

Economics 8117-8**Noncooperative Game Theory****December 2, 1997****Lecture 14****Professor Andrew McLennan**

Bargaining

I. Introduction.

- A. Edgeworth regarded bargaining as the key unsolved problem of economics: what happens if two agents (in an exchange economy perhaps) can implement any outcome from some set provided they both agree on it, there being a “status quo” or “threat point” outcome that occurs if no agreement is reached?
- B. This problem has continued to be a concern of economists.
 1. The best known example is bilateral monopoly.
 2. Edgeworth’s reasons for regarding bargaining as crucially important have not been challenged.
- C. We will consider three papers.
 1. Nash, J., “The Bargaining Problem”.
 2. Rubinstein, A., “Perfect Equilibrium in a Bargaining Model.”
 3. McLennan, A., “Bargaining Between Two Symmetrically Informed Agents”.

II. The Nash Bargaining Solution

- A. A *bargaining situation* is a pair (S, d) where $S \subset \mathbb{R}^2$ is compact and convex and $d \in S$.
 1. Here S is the *feasible set* and d is the *threat point*.
 2. Elements $u = (u_1, u_2)$ of S are interpreted as pairs of *vNM* utilities.
 3. Convexity may be motivated either by the possibility of implementing probabilistic combinations (lotteries) of pure outcomes or by the possibility of side payments if the agents are risk averse.
 4. Let \mathcal{S} be the class of all bargaining situations.

B. A *solution* is a function $c : \mathcal{S} \rightarrow \mathbb{R}^2$ with $c(S, d) \in S$ for all $(S, d) \in \mathcal{S}$. Nash imposes the following four axioms.

Optimality: $c(S, d)$ is weakly Pareto optimal in S .

Symmetry: $c(\{(u_2, u_1) \mid (u_1, u_2) \in S\}, (d_2, d_1)) = (c_2(S, d), c_1(S, d))$.

Invariance: If $A_i : u_i \mapsto a_i \cdot u_i + b_i$ with $a_i > 0, i = 1, 2$, are positive affine transformations of \mathbb{R} , and $A : (u_1, u_2) \mapsto (A_1(u_1), A_2(u_2))$, then $c(A(S), A(d)) = A(c(S, d))$.

Independence of Irrelevant Alternatives (IIA): If $T \supset S$ is compact and convex and $c(T, d) \in S$, then $c(S, d) = c(T, d)$.

1. The phrase “Independence of Irrelevant Alternatives” was coined by Luce and Raiffa, not Nash. It is similar to, but not the same as, the IIA condition of Arrow’s Theorem.
2. A choice function for an individual must obviously satisfy IIA. In fact this axiom is close to an assumption that there is a social welfare function $W(u_1 - d_1, u_2 - d_2)$ that determines c .

C. **Theorem:** If the Nash axioms are satisfied by $c : \mathcal{S} \rightarrow \mathbb{R}^2$, then

$$c(S, d) = \operatorname{argmax}_{u \in S, u \geq d} (u_1 - d_1) \cdot (u_2 - d_2)$$

for all (S, d) with $\max_{u \in S, u \geq d} (u_1 - d_1) \cdot (u_2 - d_2) > 0$.

Proof: The Invariance Axiom implies that any solution is completely determined by the restriction of the solution to the set of bargaining situations with threat point $0 = (0, 0)$.

Therefore assume that $d = 0$.

Since S is convex there is a unique maximizer u^* of $u_1 \cdot u_2$ in $S \cap \mathbb{R}_+^2$. Let

$$S' = \left\{ \left(\frac{u_1}{u_1^*}, \frac{u_2}{u_2^*} \right) \mid (u_1, u_2) \in S \right\};$$

then $(1, 1)$ maximizes the product of utilities in $S' \cap \mathbb{R}_+^2$, and Invariance implies that it suffices to show that $c(S', 0) = (1, 1)$.

Since S' and $\{u | u_1 \cdot u_2 \geq 1\}$ are convex, they are separated by a hyperplane, and the only possible hyperplane is $H = \{u | u_1 + u_2 = 2\}$. For sufficiently large $\delta > 0$ we have

$$S' \subset T = \{u | u_1, u_2 \geq -\delta \text{ and } u_1 + u_2 \leq 2\}.$$

Then Symmetry and Optimality imply that $c(T, 0) = (1, 1)$, and IIA implies that $c(S', 0) = (1, 1)$. ■

III. Rubinstein's Model.

A. Critique of the Nash Solution.

1. Nash's Theorem is beautiful, but what does it mean?
2. Viewed as a positive concept describing what *will* happen, it is inadequate in that it does not explain why. In particular it is difficult to motivate IIA as a consequence of noncooperative behavior.
3. A more subtle point is that the Nash solution is a "cardinal" concept in that the outcome is sensitive to risk aversion. (It is not invariant with respect to *monotonic* transformations of utilities.) But it is hard to see what is "risky" about actual bargaining, and the Nash solution itself is riskless.
4. Viewing the Nash solution as a normative concept describing what *should* happen, IIA is attractive as a form of collective rationality, but the framework fails to take into account factors such as the agents' wealths.

B. Rubinstein proposes an extensive form game of perfect information as a model of the negotiations.

1. The agents take turns proposing divisions of the "pie" until one is accepted.
2. There are costs of delay.
 - a. Discounting.

b. Per period utility losses.

