

Economics 8117-8**Noncooperative Game Theory****November 11, 1997****Lecture 9****Professor Andrew McLennan****More on Stability****I. Introduction**

- A. We begin with a slight generalization of the concepts introduced in the last lecture.
- B. We then explore the consequences of the different choices this technology allows in designing solution concepts, and the definitions selected by Kohlberg and Mertens.

II. A Generalization

- A. **Definition:** A *pointed space* is a pair (A, a_0) where A is a topological space and $a_0 \in A$. A *pointed map* $f : (A, a_0) \rightarrow (B, b_0)$ between pointed spaces is a continuous function $f : A \rightarrow B$ with $f(a_0) = b_0$.
- B. We take as given a normal form game $N = (S_1, \dots, S_n; u_1, \dots, u_n)$, with the usual auxiliary notation.
 1. Let E be the set of Nash equilibria.
 2. Let \mathcal{C}_0 be the space of u.s.c. convex valued correspondences $F : \Sigma \rightarrow \Sigma$ with the topology whose basic open sets are the sets of the form $\{F' \mid d(F', F) < \varepsilon\}$, where d is the notion of distance from one correspondence to another defined last time:

$$d(F', F) = \max_{(x', y') \in Gr(F')} \min_{(x, y) \in Gr(F)} \|x' - x\| + \|y' - y\|.$$

- C. **Definition:** Suppose $Q : (A, a_0) \rightarrow (\mathcal{C}_0, BR)$ is a pointed map. A nonempty compact set $K \subset E$ is *Q-stable* if for every $\varepsilon > 0$ there is a neighborhood U of a_0 such that, for all $a \in U$, $Q(a)$ has a fixed point in $\mathbf{B}(K; \varepsilon)$.
 1. If $BR \in \mathcal{C} \subset \mathcal{C}_0$ and Q is the inclusion map of \mathcal{C} in \mathcal{C}_0 , then *Q-stability* is equivalent to *C-stability*. Thus *Q-stability* is the more general concept.

D. For the results presented last time Q -stability is in no significant way more complicated than \mathcal{C} -stability. Therefore we only indicate how the arguments are changed.

Lemma: E is Q -stable.

Proof: The proof in the case of \mathcal{C} -stability is modified by replacing the sequence $\{F_r\}$ with a sequence $\{Q(a_r)\}$ where $\{a_r\} \subset A$ is a sequence converging to a_0 . ■

Lemma: If $K \subset L \subset E$ with K a Q -stable set and L compact, then L is Q -stable.

Proof: This is an immediate consequence of the definitions. ■

Definition: K is *connected* if there do not exist nonempty compact sets K_1, K_2 with $K_1 \cap K_2 = \emptyset$ and $K_1 \cup K_2 = K$.

Proposition: If $K_1 \supset K_2 \supset \dots$ and each K_j is Q -stable, then $K_\infty = \bigcap_{j=1,2,\dots} K_j$ is Q -stable. If each K_j is connected, so is K_∞ .

Proof: Exercise. ■

Definition: A *minimal Q -stable set* is a Q -stable set $K \subset E$ that does not contain a Q -stable proper subset. A *minimal connected Q -stable set* is a connected Q -stable set that does not contain a connected Q -stable proper subset.

Theorem: There exist minimal Q -stable sets, and if there is a connected Q -stable set, then there is a minimal connected Q -stable set.

Definition: A compact set $K \subset E$ is *essential* if it is \mathcal{C}_0 -stable.

Lemma: If $K \subset E$ is essential and $Q : (A, a_0) \rightarrow (\mathcal{C}_0, BR)$ is a pointed map, then K is Q -stable.

Theorem: If $K \subset E$ is essential and K_1, \dots, K_r is a partition of K into disjoint compact sets, then some K_j is essential.

Corollary: For any Q there is a connected Q -stable set and thus also a minimal one.

