma \ell + \delta m + \varepsilon r)$ such that $\gamma + \delta + \varepsilon = 1$ and $\delta \geq \frac{1}{2}$.
4. The second class of equilibria is unreasonable since agent 2's behavior is not optimal for any belief about the relative likelihood of U and C .

D. Equilibria using dominated strategies

1. Consider the normal form

$$\begin{array}{c}
 1 \setminus 2 \quad \begin{array}{cc} L & R \end{array} \\
 \begin{array}{c} U \\ D \end{array} \quad \left(\begin{array}{cc} (1, 1) & (0, 0) \\ (0, 0) & (0, 0) \end{array} \right)
 \end{array}$$

2. (U, L) and (D, R) are both equilibria, but (D, R) uses weakly dominated strategies, and seems peculiar.
3. One should retain an open mind on this, since one has to allow (D, R) if one wants the equilibrium correspondence to have a closed graph.

II. A Model of Mistakes with Small but Positive Probability

- A. The intuitive idea – each agent is forced to assign a small but positive probability to each of his pure strategies.

1. One cannot have equilibrium strategy vectors with incredible threats, since the situation in which the threat is to be carried out occurs with positive probability.
2. Bayesian posterior beliefs are defined everywhere in the game tree.
3. Weakly dominated strategies are distinctly inferior.

- B. The formal development

1. Let $N = (S_1, \dots, S_n; u_1, \dots, u_n)$ be a normal form game.
2. **Definition:** A *tremble* for N is a vector of functions $\varepsilon = (\varepsilon_i)_{i \in I}$ where each $\varepsilon_i : S_i \rightarrow (0, 1]$ satisfies $\sum_{S_i} \varepsilon_i(s_i) \leq 1$.

3. **Definition:** Given an agent i , a strategy vector σ , and a tremble ε , the ε -best response set for i is

$$BR_i^\varepsilon(\sigma) = \{\sigma'_i \mid \sigma'_i(s_i) \geq \varepsilon_i(s_i) \text{ with strict inequality only if } s_i \in BR_i(\sigma), \text{ all } s_i \in S_i\}.$$

4. **Definition:** The ε -best response correspondence is $BR^\varepsilon : \Sigma \rightarrow \Sigma$ defined by $BR^\varepsilon(\sigma) = \prod_{i \in I} BR_i^\varepsilon(\sigma)$. An ε -perfect equilibrium is a fixed point of BR^ε .

Lemma: For any tremble ε there is an ε -perfect equilibrium.

Proof: We leave it as an exercise to verify that BR^ε is an upper semicontinuous convex valued correspondence. After this fact has been verified the result follows from Kakutani's fixed point theorem. ■

5. **Definition:** A strategy vector $\sigma^* \in \Sigma$ is a *perfect equilibrium* if there is a sequence of trembles $\{\varepsilon^r\}$ with

$$(*) \quad \lim_{r \rightarrow \infty} (\max_{i \in I, s_i \in S_i} \varepsilon_i^r(s_i)) = 0$$

and a sequence $\{\sigma^r\}$, where each σ^r is an ε^r -perfect equilibrium and $\sigma^r \rightarrow \sigma^*$.

Theorem 1: There are perfect equilibria.

Proof: Let $\{\varepsilon^r\}$ be a sequence of trembles satisfying (*), and for each r let σ^r be an ε^r -perfect equilibrium. Since Σ is compact, the sequence $\{\sigma^r\}$ has a convergent subsequence. ■

Theorem 2: Perfect equilibria are Nash equilibria.

Proof: Let $\{\varepsilon^r\}$ and $\{\sigma^r\}$ be as in the definition of perfect equilibrium, with $\sigma^r \rightarrow \sigma^*$. Consider $i \in I$ and $\sigma_i \in \Sigma_i$. If σ_i is not a best response for i to σ^* , then it is not a best response to σ^r for large r , since u_i is continuous, so that $\sigma_i^r(s_i) = \varepsilon_i^r(s_i)$ and $\sigma_i^*(s_i) = 0$. Thus $\sigma_i^* \in BR_i(\sigma^*)$. ■

III. The Examples Revisited

[Reanalyze the three examples of the introduction.]

IV. Illustrating Perfection Graphically

- A. Consider the following idea: a Nash equilibrium is a point in the intersection of the graphs of the correspondences $BR_i : \Sigma_{-i} \rightarrow \Delta(S_i)$.

Lemma: Suppose, for $i = 1, \dots, n$, that $D_i \subset \mathbb{R}^{m_i}$ is nonempty, compact and convex, and $F_i : \prod_{j \neq i} D_j \rightarrow D_i$ is an uppersemicontinuous convex valued correspondence. Let $Gr(F_i) = \{\delta \in \prod_{j \in I} D_j \mid \delta_i \in F_i(\delta_{-i})\}$. Then $\bigcap_{i=1, \dots, n} Gr(F_i) \neq \emptyset$.

