

Economics 8117-8**Noncooperative Game Theory****November 4, 1997****Lecture 8****Professor Andrew McLennan****Stability****I. Introduction**

A. Consider the Kohlberg example again.

1. The equilibria are (L, ℓ) and $(R, \alpha\ell + (1 - \alpha)r)$, $\alpha \leq \frac{2}{3}$. They are all perfect.
2. The proper equilibria are (L, ℓ) , $(R, (\frac{1}{u})\ell + (\frac{3}{4})r)$ and (R, r) .
3. The qualitative properties of the equilibrium sets are unaffected by small perturbations of the payoffs.

B. We will develop a solution concept that allows only the “good” equilibria in this example.

II. Review

A. We have looked for *points* that could be justified as the limits of fixed points for some sequence of perturbed best response correspondences.

1. Nash equilibrium.
2. Perfection.
3. Properness.

B. The Kohlberg example suggests that we would like to have stability with respect to *all* perturbations.

1. In some respects this is what stability does.
2. However, the solution concept cannot be point valued.

$$\begin{array}{c}
 1 \setminus 2 \quad L \quad M \quad R \\
 U \quad \left(\begin{array}{ccc} (1, 1) & (0, 0) & (1, 0) \\ (1, 1) & (1, 0) & (0, 0) \end{array} \right) \\
 D
 \end{array}$$

3. Instead we look for *sets* of equilibria that are stable with respect to all trembles.

III. Formal Development

A. Let \mathcal{C} be a class of upper semicontinuous convex valued (u.s.c.c.v.) correspondences

$$F : \Sigma \rightarrow \Sigma.$$

B. If F, F' are two correspondences, the *distance from F' to F* is

$$d(F', F) = \max_{(x', y') \in Gr(F')} \min_{(x, y) \in Gr(F)} \|x' - x\| + \|y' - y\|.$$

1. Graph an example.

2. In words the distance from F' to F is the maximum over points in the graph of F' of the distance to the nearest point in the graph of F .

2. Define the ε -ball around F to be $\mathbf{B}(F; \varepsilon) = \{F' \mid d(F', F) < \varepsilon\}$. There is a topology on \mathcal{C} in which a set U is open if and only if, for each $F \in U$, there is some $\varepsilon > 0$ such that $\mathbf{B}(F; \varepsilon) \subset U$.

4. Note that d is not a metric, since it is not symmetric, but it does satisfy the triangle inequality $d(F'', F) \leq d(F'', F') + d(F', F)$. This implies that the balls $\mathbf{B}(F; \varepsilon)$ are open sets.

C. Let E be the set of equilibria of $N = (S_1, \dots, S_n; u_1, \dots, u_n)$.

1. Note that E is closed and therefore compact.

D. **Definition:** A set $K \subset E$ is \mathcal{C} -stable if it is nonempty, compact, and for all $\varepsilon > 0$ there exists $\delta > 0$ such that $F' \in \mathcal{C}$ and $d(F', BR) < \delta$ implies that there exists a fixed point x' of F' such that

$$\min_{x \in K} \|x' - x\| < \varepsilon.$$

1. The existence of \mathcal{C} -stable sets is an easy consequence of the Kakutani fixed point theorem.

Lemma: E is \mathcal{C} -stable.

Proof: Otherwise there is an $\varepsilon > 0$ for which there exists a sequence $\{F_r\} \in \mathcal{C}$ with

$\delta(F_r, BR) \rightarrow 0$ such that, for each r , F_r has no fixed points in $B(E; \varepsilon)$. For each r let x_r be a fixed point of F_r . Since $d(F_r, BR) \rightarrow 0$ we may choose a sequence $\{(y_r, z_r)\} \subset G_r(BR)$ with $\|x_r - y_r\| + \|x_r - z_r\| \rightarrow 0$. The sequence $\{(y_r, z_r)\}$ has a limit point, say (y, z) , since Σ is compact. We have $(y, z) \in G_r(BR)$ since BR is u.s.c, and we have $y = z$ since (y, z) is also a limit point of $\{(x_r, x_r)\}$. Thus $y \in E$, but $\min_{x \in E} \|y - x\| \geq \liminf \min_{x \in E} \|x_r - x\| \geq \varepsilon$. This contradiction completes the proof. ■

2. The following obvious fact is used frequently.

Lemma: If $K \subset L \subset E$ with K \mathcal{C} -stable and L compact, then L is \mathcal{C} -stable.

