Metrics

Definition 1 (Metric space) *Let S be a set and suppose $d : S \times S \to \mathbb{R}$, $(x,y) \to d(x,y)$. Then (S,d) is a metric space if the metric d satisfies the following properties:*

(a) $d(x,y) \geq 0$ $\forall x,y \in S$; $d(x,y) = 0$ iff $x = y$.

(b) $d(x,y) = d(y,x)$ $\forall x,y \in S$.

(c) (Triangular inequality) *$d(x,y) \leq d(x,y) + d(y,z)$ $\forall x,y,z \in S$.*

Example 2 *Suppose $S = \mathbb{R}^N$. Define $d_1(x,y) = \sum_{i=1}^N |x_i - y_i|$. Then (S,d_1) is a metric space.*

Proof. (a) $d_1(x,y) \geq 0$ $\forall x,y \in S$, this is satisfied trivially because of the properties of $|\cdot|$. Also, note that $d_1(x,y) = 0 \Leftrightarrow \sum_{i=1}^N |x_i - y_i| = 0 \Leftrightarrow |x_i - y_i| = 0$ $\forall i = \{1,...,N\} \Leftrightarrow x_i - y_i = 0$ $\forall i = \{1,...,N\} \Leftrightarrow x_i = y_i$ $\forall i = \{1,...,N\} \Leftrightarrow x = y$.

(b) $d_1(x,y) = \sum_{i=1}^N |x_i - y_i| = \sum_{i=1}^N |y_i - x_i| = d_1(y,x)$.

(c) Notice that:

$$\begin{aligned} d_1(x,y) &= \sum_{i=1}^N |x_i - z_i| = \sum_{i=1}^N |(x_i - y_i) + (y_i - z_i)| \\ &\leq \sum_{i=1}^N (|x_i - y_i| + |y_i - z_i|) \\ &= \sum_{i=1}^N |x_i - y_i| + \sum_{i=1}^N |y_i - z_i| \\ &= d_1(x,y) + d_1(y,z) \end{aligned}$$

where the second line follows from the fact that $|a+b| \leq |a| + |b|$.

Therefore, (S,d_1) is a metric space. ■

Example 3 *Suppose $S = \mathbb{R}^N$. Define $d_2(x,y) = \sqrt{\sum_{i=1}^N (x_i - y_i)^2}$. Then (S,d_2) is a metric space.*

Proof. (a) $d_2(x,y) \geq 0$ $\forall x,y \in S$, this is satisfied trivially. Also, $d_2(x,y) = 0 \Leftrightarrow \sqrt{\sum_{i=1}^N (x_i - y_i)^2} = 0 \Leftrightarrow \sum_{i=1}^N (x_i - y_i)^2 = 0 \Leftrightarrow (x_i - y_i)^2 = 0$ $\forall i = \{1,...,N\} \Leftrightarrow (x_i - y_i) = 0$ $\forall i = \{1,...,N\} \Leftrightarrow x_i = y_i$ $\forall i = \{1,...,N\} \Leftrightarrow x = y$.

(b) $d_2(x,y) = \sqrt{\sum_{i=1}^N (x_i - y_i)^2} = \sqrt{\sum_{i=1}^N (y_i - x_i)^2} = d_2(y,x)$.

(c) Setting $a_i = x_i$, $c_i = z_i$ and $b_i = y_i$, and using Corollary 3 of the previous section,

$$\begin{aligned} d_2(x,z) = \sqrt{\sum_{i=1}^N (x_i - z_i)^2} &\leq \sqrt{\sum_{i=1}^N (x_i - y_i)^2} + \sqrt{\sum_{i=1}^N (y_i - z_i)^2} \\ &= d_2(x,y) + d_2(y,z). \end{aligned}$$

Therefore, (S,d_2) is a metric space. ■

Example 4 *Suppose $S = \mathbb{R}^N$. Define $d_3(x,y) = \max_{i=1,...,N} |x_i - y_i|$. Then (S,d_3) is a metric space.*

Proof. Assigned for homework. ■

Example 5 *Suppose $S = \mathbb{R}^N$. Define $d_4(x,y) = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases}$. Then (S,d_4) is a metric space.*

Proof. (a) $d_4(x,y) \geq 0$ $\forall x,y \in S$, this is satisfied trivially by definition of d_4. Also, note that $d_4(x,y) = 0$ iff $x = y$ is also satisfied trivially.

