

Infinite Games

I. Introduction.

A. We will present several results concerning various types of infinite games.

1. Normal form games with infinite strategy spaces.
2. Infinite horizon games.
3. Games with infinitely many actions at some information sets.
4. Games with infinitely many agents.
5. Games with infinitely many types of each agent.

B. In many situations the infinite game is regarded as an idealization of a large finite game.

1. Many people would then argue that the interesting equilibria are those that are limits of equilibria, or approximate equilibria, of large finite games.
2. In many particular models the only apparent way to build ideas like perfection or stability into the analysis is to analyze finite approximations and take limits.

II. Fudenberg and Levine (1986): “Limit Games and Limit Equilibria”

A. The main result in this paper formalizes the process of approximating a normal form game with infinite strategy spaces by finite games.

B. Let $G = (S_1, \dots, S_n; \pi_1, \dots, \pi_n)$ be a normal form game.

1. We allow the sets S_i to have any cardinality.
2. We require the payoff functions $\pi_i : S = S_1 \times \dots \times S_n \rightarrow \mathbb{R}$ to be bounded.

C. We endow the spaces S_i with a natural topology that guarantees that the payoff functions are jointly continuous.

1. **Definition:** The *distance* between strategies s_i and t_i in S_i is

$$p_i(s_i, t_i) = \sup_{s_{-i} \in S_{-i}} \max_{j \in i} |\pi_j(s_i, s_{-i}) - \pi_j(t_i, s_{-i})|.$$

This is a pseudometric, clearly, so it induces a topology called the *inherent topology*.

2. The *inherent product topology* is the induced product topology on S .

a. Let $p(s, t) = \sum_i p_i(s_i, t_i)$.

- D. **Definition:** $R \subset S$ is a *restriction* or a *restricted game* if $R = \prod_i R_i$ for some sets $\emptyset \neq R_i \subset S_i, i = 1, \dots, n$.

1. It is easy to see that the inherent topology of R_i in $(R_1, \dots, R_n; \pi_1, \dots, \pi_n)$ is at least as coarse as the relative topology on R_i induced by the inherent topology on S_i .

2. The sequence of restricted games $\{R^n\}$ *approximates* S if for every subsequence $\{R^{n_k}\}, \bigcup_{k=1,2,\dots} R^{n_k}$ is dense in S .

- E. **Definition:** $s \in S$ is an ε -*equilibrium* if for all i and all $t_i \in S_i, \pi_i(t_i, s_{-i}) \leq \pi_i(s) + \varepsilon$.

1. **Proposition:** If s is an ε -equilibrium, then t is an $[\varepsilon + 2p(s, t)]$ -equilibrium.

Proof: This is a straightforward consequence of the definition of $p(s, t)$. ■

The Limit Theorem: Suppose $\{R^n\}$ approximates S and $r^n \in R^n$.

- (A) If the r^n are ε^n -equilibria relative to R^n with $\varepsilon^n \rightarrow \varepsilon$ and $r^n \rightarrow s$, then s is an ε -equilibrium.
- (B) If s is an ε -equilibrium and $r^n \rightarrow s$, then there is a sequence $\varepsilon^n \rightarrow \varepsilon$ such that each r^n is an ε^n -equilibrium relative to R^n .

Proof: (A) and (B) are immediate consequences of the Proposition and our definitions. ■

Remark: Since $\{R^n\}$ approximates S , for every $s \in S$ there is a sequence $\{r^n \in R^n\}$ with $r^n \rightarrow s$.

III. Fudenberg and Levine (1983) – “Subgame-Perfect Equilibria of Finite- and Infinite-Horizon Games”

A. The main result of this paper characterizes subgame perfect equilibria of infinite horizon games as limits of approximate perfect equilibria of finite truncations.