Proof: This is an immediate consequence of the definitions. ■

Proposition: If $K_1 \supset K_2 \supset \dots$ and each K_j is \mathcal{C} -stable, then $K_\infty = \bigcap_{j=1,2,\dots} K_j$ is \mathcal{C} -stable.

Proof: Choose $x_j \in K_j$ for all j , and let x be a limit point of $\{x_j\}$. Then compactness implies that x is an element of each K_j , so $x \in K_\infty$ and K_∞ is nonempty. Clearly K_∞ is compact.

Fix $\varepsilon > 0$. There is j such that $K_j \subset \mathbf{B}(K_\infty; \frac{\varepsilon}{2})$. (Otherwise $\{K_j - \mathbf{B}(K_\infty; \frac{\varepsilon}{2})\}$ would be a nested sequence of nonempty compact sets and have a nonempty intersection as above.) Since K_j is \mathcal{C} -stable, there is $\delta > 0$ such that $d(F', BR) < \delta$ implies the existence of x' , a fixed point of F' , with

$$x' \in \mathbf{B}(K_j; \frac{\varepsilon}{2}) \subset \mathbf{B}(K_\infty; \varepsilon).$$

Thus K_∞ is \mathcal{C} -stable. ■

E. **Definition:** A compact set K is *connected* if there do not exist nonempty compact sets K_1, K_2 with $K_1 \cap K_2 = \emptyset$ and $K_1 \cup K_2 = K$.

Proposition: If $K_1 \supset K_2 \supset \dots$ and each K_j is connected, so is $K_\infty = \bigcap_{j=1,2,\dots} K_j$.

Proof: If $K_\infty = K' \cup K''$ where K' and K'' are compact and disjoint, then there is $\varepsilon > 0$ such that $\mathbf{B}(K'; \varepsilon) \cap \mathbf{B}(K''; \varepsilon) = \emptyset$. If j is large then $K_j \subset \mathbf{B}(K'; \varepsilon) \cap \mathbf{B}(K''; \varepsilon)$ implying that K_j is disconnected. ■

F. Definition: A *minimal \mathcal{C} -stable set* is a \mathcal{C} -stable set $K \subset E$ that does not contain a \mathcal{C} -stable proper subset. A *minimal connected \mathcal{C} -stable set* is a connected \mathcal{C} -stable set that does not contain a connected \mathcal{C} -stable proper subset.

Theorem: There exist minimal \mathcal{C} -stable sets, and if there is a connected \mathcal{C} -stable set then there is a minimal connected \mathcal{C} -stable set.

Proof: Let $\{\mathbf{B}(x_1; \varepsilon_1), \mathbf{B}(x_2; \varepsilon_2), \dots\}$ be the set of balls in Σ with rational radii and centers with rational coordinates. Let n_1 be the first integer for which there is a (connected) \mathcal{C} -stable set K_1 with $K_1 \cap \mathbf{B}(x_{n_1}; \varepsilon_{n_1}) = \emptyset$. Let n_2 be the first integer greater than n_1 for which there is a (connected) \mathcal{C} -stable set $K_2 \subset K_1$ with $K_2 \cap \mathbf{B}(x_{n_2}; \varepsilon_{n_2}) = \emptyset$. In general let n_j be the first integer greater than n_{j-1} for which there is a (connected) \mathcal{C} -stable set $K_j \subset K_{j-1}$ with $K_j \cap \mathbf{B}(x_{n_j}; \varepsilon_{n_j}) = \emptyset$. By the Proposition $K_\infty = \bigcap_{j=1,2,\dots} K_j$ is a (connected) \mathcal{C} -stable set.