III. Equivalent Games

A. **Definition:** A strategy s_i is *redundant* if there are $r_{i1}, \dots, r_{ik} \in S_i - \{s_i\}$ and numbers $\alpha_1, \dots, \alpha_k > 0$ with $\sum \alpha_\ell = 1$ such that

$$u_j(s_i, s_{-i}) = \sum_{\ell=1, \dots, k} \alpha_\ell u_j(r_{i\ell}, s_{-i}) \text{ for all } j \text{ and all } s_{-i} \in S_{-i}.$$

B. **Definition:** The *reduced normal form* of a game is the normal form game obtained by eliminating all redundant strategies. Two normal form games are *equivalent* if they have the same reduced normal form.

1. There may be two strategies that are equivalent, in which case either one could be eliminated. In this definition we do not distinguish between the two games obtained from the two possible eliminations. In general we regard two games as the same if there are bijections between the sets of agents and, for each matched pair of agents, a bijection between the two pure strategy sets, such that under these bijections the payoffs in the two games are the same.
2. If N' is obtained from N by elimination of redundant strategies, then there is a natural continuous projection $\pi : \Sigma \rightarrow \Sigma'$ from the mixed strategy vectors on N to the mixed strategy vectors of N' .
 - a. If s_i in N is equivalent to $\alpha_1 r_{i1} + \dots + \alpha_k r_{ik}$ in N' , probability assigned to s_i is distributed under the map to r_{i1}, \dots, r_{ik} according to the weights $\alpha_1, \dots, \alpha_k$. Thus the map depends on a prior agreement on which mixed strategy in N' represents a redundant strategy in N .
 - b. A mixed strategy vector for N is a Nash equilibrium if and only if its image under π is a Nash equilibrium for N' . More formally:

Lemma: Let E be the set of Nash equilibria of N , and let E' be the set of Nash equilibria of N' . Then $\pi(E) = E'$ and $\pi^{-1}(E') = E$.

Proof: This follows in a straightforward way from the definition of redundancy. ■

c. Obviously π is a continuous surjection.

d. Observe that for each $\sigma' \in \Sigma'$, $\pi^{-1}(\sigma')$ is convex, hence connected.

3. There is also a natural map $\iota : \Sigma' \rightarrow \Sigma$ given by $\iota(\sigma')_i(s_i) = \sigma'_i(s_i)$ if $s_i \in S'_i$ and $\iota(\sigma')_i(s_i) = 0$ if $s_i \in S_i - S'_i$.

a. Clearly ι is continuous.

b. Observe that $\pi \circ \iota = Id_{\Sigma'}$.

C. Results concerning connectedness.

Lemma: If $D \subset \Sigma$ is connected, then $\pi(D)$ is connected.

Proof: This is a fact of general topology: if $f : X \rightarrow Y$ is a continuous surjection and $\{Y_1, Y_2\}$ is a clopen partition of Y , then $\{f^{-1}(Y_1), f^{-1}(Y_2)\}$ is a clopen partition of X . ■

Lemma: If $D' \subset \Sigma'$ is connected, then $\pi^{-1}(D')$ is connected.

Proof: Suppose $\{G_1, G_2\}$ is a clopen partition of $\pi^{-1}(D')$. We have $\iota(D') \subset \pi^{-1}(D')$, and the reasoning in the last proof implies that $\iota(D')$ is connected, so $\iota(D') \subset G_1$ or $\iota(D') \subset G_2$. Suppose that $\iota(D') \subset G_1$. Since each $\pi^{-1}(\sigma')$ is connected we have $\pi^{-1}(\sigma') \subset G_1$ for all $\sigma' \in D'$. ■

Definition: A *component* of a topological space X is a maximal connected subset of X .

Exercise: If $C \subset X$ is connected, then the union of all connected supersets of C is connected, hence a component. Thus the components of X constitute a partition of X .

Lemma: If K is a component of E , then $\pi(K)$ is a component of E' .

Proof: Our results above imply that $\pi(K)$ is connected, and if $K' \subset E'$ is a connected proper superset of $\pi(K)$, then $\pi^{-1}(K') \subset E$ is a connected proper superset of K . ■

Lemma: If K' is a component of E' , then $\pi^{-1}(K')$ is a component of E .