Proof: Apply the Kakutani fixed point theorem to $F : \prod_{j \in I} D_j \rightarrow \prod_{j \in I} D_j$ defined by $F(\delta) = \prod_{i \in I} F_i(\delta_{-i})$. ■

- B. Consider the intersection in $\Delta(\{U, D\}) \times \Delta(\{L, R\})$ of the graphs of the agents' best response correspondences, and the intersections of the agents' ε -best response correspondences, for Incredible Threat.

V. Superperfection

- A. Consider the following question. Can one find a strategy vector σ^* such that for any sequence $\{\varepsilon^r\}$ of trembles satisfying (*) there is a sequence $\{\sigma^r\} \subset \Sigma$ where each σ^r is an ε^r -perfect equilibrium and a subsequence that converges to σ^* .
1. An affirmative answer would define a more powerful solution concept.
 2. One could argue that such superperfect equilibria are more reasonable than equilibria that are merely perfect for some sequence of trembles.
- B. In any event one cannot guarantee the existence of a superperfect equilibrium.

1. Consider the following normal form:

$$\begin{array}{c}
 1 \setminus 2 \quad L \quad R \\
 U \quad \left(\begin{array}{cc} (1, 1) & (1, 1) \\ (0, 1) & (0, 0) \\ (0, 0) & (0, 1) \end{array} \right) \\
 C \\
 D
 \end{array}$$

2. When $\varepsilon_1(C) > \varepsilon_1(D)$, all ε -perfect equilibria have agent 2 assigning probability $1 - \varepsilon_2(R)$ to L , while if $\varepsilon_1(C) < \varepsilon_1(D)$ then all ε -perfect equilibria have agent 2 assigning probability $1 - \varepsilon_2(L)$ to R .

C. There are results for generic normal form payoffs.

1. They are derived from Sard's Theorem.
2. Generic extensive form payoffs do not map into a generic set of normal form payoffs.

VI. Perfection for Games with Infinite Strategy Spaces

A. Leo Simon ("Local Perfection," *Journal of Economic Theory* **43** (1987), 134–156) has considered the issue of extending the notion of perfection to games with compact metric spaces of strategies.

1. There is a conceptual difficulty concerning the appropriate notion of a tremble in this context. In general it is not possible to insist on anything as well behaved as absolute continuity with respect to Lebesgue measure.
2. Given a class of trembles, one can follow the lines laid out by Selten, applying the Fan-Glicksberg fixed point theorem to get existence of trembling-hand perfect equilibrium, then using compactness (in the weak* topology) of the space of probability measures on a compact metric space to extract convergent subsequences.

B. In addition, Simon develops a notion akin to properness (next lecture) by imposing a condition that, intuitively, amounts to a requirement that "mistakes" near the

supports of the agents equilibrium mixed strategies are much more probable than distant mistakes.

VII. Imperfections of Perfection

A. Adding strictly dominated strategies can make equilibria perfect that were not perfect before.

$$\begin{array}{c}
 1 \setminus 2 \\
 \begin{array}{ccc}
 L & R & M \\
 U & \left(\begin{array}{cc} (1, 1) & (0, 0) \end{array} \right. & \left. \begin{array}{c} (1, -1) \\ (0, 0) \\ (-1, 1) \end{array} \right) \\
 D & \left(\begin{array}{cc} (0, 0) & (0, 0) \end{array} \right. & \left. \begin{array}{c} (2, -1) \\ (-1, 2) \\ (-1, -1) \end{array} \right) \\
 C & \left(\begin{array}{cc} (-1, 1) & (-1, 2) \end{array} \right. & \left. \begin{array}{c} (-1, -1) \\ (-1, -1) \\ (-1, -1) \end{array} \right)
 \end{array}
 \end{array}$$

1. $(\alpha U + (1 - 3\alpha)D + 2\alpha C, \alpha L + (1 - 3\alpha)R + 2\alpha M)$ is an ε -perfect equilibrium for an appropriate tremble ε .

B. The Kohlberg Example

1. Draw the extensive form.
2. The normal form is

$$\begin{array}{c}
 1 \setminus 2 \\
 \begin{array}{cc}
 \ell & r \\
 U & \left(\begin{array}{cc} (3, 3) & (0, 0) \end{array} \right) \\
 M & \left(\begin{array}{cc} (0, 0) & (1, 1) \end{array} \right) \\
 D & \left(\begin{array}{cc} (2, 2) & (2, 2) \end{array} \right)
 \end{array}
 \end{array}$$

3. Here M is a dominated action, but adding it to the game makes (D, r) a perfect equilibrium whereas it was not perfect before.