Suppose that K_∞ is not minimal, so that there is a nonempty compact proper subset K that is (connected and) \mathcal{C} -stable. Then there is n such that $\mathbf{B}(x_n; \varepsilon_n) \cap K_\infty \neq \emptyset$ and $\mathbf{B}(x_n; \varepsilon_n) \cap K = \emptyset$, but the choice of n_j implies that we cannot have $n_{j-1} < n < n_j$ for any j . ■

IV. Remarks

A. Nothing of great significance has really been proved so far, since we have not even proved the existence of a connected \mathcal{C} -stable set, and \mathcal{C} -stable sets that were not connected would be difficult to interpret.

- B. Enlarging \mathcal{C} makes it more difficult to be stable, but easier to be minimal.
- C. If \mathcal{C} includes at least the correspondences BR^ε , ε a tremble, then we can handle the Kohlberg example.

V. The Main Result

A. **Definition:** A compact set $K \subset E$ is *essential* if it is \mathcal{C}_0 -stable, where \mathcal{C}_0 is the set of all u.s.c.c.v. correspondences $F : \Sigma \rightarrow \Sigma$.

Theorem: If $K \subset E$ is essential and K_1, \dots, K_r is a partition of K into disjoint compact sets, then some K_j is essential.

Corollary 1: Minimal essential sets are connected.

Proof: Any disconnected essential set has a nontrivial compact partition, by the definition of connectedness, and therefore cannot be minimal. ■

Corollary 2: For any \mathcal{C} there is a connected \mathcal{C} -stable set and thus also a minimal one.

Proof: A minimum essential set is \mathcal{C} -stable for any \mathcal{C} . ■

B. Convex combinations of correspondences are defined as follows: if

$F_0, F_1 : \Sigma \rightarrow \Sigma$ are u.s.c.c.v. correspondences and $\alpha : \Sigma \rightarrow [0, 1]$ is continuous, let

$$[(1 - \alpha) \cdot F_0 + \alpha \cdot F_1](x) = \{(1 - \alpha(x)) \cdot y_0 + \alpha(x) \cdot y_1 \mid y_0 \in F_0(x), y_1 \in F_1(x)\}.$$

Exercise: $(1 - \alpha) \cdot F_0 + \alpha \cdot F_1$ is u.s.c.c.v.

Lemma: Given an u.s.c.c.v. correspondence F and $\gamma > 0$, there is $\delta > 0$ such that for all F', F'' and $\alpha : \Sigma \rightarrow [0, 1]$,

$$d(F', F) < \delta \text{ and } d(F'', F) < \delta \text{ implies } d((1 - \alpha) \cdot F' + \alpha \cdot F'', F) < \gamma.$$

Proof: Otherwise one could choose F , γ , and sequences $\{F'_n\}$, $\{F''_n\}$, $\{\alpha_n\}$, $\{x_n\}$, $\{y'_n\}$, $\{y''_n\}$ such that

$$d(F'_n, F) \rightarrow 0, d(F''_n, F) \rightarrow 0, y'_n \in F'_n(x_n), y''_n \in F''_n(x_n), \text{ and}$$

$$\min_{(x, y) \in Gr(F)} \|x_n - x\| + \|(1 - \alpha_n(x_n)) \cdot y'_n + \alpha_n(x_n) \cdot y''_n - y\| > \gamma.$$

Taking subsequences, we may assume that $x_n \rightarrow x$, $y'_n \rightarrow y'$, $y''_n \rightarrow y''$, and $\alpha_n(x_n) \rightarrow \alpha$. Then $y', y'' \in F(x)$, since the graph of F is closed, and $\alpha \cdot y' + (1 - \alpha) \cdot y'' \in F(x)$ since $F(x)$ is convex. This contradicts the inequality above. ■

Proof of the Theorem: Suppose no K_j is essential. Then there is $\varepsilon > 0$ such that for each j we can find correspondences F_j with $d(F_j, BR)$ arbitrarily small that have no fixed points in $\mathbf{B}(K_j; \varepsilon)$. Fix such an ε small enough that the sets $\mathbf{B}(K_j; \varepsilon)$ are pairwise disjoint.