Proof: Our results above imply that $\pi^{-1}(K')$ is connected, and if $K \subset E$ is a connected proper superset of $\pi^{-1}(K')$, then $\pi(K) \subset E'$ is a connected proper superset of K' . ■

Proposition: π induces a one-to-one correspondence between the components of E and the components of E' .

Proof: This simply summarizes the discussion above. ■

Lemma: If C is a component of X , then C is closed.

Proof: If $x \in \bar{C} - C$, then $C \cup \{x\}$ is easily seen to be connected, so the maximality of C implies that $x \in C$. ■

Corollary: If X has finitely many connected components C_1, \dots, C_r , then each C_j is clopen.

Proof: Since each C_j is closed, each $C_j = X - (C_1 \cup \dots \cup C_{j-1} \cup C_{j+1} \cup \dots \cup C_r)$ is also open. ■

D. A Deep Result.

1. A *semialgebraic set* is a set $K \subset \mathbb{R}^n$ defined by finitely many inequalities of the form $P(x) \geq 0$, P a polynomial in x_1, \dots, x_n .

a. Cute trick: Insofar as “ $P_1(x) = 0$ and \dots and $P_k(x) = 0$ ” is equivalent to

“ $P_1(x)^2 + \dots + P_k(x)^2 = 0$ ”, no generality is lost in requiring $k = 1$.

Theorem: A compact semialgebraic set is the union of finitely many connected components, each of which is path connected. (Though intuitive, this is a very deep result, and its proof will not be discussed here.)

2. Note that the set of Nash equilibria is the subset of $\mathbb{R}^{S_1} \cup \dots \cup S_n$ defined by the inequalities $\sigma_i(s_i) \geq 0$, $\sum_{S_i} \sigma_i(s_i) = 1$, $\sigma_i(s_i) \cdot [u_i(s_i, \sigma_{-i}) - u_i(t_i, \sigma_{-i})] \geq 0$. Thus the set of Nash equilibria is a semialgebraic set.

IV. Kohlberg and Merten's Definitions

A. Hyperstability

1. Let $A_H = (\mathbb{R}^S)^I$ be the set of possible payoffs for the given sets of agents and pure strategies. Let $a_H \in A_H$ be the given payoff. Let $Q_H : (A_H, a_H) \rightarrow (C_0, BR)$ be the function that assigns to each payoff the corresponding best response correspondence.
 - a. Although these definitions are presented with reference to the given normal form, they will be applied below to various equivalent normal forms, so they are to be understood as defined simultaneously for all normal form games, even though we do not make the dependence of the objects on the normal form explicit in the notation.
2. A set K of equilibria of a game N^R in reduced normal form is *hyperstable* if it is minimal with respect to the following property:

Property H : K is compact, and for any equivalent game N , $\pi^{-1}(K)$ is Q_H -stable.

- a. Note that connectedness is not required, though a perfectly reasonable concept is defined if it is required.
- b. We prove a result of some interest for both versions of the concept.

Proposition: (This is essentially due to Kohlberg and Mertens.) There is a connected component of the set of Nash equilibria that has property H .

Proof: Our results above imply that each component of the set of Nash equilibria is closed, hence compact. Since the components constitute a finite compact partition of E , and E is essential, some component must be essential, and any such component is necessarily Q_H -stable.

Suppose that N' is obtained from N by eliminating redundant strategies, let E' and E be the respective sets of Nash equilibria, and let $\pi : \Sigma \rightarrow \Sigma'$ be as above. At an informal level it is quite clear that if C is a Q_H -stable component of E , then $\pi(C)$ is Q_H -stable, since any perturbation of the payoffs of N' can be “reproduced” by perturbing the payoffs of N . Formal details of this argument are left to the reader.

