Economics 8117–8 **Noncooperative Game Theory**

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Auction Theory

I. Introduction.

A. Milgrom and Weber begin with the antiquity of auctions.

1. Auctions are mentioned by Herodotus.
2. China $\sim$ 7th century A.D.
3. Of course it is common to begin papers with "Since time immemorial ...", but here there is an important point: in spite of ample opportunity for "evolution," auctions have persisted.

B. Many markets utilize auctions.

1. Mineral rights.
2. Satellite channels.
3. Art, flowers.
4. All sorts of "unique" objects.

C. Questions.

1. Equilibrium strategies and outcomes.
2. Optimality.
   a. Pareto.
   b. For the seller.
3. Do auction outcomes resemble market outcomes?
4. Aggregation of information.

II. Types of Auction.

A. Dutch – the price descends continuously until a bidder yells "stop".

B. First price sealed bid – the highest bidder pays his or her bid.

C. Second price sealed bid – the highest bidder pays the second highest bid.

D. English – bids are announced until there is a bid that no one wants to top.

III. Independent Private Values Model.

A. Each player is risk neutral and knows his or her own valuation for the object.

1. For example the good in question may be wine, so that resale value is not a factor.

B. Player's values are statistically independent.

C. Results.

1. The English auction is strategically equivalent to the second price sealed bid auction. (This result depends on the assumption that each bidder's valuatin does not depend on others' information.)

2. Bidding one's valuation is a dominant strategy, so the bidder with the highest valuation wins and the outcome is Pareto optimal.

3. The Dutch auction is strategically equivalent to the first price sealed bid auction.

4. If the valuations are i.i.d. and the equilibrium is symmetric, then the outcome is Pareto optimal, since equilibrium bidding functions are monotonic functions of the valuation.

5. All auction forms have the same expected revenue for the seller.

a. Let $e(p)$ be the expected payment of the least cost strategy that wins the object with probability p.

b. Let $p^*(v)$ be the equilibrium probability of winning the object as a function of v.

c. The marginal condition for optimization is

$$\frac{d[pv - e(p)]}{dp}\Big|_{p=p^*(v)} = 0 \quad \text{or} \quad v = e'(p^*(v)), \quad \text{so}$$

$$\frac{d[e(p^*(v))]}{dv} = v\left[\frac{dp^*}{dv}\right](v).$$

d. We now have the following computation

$$e(p^*(v)) = e(p^*(0)) + \int_0^v \left(\frac{d[e(p^*(v))]}{dv} \right) dv \quad \text{or}$$

$$e(p^*(v)) = e(p^*(0)) + \int_0^v w \, dp^*(w).$$

e. All the auction forms discussed above give the object to the agent with the highest valuation, so they all have the same equilibrium p^* function and, consequently, the same e function.

5. Milgrom and Weber assert that "for many common distributions" these auctions are optimal for the seller.

IV. Mineral Rights Model.

A. Assumptions.

1. The unknown "true value" v is the same for everyone.

2. Conditional on v, x_i and x_j are statistically independent.

B. In the first price sealed bid auction agent i chooses b_i to maximize $E[(v - b_i) \cdot 1_{b_i = \max\{b_j\}} | x_i]$.

1. Setting the derivative of this expression with respect to b_i equal to 0 yields a first order differential equation for b_i as a function of x_i.

C. In this setting one has the so–called "winner's curse" – the expected value of the object conditional on x_i may be much higher than the expected value conditional on x_i and $x_i = \max\{x_j\}$.

D. In many settings the first price sealed bid auction is an effective aggregator of information in the sense that the winning bid closely approximates the value of the object conditional on all x_i.

V. The More General Assumptions.

A. For some markets neither the independent private values model nor the mineral rights model seems appropriate. For instance consider art.

1. The IPV model fails to take into account that the investment value of a painting will be positively correlated with others' valuations.
2. The MR model fails to consider individual tastes.
3. Milgrom and Weber construct a model that has both IPV and MR as special cases.

B. Variables.

1. $X = (X_1, \ldots, X_n)$ – real valued informational variables, one for each bidder.
2. S – another random variable, possibly observed by the seller.
3. $V_i = u_i(S, X)$ – the full information value to bidder i.