Let $\varphi_1, \dots, \varphi_r, \varphi : \Sigma \rightarrow [0, 1]$ be continuous with $\text{supp } \varphi_j \subset \mathbf{B}(K_j; \varepsilon)$, $\text{supp } \varphi \subset \Sigma - \bigcup_{j=1, \dots, r} \mathbf{B}[K_j; \frac{\varepsilon}{2}] = \Sigma - \mathbf{B}[K; \frac{\varepsilon}{2}]$, (here $\mathbf{B}[K; \frac{\varepsilon}{2}]$ denotes the *closed* $\frac{\varepsilon}{2}$ -ball around K) and $\varphi_1 + \dots + \varphi_r + \varphi = \chi_\Sigma$ (χ_Σ is the the *characteristic function* of $\Sigma : \chi_\Sigma(x) = 1$ for all $x \in \Sigma$). Such a collection of functions is a *partition of unity* subordinate to the open cover

$$\mathbf{B}(K_1, \varepsilon), \dots, \mathbf{B}(K_r; \varepsilon), \Sigma - \bigcup_{j=1, \dots, r} \mathbf{B}[K; \frac{\varepsilon}{2}].$$

In general partitions of unity exist by virtue of Urysohn's Theorem, and in our setting (since there is an explicit metric) more direct methods of construction can be employed (Exercise).

Fix $\gamma > 0$. The Lemma implies that there is $\delta > 0$ with the property that if

$$d(F_j, BR) < \delta \text{ for } j = 1, \dots, r \text{ then } d((1 - \sum \varphi_j) \cdot BR + \sum \varphi_j \cdot F_j, BR) < \gamma.$$

If no K_j is essential then there are u.s.c.c.v. correspondences F_j with $d(F_j, BR) < \delta$ that have no fixed point in $\mathbf{B}(K_j; \varepsilon)$. In this case

$$d((1 - \sum \varphi_j) \cdot BR + \sum \varphi_j \cdot F_j, BR) < \gamma$$

and $(1 - \sum \varphi_j) \cdot BR + \sum \varphi_j \cdot F_j$ has no fixed points in

$$\bigcup \mathbf{B}(K_j; \frac{\varepsilon}{2}) = \mathbf{B}(K; \frac{\varepsilon}{2}).$$

Since γ may be arbitrarily small, this contradicts the assumption that K is essential. ■

VI. Possible Choices for \mathcal{C}

A. The space of all u.s.c.c.v. correspondences $F : \Sigma \rightarrow \Sigma$.

1. This is the most stringent notion of stability.
2. As we will see, minimal essential sets can be large.
3. For “generic” extensive form games this provides the most refined selection of equilibrium “paths.”

B. $\{F | F(\sigma) = \pi \in \prod_i F_i(\sigma_{-i}) \text{ where each } F_i : \Sigma_{-i} \rightarrow \Delta(S_i) \text{ is u.s.c.c.v.}\}$.

1. Intuitively this disallows perturbations in which the agents “collude.”

C. Best response correspondences of normal form games with the same pure strategies and variable payoffs.

D. Each agent is restricted to a convex closed polyhedron in the interior of $\Delta(S_i)$.

E. $\{BR^\varepsilon | \varepsilon \text{ is a tremble}\}$.

1. This class of perturbations is contained in the classes given in C and D .

This is obvious [INFACT IT IS FALSE! - the underlying point, namely implications for robustness, is correct, but requires more care] for C , and for D we consider the system of payoffs given by the formula $u^i(\sigma) = u_i([(1 - \sum_{s_j \in S_j} \varepsilon_j(s_j))\sigma_j + \varepsilon_j]_{j \in I})$, $\sigma \in \Sigma$.