B. The Setup

1. In each period each agent chooses a point in \mathbb{R}^M .
 - a. Let B be $\prod_{t=1,2,\dots} \mathbb{R}^{MN}$. Generic elements of B are written $x = (x_1, x_2, \dots)$ where $x_t = (x_t^1, \dots, x_t^N)$.
 - b. A game is determined by a space $X \subset B$ of possible *outcomes* and *utility functions* $V^i : X \rightarrow \mathbb{R}, i = 1, \dots, N$.
2. Properties of X .
 - a. For each t let $X(t) = \{(x_1, \dots, x_t) | x \in X\}$ be the space of possible t period histories.
 - b. For $x \in X(t-1)$ let $X(x, t) = \{y \in \mathbb{R}^{MN} | (x, y) \in X(t)\}$.
 - c. For $x \in X(t-1)$ and $i = 1, \dots, N$ let $X^i(x, t) = \{y^i | y \in X(x, t)\}$ be the set of feasible strategies for agent i after x .
 - d. **Assumption 1:** For all t and $x \in X(t-1)$, $X(x, t) = \prod_i X^i(x, t)$.
 - i. Of course this is necessary if we are to regard the sets $X^i(x, t)$ as the sets of feasible choices for the various agents.
 - e. **Assumption 2:** If $(x_1, \dots, x_t) \in X(t)$ for all t , then $x \in X$.
 - f. **Assumption 3:** $0 \in X^i(x, t)$ for all i, t , and $x \in X(t-1)$.
3. A T -period pure strategy for agent i is $g^i = (g_1^i, \dots, g_t^i)$ where $g_1^i \in X^i(0, 1)$ and $g_s^i : X(s-1) \rightarrow \mathbb{R}^M$ satisfies $g_s^i(x) \in X^i(x, s), s = 2, \dots, t$. Let $S^i(T)$ be the set of T -period pure strategies for i , and let $S(T) =$

$\prod_i S^i(T)$.

- a. Note that this definition remains reasonable when $t = \infty$.
- b. For $g \in S(\infty)$ and $x \in X(s-1)$, let $x^s(g)$ be the outcome when strategy g is played beginning in period s at x .
- c. Define $V_T^i : S(T) \rightarrow \mathbb{R}$ by

$$V_T^i(x_1, \dots, x_T) = V^i(x_1, \dots, x_T, 0, 0, \dots).$$

C. Definition: $g^* \in S(T)$ is a *subgame-perfect ε -Nash equilibrium* (or simply ε -perfect) if for each $s \geq 0$, history $x \in X(s-1)$, strategy $g \in S(T)$, and player i ,

$$V_T^i(x^s(g^i, g^{*-i})) - V_T^i(x^s(g^*)) \leq \varepsilon.$$

1. For each T , let w^T be the supremum of $|V^i(x) - V^i(z)|$ over all i and all $x, z \in X$ with $(x_1, \dots, x_{T-1}) = (z_1, \dots, z_{T-1})$.
 - a. We say that (X, V) is *continuous at infinity* if $w^T \rightarrow 0$ as $T \rightarrow \infty$.

2. Lemma 1:

- (A) If h^* is ε -perfect in $S(T)$, then h^* is $(\varepsilon + w^T)$ -perfect in $S(\infty)$.
- (B) If g^* is ε -perfect in $S(\infty)$, then (g_1^*, \dots, g_T^*) is $(\varepsilon + 2w^T)$ -perfect in $S(T)$.

Proof: Proceed in the obvious manner. ■

D. Let $B = \prod_t \mathbb{R}^{MN}$ be endowed with the product topology, where each copy of \mathbb{R}^{MN} has its customary topology.

1. This topology has the metric

$$d(x, z) = \sup_T \left(\frac{1}{T} \right) \min\{\|x_T - z_T\|, 1\}.$$

2. In the product topology the distant future does not matter much.
3. **Definition:** V is uniformly continuous if for all sequences $\{x^n\}, \{z^n\} \in X$, $d(x^n, z^n) \rightarrow 0$ implies $\|V(x^n) - V(z^n)\| \rightarrow 0$.

a. **Remark:** Clearly (X, V) is continuous at infinity whenever V is uniformly continuous by virtue of our definition of d .

4. For $f, g \in S(T)$ we define the distance between f and g to be

$$d(f, g) = \sup_{x \in X(t)} \{d(x^t(f), x^t(g)), \sup_{i, h^i \in S^i(\infty)} d(x^t(h^i, f^{-i}), x^t(h^i, g^{-i}))\}.$$

a. In words, f and g are close if they generate similar outcomes for any deviation by a single agent after any partial history.

E. Results.

Lemma 2: Let g_n be ε -perfect in $S(\infty)$ and $g_n \rightarrow g$ in a continuous game. Then g is also ε -perfect.

Proof: Suppose g is not ε -perfect so that for some t , some $x \in X$, some $h^i \in S^i(\infty)$, and some $\delta > 0$,

$$V^i(x^t(h^i, g^{-i})) - V^i(x^t(g)) \geq \varepsilon + 3\delta.$$

Since $g_n \rightarrow g$, for large n , $x^t(g)$ is near $x^t(g_n)$, and $x^t(h^i, g^{-i})$ is near $x^t(h^i, g_n^{-i})$.

