Elementary theory of metric spaces

Theorem 1 (Schwarz inequality) *Let* $x = (x_1, ..., x_N)$ *and* $y = (y_1, ..., y_N)$ *be elements of* $\mathbb{R}^N$. *Then:*

$$\left|\sum_{i=1}^{N} x_i y_i\right| \leq \left(\sum_{i=1}^{N} x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{N} y_i^2\right)^{\frac{1}{2}}.$$

Proof. Define $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(\lambda) = \sum_{i=1}^{N}(y_i - \lambda x_i)^2$. Note that $f(\lambda) \geq 0\ \forall\lambda$. Also,

$$f(\lambda) = \sum_{i=1}^{N} y_i^2 - \lambda \cdot 2\sum_{i=1}^{N} x_i y_i + \lambda^2 \sum_{i=1}^{N} x_i^2.$$

Note that since the equation depends on λ and not x or y, we can rearrange the equation as:

$$f(\lambda) = a\lambda^2 - b\lambda + c$$

and find the λ^* that minimizes f. There are two possible cases:
(a) $a = 0$. If $a = 0$ then $\sum_{i=1}^{N} x_i^2 = 0$, which is equal to having $x_i = 0\ \forall i$. Then,

$$\left|\sum_{i=1}^{N} x_i y_i\right| \leq \left(\sum_{i=1}^{N} x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{N} y_i^2\right)^{\frac{1}{2}}$$

$$\left|\sum_{i=1}^{N} 0 \cdot y_i\right| \leq \left(\sum_{i=1}^{N} 0\right)^{\frac{1}{2}} \left(\sum_{i=1}^{N} y_i^2\right)^{\frac{1}{2}}$$

$$0 \leq 0$$

And the expression is satisfied.
(b) $a \neq 0$. The value of λ^* that minimizes f satisfies $f'(\lambda^*) = 0$. Then,

$$f'(\lambda) = 2a\lambda - b$$

$$2a\lambda^* - b = 0$$

$$\lambda^* = \frac{b}{2a}$$

It follows that $f(\lambda^*)$ is:

$$\begin{aligned} f(\lambda^*) &= a\left(\frac{b}{2a}\right)^2 - b\left(\frac{b}{2a}\right) + c \\ &= \frac{ab^2}{4a^2} - \frac{b^2}{2a} + c \\ &= \frac{b^2}{4a} - \frac{b^2}{2a} + c \\ &= -\frac{b^2}{4a} + c \end{aligned}$$

But since $f(\lambda) \geq 0$ then $-\frac{b^2}{4a} + c \geq 0$ and $\frac{-b^2+4ac}{4a} \geq 0$; therefore, $b^2 \leq 4ac$. Recalling the values of a, b, and c:

$$\left[2\sum_{i=1}^{N} x_i y_i\right]^2 \leq 4\left(\sum_{i=1}^{N} x_i^2\right)\left(\sum_{i=1}^{N} y_i^2\right)$$

Taking square roots, and noting that $\sqrt{x^2} = |x|$,

$$\left|\sum_{i=1}^{N} x_i y_i\right| \leq \left(\sum_{i=1}^{N} x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{N} y_i^2\right)^{\frac{1}{2}}.$$

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There are two useful derivations of this result.

Corollary 2 (Triangle inequality) *Let* $x = (x_1, ..., x_N)$ *and* $y = (y_1, ..., y_N)$ *be elements of* $\mathbb{R}^N$. *Then:*

$$\sqrt{\sum_{i=1}^{N} (x_i + y_i)^2} \leq \sqrt{\sum_{i=1}^{N} x_i^2} + \sqrt{\sum_{i=1}^{N} y_i^2}.$$

Proof. Note that:

$$\sum_{i=1}^{N} (x_i + y_i)^2 = \sum_{i=1}^{N} x_i^2 + 2\sum_{i=1}^{N} x_i y_i + \sum_{i=1}^{N} y_i^2.$$

From Theorem 1 we know that $\left|\sum_{i=1}^{N} x_i y_i\right| = \left(\sum_{i=1}^{N} x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{N} y_i^2\right)^{\frac{1}{2}}$. Substituting:

$$\sum_{i=1}^{N} (x_i + y_i)^2 = \sum_{i=1}^{N} x_i^2 + 2\sum_{i=1}^{N} x_i y_i + \sum_{i=1}^{N} y_i^2$$

$$\leq \sum_{i=1}^{N} x_i^2 + 2\left(\sum_{i=1}^{N} x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{N} x_i^2\right)^{\frac{1}{2}} + \sum_{i=1}^{N} y_i^2$$

$$= \left(\sqrt{\sum_{i=1}^{N} x_i^2} + \sqrt{\sum_{i=1}^{N} y_i^2}\right)^2,$$

since $\sum_{i=1}^{N} x_i y_i$ is just a particular (and valid) case of $\left|\sum_{i=1}^{N} x_i y_i\right|$. Taking square roots:

$$\sqrt{\sum_{i=1}^{N} (x_i + y_i)^2} \leq \sqrt{\sum_{i=1}^{N} x_i^2} + \sqrt{\sum_{i=1}^{N} y_i^2}.$$

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Corollary 3 *Let* $a = (a_1, ..., a_N)$, $b = (b_1, ..., b_N)$, $c = (c_1, ..., c_N)$ *be elements of* $\mathbb{R}^N$. *Then,*

$$\sqrt{\sum_{i=1}^{N} (a_i - c_i)^2} \leq \sqrt{\sum_{i=1}^{N} (a_i - b_i)^2} + \sqrt{\sum_{i=1}^{N} (b_i - c_i)^2}.$$

Proof. Set $x_i = a_i - b_i$, $y_i = b_i - c_i$. Then, by Corollary 2,

$$\sqrt{\sum_{i=1}^{N} ((a_i - b_i) + (b_i - c_i))^2} \leq \sqrt{\sum_{i=1}^{N} (a_i - b_i)^2} + \sqrt{\sum_{i=1}^{N} (b_i - c_i)^2}.$$

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Definition 4 (Cartesian product) *Let* S *and* T *be sets. The cartesian product of* S *and* T, *denoted as* $S \times T$, *is the set of all ordered pairs* (p, q) *in which* p *belongs to* S *and* q *belongs to* T:

$$S \times T = \{(p, q) | p \in S, q \in T\}.$$

Remark 5 *Almost generally,* $S \times T \neq T \times S$.

Example 6 *Let* $S = \{1, 2\}$ *and* $T = \{1, 3\}$. *Then* $S \times T = \{(1, 1), (1, 3), (2, 1), (2, 3)\}$. *However, the product* $T \times S$ *is different, since* $T \times S = \{(1, 1), (1, 2), (3, 1), (3, 2)\}$.

Definition 7 *The cartesian product of any finite number of sets* $S_1, ..., S_M$ *is the set of ordered N-tuples* $(p_1, ..., p_N)$ *in which* $p_i \in S_i$, $\forall i = \{1, ..., N\}$. *This is denoted by:*

$$\bigotimes_{i=1}^{N} S_i.$$