

Economics 8117-8**Noncooperative Game Theory****April 4, 1998****Lecture 17A****Professor Andrew McLennan****Generic Finiteness of Equilibrium Outcome Distributions in Game Forms****A Game Form with Infinitely Many Equilibria
on an Open Set of Payoffs**

by

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This note, which is intended for eventual inclusion in a larger paper, provides an example of a game form for which there is an open set of payoffs, as functions of outcomes, such that each payoff in the set induces a game with a continuum of equilibrium distributions over outcomes.

Consider the following game form in which agent 1 is the row player, agent 2 is the column player, and agent 3 is the matrix player:

$$\begin{array}{cc}
 & \begin{array}{cc} U & D \end{array} \\
 \begin{array}{c} L \quad R \\ N \left(\begin{array}{cc} a & a \\ E \left(\begin{array}{cc} b & b \\ S \left(\begin{array}{cc} a & b \\ W \left(\begin{array}{cc} e & f \end{array} \right) \end{array} \right) \end{array} \right) \end{array} \end{array} & \begin{array}{c} L \quad R \\ N \left(\begin{array}{cc} c & c \\ E \left(\begin{array}{cc} d & d \\ S \left(\begin{array}{cc} c & d \\ W \left(\begin{array}{cc} e & f \end{array} \right) \end{array} \right) \end{array} \right) \end{array} \end{array}
 \end{array}$$

In our discussion we will attach subscripts to the outcomes to indicate payoffs, so that a_1 is the payoff to agent 1 when outcome a occurs, etc. Let p , q , r , and s be the probabilities of N , E , S , and W , respectively; of course $p + q + r + s = 1$. Let y be the probability of L , and let z be the probability of U .

We now derive necessary conditions for a totally mixed equilibrium. To begin with note that agent 1's indifference between N and E implies that

$$za_1 + (1 - z)c_1 = zb_1 + (1 - z)d_1,$$

so that

$$z = \frac{d_1 - c_1}{a_1 - b_1 - c_1 + d_1}.$$

It is a very important point that the lottery over outcomes resulting from S is a convex combination of the lotteries resulting from N and E , so that whenever agent 1 is indifferent between N and E , he or she is also indifferent between either of these strategies and S . On the other hand indifference between N and W amounts to

$$ye_1 + (1 - y)f_1 = za_1 + (1 - z)c_1$$

so that

$$y = \frac{c_1 - f_1 + z(a_1 - c_1)}{e_1 - f_1}.$$

For agent 2 indifference between L and R is the condition

$$r(za_2 + (1 - z)c_2) + se_2 = r(zb_2 + (1 - z)d_2) + sf_2$$

so that

$$r = s \cdot \frac{(f_2 - e_2)}{z(a_2 - b_2) + (1 - z)(c_2 - d_2)}.$$

For agent 3 indifference between U and D is the condition

$$pa_3 + qb_3 + r(ya_3 + (1 - y)b_3) = pc_3 + qd_3 + r(yc_3 + (1 - y)d_3)$$

so that

$$p(a_3 - c_3) + q(b_3 - d_3) + r(y(a_3 - c_3) + (1 - y)(b_3 - d_3)) = 0.$$

We now see that we can easily produce the desired counterexample. So long as the parameters avoid problems of division by zero in the formulas above for z , y , agent 1's

indifference conditions determine y and z . Once these numbers are fixed, there are two linear conditions (in addition to $p + q + r + s = 1$) on agent 1's strategy, with parameters given by the utilities of agents 2 and 3. Except on an obvious set, with positive codimension, of special utilities for these two agents, there will be a one dimensional space of (p, q, r, s) satisfying all conditions. There is no great art to choosing particular values of the parameters such that $y, z \in (0, 1)$ and this line passes through the interior of the simplex $\{(p, q, r, s) \in \mathbb{R}_+^4 : p + q + r + s = 1\}$, and we leave this aspect of the argument to the reader. The set of solutions varies continuously, in the obvious sense, on the open set of parameters that do not give rise to some division by zero in the equations above, so we have the desired open set of utilities for which there are a continuum of totally mixed equilibria.

To see that the one dimensional continuum of totally mixed equilibria maps to a one dimensional set of distributions over outcomes, it is only necessary to observe that the distribution over outcomes necessarily varies so long as s varies in the set of equilibria. To see that the continuum of totally mixed equilibria maps to a one dimensional set of equilibrium payoffs, note that the equilibrium conditions make no reference to e_3 and f_3 , so again it suffices that s is variable.