C. The extensive form.

1. $T = \{w\} \cup (\bigcup_{n=1,2,\dots} [0, 1]^n) \cup (\bigcup_{n=1,2,\dots} [0, 1]^n \times Y)$.
2. $Z = \bigcup_{n=1,2,\dots} [0, 1]^n \times Y$.
3. $x = (x_1, \dots, x_m) \prec z = (z_1, \dots, z_n, Y)$ if and only if $m \leq n$ and $x_j = z_j, j = 1, \dots, m$.
4. $x = (x_1, \dots, x_m) \prec y = (y_1, \dots, y_n)$ if and only if $m < n$ and $x_j = y_j, j = 1, \dots, m$.
5. $w \prec t$ for all $t \in T - \{w\}$.
6. For $x \in X$ define $\tau(x)$ by $\tau(w) = 0$ and $\tau(x) = n$ if $x \in [0, 1]^n$.
7. For $x \in X$ let $\iota(x) = 1$ if $\tau(x)$ is odd and $\iota(x) = 2$ if $\tau(x)$ is even.

D. Discounting.

1. The payoffs are given by

$$u_1(z) = \delta_1^n \cdot v_1(z_n) \quad \text{and}$$

$$u_2(z) = \delta_2^n \cdot v_2(1 - z_n) \quad (z = (z_1, \dots, z_n, V) \in Z),$$

where $0 < \delta_1, \delta_2 < 1$ and $v_i : [0, 1] \rightarrow [0, 1], i = 1, 2$, are strictly increasing concave (hence continuous) functions with $v_i(0) = 0$ and $v_i(1) = 1$.

2. For $u_2 \in [0, 1]$ let $U_1(u_2) = v_1(1 - v_2^{-1}(u_2))$, and for $u_1 \in [0, 1]$ let $U_2(u_1) = v_2(1 - v_1^{-1}(u_1))$.
3. Let \underline{u}_1 be the infimum over all subgame perfect equilibria and all nodes $x = (x_1, \dots, x_n) \in X$ with n odd of the supremum of utilities agent 1 can obtain by making a proposal. Let \bar{u}_2 be the supremum over all subgame perfect equilibria and all nodes $x = (x_1, \dots, x_n) \in X$ with n even of the supremum of utilities agent 2 can obtain by making a proposal.
4. Then $\underline{u}_1 \geq \delta_1 \cdot U_1(\bar{u}_2)$ and $\bar{u}_2 \leq \delta_2 \cdot U_2(\underline{u}_1)$, so $\underline{u}_1 \geq \delta_1 \cdot U_1(\delta_2 \cdot U_2(\underline{u}_1))$.
5. As u_1 increases from 0 to 1, $\delta_1 \cdot U_1(\delta_2 \cdot U_2(u_1))$ increases from a number greater than 0 to a number less than 1. Let w_1 be the least number such that

$w_1 = \delta_1 \cdot U_1(\delta_2 \cdot U_2(w_1))$. Then $\underline{u}_1 \geq w_1$.

6. However, it is easy to construct a subgame perfect equilibrium in which agent 1 receives w_1 if it is his turn to move and he does not wish to accept agent 2's proposal.
7. Letting $\delta_1 = \delta_2 = 1 - r$, it can be shown that as $r \rightarrow 0$ the set of subgame perfect equilibria converges to the Nash solution $c(S, 0)$ where $S = \{u \in \mathbb{R}_+^2 \mid u_1 \leq U_1(u_2)\}$.

E. Per period bargaining costs.

1. The payoffs are given by

$$u_1(z) = v_1(z_n) - n \cdot c_1 \quad \text{and}$$

$$u_2(z) = v_2(1 - z_n) - n \cdot c_2 \quad (z = (z_1, \dots, z_n, Y) \in Z),$$

where $c_1, c_2 > 0$ and $v_i : [0, 1] \rightarrow [0, 1], i = 1, 2$, are strictly increasing concave functions with $v_i(0) = 0$ and $v_i(1) = 1$.

2. Here the agent with the lower bargaining cost receives essentially all the surplus.

IV. Generalization's of Rubinstein's Model.

A. Critique of Rubinstein's model.

1. The preferences considered by Rubinstein are quite restrictive. In particular they are stationary in that share x at time m is preferred to share y at time n if and only if share x at time $m + p$ is preferred to share y at time $n + p$.
2. There is no consideration of "outside options" as threats.

B. The environment of the general model.

1. $S : \mathbb{R}_+ \rightarrow \mathbb{R}^2$ is a continuous convex valued *feasibility correspondence* with $S(t)$ convex and closed and $S(t) - \mathbb{R}_+^2 \subset S(t)$ for all t and $S(t_1)$ a strict superset of $S(t_2)$ if $t_1 < t_2$.

2. $\theta_1, \theta_2 : \mathbb{R}_+ \rightarrow \mathbb{R}$ are continuous *threat point* functions.
3. There is a date T such that $(\theta_1(t), \theta_2(t)) \in \text{int } S(t)$ for all $t < T$ and $(\theta_1(t), \theta_2(t)) \in \mathbb{R}^2 - S(t)$ if $t > T$.

C. The rules are as in Rubinstein's game with the time between offers being Δt , except that in addition to being able to accept and make a counterproposal, it is also possible to bring about the threat utility pair $(\theta_1(t), \theta_2(t))$ when one chooses a move at time t .

1. For the case $\theta_1(t) = 0 = \theta_2(t) (t \in [0, T])$ the solution by backwards induction is shown in Figure 1.
2. Taking the limit as $\Delta t \rightarrow 0$ leads to a solution path described by differential equations.
3. With threats the equilibrium payoffs must lie in the set of utility pairs that are Pareto optimal and individually rational. The effect of the threats on the solution method is shown in Figure 2.
4. In an important special case the Nash solution can be recovered.