Economics 8117-8 **Noncooperative Game Theory**

November 18, 1997 **Lecture 11**

Professor Andrew McLennan

Persistent Equilibrium

I. Motivation

A. Consider the battle of the sexes.

$$\begin{array}{ccc} 1\backslash 2 & f & b \\ F & (1,1) & (0,0) \\ B & (0,0) & (1,1) \end{array}$$

1. The mixed equilibrium is not dynamically stable.
2. Nonetheless it is stable in the sense of Kohlberg and Mertens, and in fact it is an essential set of equilibria.

B. The mixed equilibrium mixes over several strategies. The delicacy of the equilibrium is reflected in the fact that there are equilibria using subsets of the set of strategies used in the mixed equilibrium.

C. Persistence looks at equilibria using minimal sets of pure strategies.

II. Definitions

A. A set $R \subset \Sigma$ is a *retract* if there are nonempty convex closed sets $R_i \subset \Delta(S_i)$ such that $R = \Pi_{i \in I} R_i$.

B. If R is a retract and $A \subset \Sigma$, we say that R *absorbs* A if $BR(\sigma) \cap R \neq \emptyset$ for all $\sigma \in A$.

C. R is a *Nash retract* if it absorbs itself.

Proposition: The minimal Nash retracts are exactly the sets of the form $\{\sigma^*\}, \sigma^*$ a Nash equilibrium.

Proof: Clearly $\{\sigma^*\}$ is a Nash retract if σ^* is a Nash equilibrium, and it must be minimal. The converse is a consequence of ...

Lemma 1: Every Nash retract contains a Nash equilibrium.

Proof: The correspondence $BR^R : R \to R$ defined by $BR^R(\sigma) = R \cap BR(\sigma)$ has nonempty convex values and is u.s.c., so the Kakutani fixed point theorem applies. ∎

D. R is an *essential Nash retract* if it absorbs a neighborhood of itself.

E. R is a *persistent retract* if it is an essential Nash retract and is minimal with respect to this property.

F. A *persistent equilibrium* is a Nash equilibrium contained in a persistent retract.

G. Outline

1. Characterize persistent retracts, and prove existence.
2. Prove that every persistent retract contains a proper equilibrium.
3. Prove that an essential Nash retract contains a stable set of Nash equilibria.

III. Characterization of Persistent Retracts

A. **Definition:** Two strategies s_i and t_i are *equivalent for* i if

$$u_i(s_i, \sigma_{-i}) = u_i(t_i, \sigma_{-i}) \text{ for all } \sigma_{-i} \in \Sigma_{-i}.$$

1. For each i let $Q_i^1, \ldots, Q_i^{k(i)}$ be the equivalence classes of strategies for player i.

Lemma 2: Let $\mathcal{O} \subset \Sigma_{-i}$ be open and nonempty. If

$$u_i(s_i, \sigma_{-i}) = u_i(t_i, \sigma_{-i}) \text{ for all } \sigma_{-i} \in \mathcal{O},$$

then s_i and t_i are equivalent for i.

Proof: $u_i(s_i, \cdot) : \Sigma_{-i} \to \mathbb{R}$ and $u_i(t_i, \cdot) : \Sigma_{-i} \to \mathbb{R}$ are polynomials, so they agree everywhere if they agree on any open set. ∎

Lemma 3: If $\mathcal{O} \subset \Sigma$ is nonempty and open then it contains σ such that for all $i \in I$, $BR_i(\sigma) = \Delta(Q_i^j)$ for some j.

Proof: Suppose r_i and s_i are inequivalent. By Lemma 2 there is $\sigma \in \mathcal{O}$ such that $u_i(r_i, \sigma_{-i}) \neq u_i(s_i, \sigma_{-i})$, and since u_i is continuous, we can choose $\mathcal{O}_1 \subset \mathcal{O}$ with $\sigma \in \mathcal{O}_1$ and $u_i(r_i, \sigma'_{-i}) \neq u_i(s_i, \sigma'_{-i})$ for all $\sigma' \in \mathcal{O}_1$. Taking a descending sequence of nonempty open subsets, we get a nonempty open subset $\mathcal{O}^*$ with $u_i(r_i, \sigma_{-i}) \neq u_i(s_i, \sigma_{-i})$ whenever r_i and s_i are inequivalent and $\sigma \in \mathcal{O}^*$. ∎

B. Strategy selections and selection retracts.

1. **Definition:** $F_i \subset \Delta(S_i)$ is a *strategy selection* if

$$\emptyset \neq F_i \subset \bigcup_{j=1,\ldots,k(i)} \Delta(Q_i^j)$$

   and for each $j = 1, \ldots, k(i)$, $F_i \cap \Delta(Q_i)$ is empty or a singleton.

2. **Definition:** A retract R is a *selection retract* if $R = \Pi_{i \in I} R_i$, where each R_i is the convex hull of a strategy selection F_i.