C. Assumptions

1. **Assumption 1:** There is $u : I\!R^{n+1} \to I\!R$ such that

$$u_i(S, X) = u(S, X_i, \{X_j\}_{j \neq i})$$

2. **Assumption 2:** u is nonnegative, continuous, and nondecreasing.
3. **Assumption 3:** $E(V_i) < \infty$.
4. **Assumption 4:** $(S, X_1, \ldots, X_n)$ are jointly distributed with density f, where f is symmetric in its last n arguments.
5. **Assumption 5:** S and $X_1 \ldots, X_n$ are affiliated.

Definition: For $z, z' \in I\!R^{n+1}$, let

$$z \vee z' = (\max\{z_0, z_0'\}, \ldots, \max\{z_n, z_n'\}) \quad \text{and} \quad \text{let}$$

$$z \wedge z' = (\min\{z_0, z_0'\}, \ldots, \min\{z_n, z_n'\}).$$

The density f is *affiliated* if, for almost all z, z',

$$f(z \wedge z')f(z \vee z') \geq f(z)f(z').$$

D. Connections with other models.

1. In the IPV model we have $f(z \wedge z')f(z \vee z') = f(z)f(z')$.

2. In the MR model we have $f(s, x) = h(s)g(x_1|s) \ldots, g(x_n|s)$ where g satisfies the *monotone likelihood ratio property*:

$$g(x|s)g(x'|s') \geq g(x|s')g(x'|s), \quad \text{or} \quad \frac{g(x|s)}{g(x'|s)} \geq \frac{g(x|s')}{g(x'|s')},$$

when $s' > s$ and $x' > x$.

VI. Consequences of Affiliation

A. **Theorem 1:** (i) If $f : I\!R^k \to I\!R$ is C^2, then it is affiliated if and only if

$$\frac{\partial^2(\ell n\, f)}{\partial z_i \partial z_j} \geq 0$$

whenever $i \neq j$. (ii) If $f(z) = g(z)h(z)$, where g and h are affiliated, then f is affiliated.

B. Let $Y_1, \ldots, Y_{n-1}$ be that largest, second largest, $\ldots$, smallest of $X_2, \ldots, X_n$.

Theorem 2: If f is affiliated and symmetric in $X_2, \ldots, X_n$, then $(S, X_1, Y_1, \ldots, Y_{n-1})$ are affiliated.

Proof: Apply Theorem 1 to the density for $(s, x_1, y_1, \ldots, y_{n-1})$ which is $(n-1)! \cdot f(s, x_1, y_1, \ldots, y_{n-1}) \cdot 1_{\{y_1 \geq \ldots \geq y_{n-1}\}}$. ∎

Theorem 3: If $Z_1, \ldots, Z_k$ are affiliated and $g_1, \ldots, g_k : I\!R \to I\!R$ are all increasing function, then $g_1(z_1), \ldots, g_k(z_k)$ are affiliated.

Theorem 4: If $Z_1, \ldots, Z_k$ are affiliated, then $Z_1, \ldots, Z_{k-1}$ are affiliated.

Theorem 5: If $Z_1, \ldots, Z_k$ are affiliated and $H : I\!R^k \to I\!R$ is nondecreasing, then

$$h(a_1, b_1; \ldots; a_k, b_k) = E[H(z_1, \ldots, z_k)|a_1 \leq z_1 \leq b_1, \ldots, a_k \leq z_k \leq b_k]$$

is nondecreasing in all arguments. In particular, $E[V_1|X_1 = x_1, Y_1 = y_1, \ldots, Y_{n-1} = y_{n-1}]$ is nondecreasing in all arguments.

VII. Equilibrium in Second Price Auctions.

A. We assume that all valuations are monetary and that all agents are risk neutral.

B. We assume that agents $j \neq 1$ are following bidding strategies $b_j : X_j \mapsto b_j$, and we analyze the behavior of agent 1.

1. Set $W = \max_{j \neq i} b_j(X_j)$.
2. Agent 1's problem is max $E[(V_1 - W) \cdot 1_{\{W<b\}} | X_1]$.

C. Set $v(x, y) = E[V_1 | X_1 = x, Y_1 = y]$. By the Theorems above, v is nondecreasing in both variables. Let $b^*(x) = v(x, x)$.