Suppose N' can be obtained from either N_1 or N_2 by eliminating redundant strategies. Then an obvious and tedious construction (details are again left to the reader) leads to a game N such that both N_1 and N_2 can be obtained from N by eliminating redundant strategies. Since some component of the set of equilibria of such an N is Q_H -stable, this implies that it *cannot* be the case that for each component of the set of equilibria of the given game N^R there is an equivalent game for which the corresponding component of the set of equilibria is not Q_H -stable. Thus some component of the set of equilibria of N^R has property H . ■

B. Fully stable sets of equilibria.

1. A *polyhedron* in \mathbb{R}^n is the intersection of finitely many closed half spaces.
2. The *Hausdorff distance* between two compact sets K and L is

$$\max\left\{\max_{x \in K} \min_{y \in L} \|x - y\|, \max_{y \in L} \min_{x \in K} \|x - y\|\right\}.$$

- a. **Exercise:** the Hausdorff distance is a metric.
3. For each i let \mathcal{P}_i be the set of nonempty convex polyhedra contained in the interior of $\Delta(S_i)$. Let $A_F = \{(\Delta(S_1), \dots, \Delta(S_n))\} \cup \mathcal{P}_1 \times \dots \times \mathcal{P}_n$, and let $a_F = (\Delta(S_1), \dots, \Delta(S_n))$. Define $Q_F : (A_F, a_F) \rightarrow (C_0, BR)$ by letting

$Q_F(a)$ be the perturbation of the best response correspondence obtained by constraining each agent to choose optimally in his or her component of a .

a. **Exercise:** Prove that $Q_F : A_F \rightarrow \mathcal{C}_0$ is continuous.

4. **Definition:** A set K of Nash equilibria is *fully stable* if it is minimally Q_F -stable.

a. As was the case with hyperstability, we could also require connectedness.

5. Requiring K to be fully stable not only in the original game but also in all equivalent games would not lead to a new concept. Any convex polyhedron of totally mixed strategies in an equivalent game is equivalent to a polyhedron in the original game.

Theorem: Every game has a fully stable set of equilibria.

Proof: This follows from our general results above. ■

6. Kohlberg and Mertens claim that every hyperstable set of equilibria contains a fully stable set of equilibria, but their rather sketchy argument does not seem correct to me. They point out that the game given by any collection of convex polyhedra in the interiors of the agents mixed strategy sets corresponds to some perturbation of the payoffs of some equivalent game. This is not enough, since the definition of hyperstability (as I understand it) is that for any equivalent game and any $\varepsilon > 0$, there is $\delta > 0$ such that ... , not that for any $\varepsilon > 0$ there is $\delta > 0$ such that for all equivalent games ...

C. Stable sets of equilibria.

1. For each i let $\mathcal{T}_i \subset \mathcal{P}_i$ be the set of nonempty convex polyhedra derived from trembles for agent i . Let

$A_S = \{(\Delta(S_1), \dots, \Delta(S_n))\} \cup \mathcal{T}_1 \times \dots \times \mathcal{T}_n$, and let

$a_S = (\Delta(S_1), \dots, \Delta(S_n))$. Let $Q_S : (A_S, a_S) \rightarrow (\mathcal{C}_0, BR)$ be the restriction of Q_F to A_S .

2. **Definition:** A set K of Nash equilibria is *stable* if it is minimally Q_S -stable.
3. Requiring K to be stable not only in the original game but also in all equivalent games leads to nothing new, since the constraint that at least probability δ_i be assigned to a mixed strategy σ_i in an equivalent game would be equivalent to the requirement that probability δ_i be assigned to the equivalent of σ_i in the original game.

Theorem: Every game has a stable set of equilibria.

Proof: As above, this follows from our general results. ■

V. Summary of Techniques for Defining Refinements

A. Clever trembles and other special classes of perturbations.

B. Essentiality

1. This is the most restrictive concept for connected components of the set of Nash equilibria.
2. Insofar as it is a well defined mathematical “extreme”, this concept seems the least arbitrary.

C. Q -stability for various classes of perturbations Q .

1. By insisting on minimal Q -stable sets, restricting the domain of Q leads to smaller solution sets.
2. One can combine Q -stability with a requirement of connectedness.

D. One can ask for Q -stability not just in the given game, but also in all normal form games with the same reduced normal form.