As V^i is continuous, for large enough n we have that

$$V^i(x^t(g_n)) - V^i(x^t(g)) < \delta \quad \text{and}$$

$$V^i(x^t(h^i, g^{-i})) - V^i(x^t(h^i, g_n^{-i})) < \delta.$$

Combining these inequalities, for large enough n we have

$$V^i(x^t(h^i, g_n^{-i})) - V^i(x^t(g_n)) > \varepsilon + \delta.$$

As δ can be taken to be arbitrarily small, this contradicts the assumption that g_n is ε -perfect. ■

Theorem: (Limit Theorem) Suppose V is uniformly continuous. The following are equivalent.

- (A) g^* is a perfect equilibrium in $S(\infty)$.
- (B) There are sequences $\{T(n)\}$ and $\{g_n\}$ with each g_n a $2w^{T(n)}$ -perfect equilibrium in $S(T(n))$ such that $T(n) \rightarrow \infty$ and $g_n \rightarrow g^*$ as $n \rightarrow \infty$. (By the convergence of g_n to g^* we mean that $(g_{n1}, \dots, g_{nT(n)}, 0, 0, \dots) \rightarrow g^*$.)
- (C) There are sequences $\varepsilon_n, T(n)$, and g_n such that g_n is a $2w^{T(n)}$ -perfect equilibrium in $S(T(n))$ and $\varepsilon_n \rightarrow 0, T(n) \rightarrow \infty$, and $g_n \rightarrow g^*$ as $n \rightarrow \infty$.

Proof: (A) \Rightarrow (B) We claim the sequence $\{g^*(n) = (g_1^*, \dots, g_n^*, 0, 0, \dots)\}$ with $T(n) = n$ has the requisite property. First, since $g^*(n)$ and g^* exactly agree in the first n periods, $d(g^*(n), g^*) \leq \frac{1}{(n+1)}$. Thus $g^*(n) \rightarrow g^*$. Part (B) of Lemma 1 implies that $g^*(n)$ is $2w^{T(n)}$ -perfect in $S(n)$.

(B) \Rightarrow (C) Obvious.

(C) \Rightarrow (A) By part (A) of Lemma 1, g_n is $(\varepsilon_n + w^{T(n)})$ -perfect in $S(\infty)$. Since $\varepsilon_n + w^{T(n)} \rightarrow 0$, for each $\delta > 0$ there is an N such that $w^{T(n)} + \varepsilon_n < \delta$ whenever $n > N$. Thus by Lemma 2 g^* is δ -perfect. Since this is true for every $\delta > 0$, g^* is in fact perfect. ■

E. Harris (1985) – “Existence and Characterization of Perfect Equilibrium in Games of Perfect Information”

1. This paper considers infinite horizon games of perfect information in which the sets of actions in each period need not be finite and the space of histories need not be compact in the product topology.
2. The gist of it is that an outcome is a perfect equilibrium outcome if, at each stage, there are punishments (perfect equilibrium outcomes of the relevant subgames) that are sufficient to deter any deviation.

IV. Mas-Colell (1984) – “On a Theorem of Schmeidler”

- A. This paper is concerned with normal form games with a continuum of agents.

1. Naively one thinks of a strategy vector as a (measurable) function from the set of agents of the set of actions. Schmeidler's analysis is in this framework.
2. Mathematically it is much simpler to work in the space of joint distributions on characteristics and actions.

B. The Framework.

1. Let A be a nonempty compact metric space of *actions*.
2. Let $\mathcal{M} = \Delta(A)$ be endowed with the weak convergence topology.
 - a. $\nu_n \rightarrow \nu$ weakly if $\int f d\nu_n \rightarrow \int f d\nu$ for every bounded continuous $f : A \rightarrow \mathbb{R}$.
 - b. The Banach–Alaoglu theorem of functional analysis implies that \mathcal{M} is compact.
3. Let \mathcal{U}_A be the space of continuous utility function $u : A \times \mathcal{M} \rightarrow \mathbb{R}$ endowed with the supremum norm: $\|u\| = \max |u(a, \nu)|$.
 - a. Standard results from functional analysis imply that \mathcal{U}_A is a separable Banach (complete normed linear) space.
 - b. A *game* is a Borel measure μ on \mathcal{U}_A .