Theorem 6: $(b^*, \ldots, b^*)$ is an equilibrium.

Proof: The consequences of, say, increasing the bid over b^* is that one wins in situations where one would have lost before. If $Y_1 = X$ and everybody else is following b^*, one can only do worse by departing from b^*.

VIII. Equilibrium in English Auctions.

A. There are many variants of the English auction. We consider one in which the information people have is easily specified. Each bidder has a button, and at the beginning all bidders are depressing their buttons. As time passes the posted price rises continuously, and a bidder drops out by releasing the button. The remaining bidders know how many bidders have dropped out and the prices at which they departed. The winner is the last bidder, and he or she pays the price at which the last bidder departed.

B. A strategy is described by functions $b_{ik}(x_i | p_1, \ldots, p_k)$, $p_1 \leq \ldots \leq p_k$ describing the price at which agent i drops out when his signal is x_i and k other bidders drop out at prices $p_1, \ldots, p_k$.

C. Define the functions b_k^* inductively by

$$b_0^*(x) = E[V_1 | X_1 = x,\, Y_1 = x,\, \ldots,\, Y_{n-1} = x] \text{ and}$$

$$b_k^*(x | p_1,\, \ldots,\, p_k) = E[V_1 | X_1 = x,\, Y_1 = x,\, \ldots,\, Y_{n-k-1} = x,$$
$$b_{k-1}^*(Y_{n-k} | p_1,\, \ldots,\, p_{k-1}) = p_k,\, \ldots,\, b_0^*(Y_{n-1}) = p_1].$$

1. Let $b^* = (b_0^*,\, \ldots,\, b_{n-2}^*)$.

Theorem 10: $(b^*,\, \ldots,\, b^*)$ is an equilibrium of the button auction.

Proof: If all other agents follow b^*, it is again the case that by following b^* agent 1 wins if and only if that is what he wants to do conditional on others' information. ∎

IX. First Price Auctions.

A. This is the traditional sealed bid auction in which the winner pays his bid price.

B. Again we wish to determine when $(b^*,\, \ldots,\, b^*)$ is an equilibrium. Assume b^* is increasing and differentiable. (Even with smooth densities it is not necessarily the case that b^* is in fact differentiable.)

C. Assume all $j \neq 1$ play b^* while bidder 1 observes x and bids b. His expected payoff is

$$\begin{aligned}
\Pi(b;\, x) &= E[(V_i - b) \cdot 1_{\{b^*(Y_1) < b\}} | X_1 = x] \\
&= E[E[(V_i - b) \cdot 1_{\{b^*(Y_1) < b\}} | X_1,\, Y_1]\, | X_1 = x] \\
&= E[(v(X_1,\, Y_1) - b) \cdot 1_{\{b^*(Y_1) < b\}} | X_1 = x] \\
&= \int_0^{b^{*-1}(b)} (v(x,\, \alpha) - b)\, f_{Y_1}(\alpha | x)\, d\alpha
\end{aligned}$$

Here $f_{Y_1}(\cdot | x)$ is the density of Y_1 given x. Let $F_{Y_1}(\cdot | x)$ be the cumulative distribution function.

D. If $b = b^*(x)$ is optimal then (after some work) the first order condition yields the differential equation

$$(*) \qquad b^{*\prime}(x) = (v(x,\, x) - b^*(x)) \left[\frac{f_{Y_1}(x|x)}{F_{Y_1}(x|x)} \right]$$

1. Necessarily $v(x, x) - b^*(x)$ is nonnegative for all x, since otherwise one is bidding when one prefers to not win.
2. Let $\underline{x}$ be the lowest possible signal. If $v(\underline{x}, \underline{x}) - b^*(\underline{x}) > 0$ then one does better by replacing $b^*(\underline{x})$ with $b^*(\underline{x}) + \varepsilon$, , so

$$(**) \qquad b^*(\underline{x}) = v(\underline{x}, \underline{x}).$$

Theorem 14: If b^* satisfies $(*)$ and $(**)$ then it is an equilibrium.

X. Milgrom and Weber consider several questions for the solutions derived above. In particular they are interested in the seller's preferences over auctions and whether the seller should reveal his private information.