Proposition: Suppose R is an essential Nash retract, and let T be a neighborhood of R that is absorbed by R. Let $R^* \subset R$ be a selection retract that is not a proper subset of some other selection retract that is contained in R. Then R^* absorbs T.

Proof: Assuming that, for each i, R_i^* is the convex hull of the strategy selection F_i, if, for some i and $s \in T$, we had $BR_i(\sigma) \cap F_i = \emptyset$, then $BR_i(\sigma') \cap F_i = \emptyset$ for all σ' in a neighborhood of σ, and Lemma 3 implies that $BR_i(\sigma') = \Delta(Q_i^j)$ for some σ' in this neighborhood. Choosing $\sigma_i \in BR_i(\sigma') \cap R$, in the obvious way we can construct a selection retract $R^{**} \neq R^*$ with $R^* \subset R^{**} \subset R$, contrary to our hypotheses. ∎

Proposition: Every essential Nash retract contains an essential Nash selection retract.

Proof: Let $R = \Pi_{i \in I} R_i$ be an essential Nash retract. For each i let

$$J_i = \{j | R_i \cap \Delta(Q_i^j) \neq \emptyset\}.$$

Since R absorbs a neighborhood of itself, Lemma 3 implies that J_i is nonempty. For each $j \in J_i$, choose $s_i^j \in \Delta(Q_i^j) \cap R_i$, and let $F_i = \{\sigma_i^j | j \in J_i\}$. Let

$$R_i^* = \text{con}(F_i), \text{ and let } R^* = \Pi_{i \in I} R_i^*.$$

Then R^* is a selection retract contained in R, and R^* is clearly maximal with respect to these properties, so the preceding result implies that R^* is an essential Nash retract. ∎

Corollary 1: Every persistent retract is a selection retract.

Corollary 2: There exists a persistent retract.

Proof: Σ is an essential Nash retract. By the Proposition Σ contains an essential selection retract R. Let $\mathcal{R}$ be the set of essential Nash selection retracts contained in R. Clearly $\mathcal{R}$ is finite, so it has a minimal element, and the Proposition implies that this minimal element is persistent. ∎

Theorem: There exists a persistent equilibrium.

IV. Properties of Persistent Retracts

A. Recall that if (X, δ) is a metric space and $\emptyset \neq K, L \subset X$ are compact, then the *Hausdorff distance* between K and L is

$$\delta_{\mathcal{H}}(K, L) = \max\{\max_{x \in K} \min_{y \in L} \delta(x, y), \max_{y \in L} \min_{x \in K} \delta(x, y)\}.$$

Lemma: Let R be an essential Nash retract, and let T be a neighborhood of R absorbed by R. Then there is $\gamma > 0$ such that any u.s.c.c.v. correspondence $f : \Sigma \to \Sigma$ with

$$(*) \qquad \delta_{\mathcal{H}}(BR(\sigma), f(\sigma)) < \gamma \, (\sigma \in \Sigma)$$

has a fixed point in T.

Proof: There is $\gamma > 0$ such that the closed ball $\bar{\mathbf{B}}(R;\gamma) \subset T$, and since R is convex, so is this ball. (Exercise) If f satisfies $(*)$, we define $f_\gamma : \bar{\mathbf{B}}(R;\gamma) \to \bar{\mathbf{B}}(R;\gamma)$ by $f_\gamma(\sigma) = f(\sigma) \cap \bar{\mathbf{B}}(R;\gamma)$. Clearly f_γ is u.s.c.c.v., and if $\sigma \in \bar{\mathbf{B}}(R;\gamma)$ then there is $\sigma' \in BR(\sigma) \cap R$ and then $\sigma'' \in f(\sigma) \cap \mathbf{B}(\sigma';\gamma)$, so $f_\gamma(\sigma) \neq \emptyset$. Thus the correspondence f_γ satisfies the hypotheses of the Kakutani fixed point theorem. ∎

Corollary: An essential Nash retract is $\{BR^\varepsilon | \varepsilon$ is a tremble $\}$–stable.

Corollary: An essential Nash retract contains a proper equilibrium.

V. Discussion.

A. Persistence is acceptably defined insofar as there is a universal existence theorem.

B. It is a "low cost" concept in that it is possible to impose other concepts simultaneously. In particular there are stable sets of equilibria in persistent retracts.

C. It perform well for some examples, especially "unanimity games" of which the battle of the sexes is the prototype.

D. It does not always do what it was intended to do.

1. Consider the battle of the sexes with one of the "bad" outcomes replaced by matching pennies.
2. The normal form is as follows.

$1\backslash 2$	f	bu	bd
FU	$(1,1)$	$(4,-4)$	$(-4,4)$
FD	$(1,1)$	$(-4,4)$	$(4,-4)$
B	$(0,0)$	$(1,1)$	$(1,1)$

a. No two strategies are equivalent.

b. Let R be a persistent retract. Then one can show that $R = \Sigma$.

c. Thus in this example persistence does not eliminate any equilibria.