C. **Definition:** Given a game μ , a Borel measure τ on $\mathcal{U}_A \times A$ is a *Nash equilibrium distribution* if, letting $\tau_{\mathcal{U}}$ and τ_A denote the marginals of τ on \mathcal{U}_A and A respectively, we have:

- (i) $\tau_{\mathcal{U}} = \mu$;
- (ii) $\tau(\{(u, a) | u(a, \tau_A) = \max_{a' \in A} u(a', \tau_A)\}) = 1$.

D. **Theorem:** Every game μ on \mathcal{U}_A has a Nash equilibrium distribution.

Proof: Let \mathcal{T} be the set of probability measures τ on $\mathcal{U}_A \times A$ with $\tau_{\mathcal{U}} = \mu$. The map $\tau \mapsto \tau_{\mathcal{U}}$ is continuous (exercise), so \mathcal{T} is a closed subset of the set of measures on $\mathcal{U}_A \times A$ with total measure $\mu(\mathcal{U}_A)$, whence the Banach–Alaoglu theorem implies that \mathcal{T} is compact. Given $\tau \in \mathcal{T}$, let $B_\tau = \{(u, a) | u(a, \tau_A) \geq u(A, \tau_A)\}$. Define $\Phi : \mathcal{T} \rightarrow \mathcal{T}$

by $\Phi(\tau) = \{\tau' \in \mathcal{T} | \tau'(B_\tau) = 1\}$. Obviously $\Phi(\tau)$ is convex for all τ . In the usual way it is easy to verify that Φ is upper semicontinuous and convex valued. Therefore the conditions of the Fan–Glicksberg fixed point theorem are satisfied. ■

E. Mas–Colell goes on to show that if A is finite and μ is atomless, then there is a Nash equilibrium distribution that assigns all probability to the graph of a function from \mathcal{U}_A to A . That is, players with identical characteristics choose the same action.

V. Milgrom and Weber (1985) – “Distributional Strategies for Games with Incomplete Information”

A. The given elements of the model are as follows:

1. $N = \{1, \dots, n\}$ – the set of *players*;
2. For each $i \in N$, T_i is a Polish (complete separable metric) space of *types*;
3. For each $i \in N$, A_i is a compact metric space of *actions*;
4. T_0 is a complete separable metric space of *states*. Let $T = T_0 \times T_1 \times \dots \times T_n$, and let $A = A_1 \times \dots \times A_n$;
5. For each i , $U_i : T \times A \rightarrow \mathbb{R}$ is measurable and bounded;
6. The *information structure* is h , a probability measure on T .
 - a. Let η_i be the marginal distribution on T_i induced by h .

B. Strategies and Equilibrium

1. **Definition:** A *distributional strategy* for agent i is a probability measure μ_i on $T_i \times A_i$ whose marginal distribution on T_i is η_i .
2. Given distributional strategies μ_1, \dots, μ_n , the expected payoff for agent i is

$$\pi_i(\mu_1, \dots, \mu_n) = \int U_i(t, a) \mu_1(da_1|t_1) \dots \mu_n(da_n|t_n) \eta(dt).$$

- a. In order to make sense of this we must show that the conditional probability $\mu_i(\cdot|\cdot)$ exists and, if there is more than one, that the choice does not effect

the integral. This is advanced measure theory.

3. Definition: An *equilibrium* is a vector of distributional strategies (μ_1, \dots, μ_n) such that $\pi_i(\mu) \geq \pi_i(\mu'_i, \mu_{-i})$ for all i and all distributional strategies μ'_i .

a. In order for this definition to be sensible it must be the case that if any nonnull set of types of an agent can improve their expected payoffs, then the agent has an improving distributional strategy. This is easily demonstrated.

C. Milgrom and Weber prove the existence of equilibria when certain technical conditions are satisfied.

1. They prove that the equilibrium correspondence is upper hemicontinuous, and that under certain circumstances one can insure the existence of pure strategy equilibria.

D. Recently Balder has improved and generalized the existence result.

1. Balder works with behavior strategies: a *behavior strategy* for i is a transition probability from T_i to A_i . (Recall that a function $P : T_i \rightarrow \Delta(A_i)$ is a transition probability if, for all measurable $E \subset A_i$, $P(\cdot; E) : T_i \rightarrow [0, 1]$ is measurable.