(b) $d_4(x,y) = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases} = \begin{cases} 0 & \text{if } y \neq x \\ 1 & \text{if } y = x \end{cases} = d_4(y,x)$.

(c) We have two cases:

(c.1) $x = z$. Then $d_4(x,z) = 0$ and the property is satisfied.

(c.2) $x \neq z$. Then $d_4(x,z) = 1$. But in this case, we can rule out the possibility that $y = x = z$ (since $x \neq z$), and therefore $y = x$, or $y = z$ or $y \neq x \neq z$. It follows that:

(c.2.1) If $y = x$, $d_4(x,y) + d_4(y,z) = 0 + 1 = 1$.

(c.2.2) If $y = z$, $d_4(x, y) + d_4(y, z) = 1 + 0 = 1$.
(c.2.3) If $y \neq x \neq z$, $d_4(x, y) + d_4(y, z) = 1 + 1 = 2$.
And in every case, the inequality is satisfied.
Therefore, (S, d_4) is a metric space. ∎

Example 6 *Suppose $S = \{f : [0, 1] \longrightarrow \mathbb{R} | f$ is continuous$\}$. Define $d(f, g) = \max_{x\in[0,1]} |f(x) - g(x)|$. Then (S, d) is a metric space.*

Proof. (a) $d(f, g) \geq 0\ \forall f, g \in S$, this is satisfied trivially because of the properties of $|\cdot|$. Also, note that $d(f, g) = 0 \Leftrightarrow \max_{x\in[0,1]} |f(x) - g(x)| = 0 \Leftrightarrow |f(x) - g(x)| = 0\ \forall x \in [0, 1] \Leftrightarrow f(x) = g(x)\ \forall x \in [0, 1] \Leftrightarrow f = g$.

(b) $d(f, g) = \max_{x\in[0,1]} |f(x) - g(x)| = \max_{x\in[0,1]} |g(x) - f(x)| = d(g, f)$.

(c) Notice that:

$$\begin{aligned} d(f, h) &= \max_{x\in[0,1]} |f(x) - h(x)| \\ &= \max_{x\in[0,1]} |f(x) - g(x) + g(x) - h(x)| \\ &\leq \max_{x\in[0,1]} (|f(x) - g(x)| + |g(x) - h(x)|) \end{aligned}$$

where the last line follows from the fact that $|a + b| \leq |a| + |b|$. Note that the last expression maximizes to x with one degree of freedom, while

$$\max_{x\in[0,1]} |f(x) - g(x)| + \max_{x\in[0,1]} |g(x) - h(x)|$$

has two degrees of freedom, therefore allowing for a greater amount. So it must be the case that

$$\max_{x\in[0,1]} (|f(x) - g(x)| + |g(x) - h(x)|) \leq \max_{x\in[0,1]} |f(x) - g(x)| + \max_{x\in[0,1]} |g(x) - h(x)|$$

so that

$$\begin{aligned} d(f, h) = \max_{x\in[0,1]} |f(x) - h(x)| &\leq \max_{x\in[0,1]} |f(x) - g(x)| + \max_{x\in[0,1]} |g(x) - h(x)| \\ &= d(f, g) + d(g, h). \end{aligned}$$

∎

Definition 7 (Equivalency of metric spaces) *Let S be a set and (S, d_1), (S, d_2) be two different metric spaces. We say that d_1 and d_2 are equivalent if there exists c and C such that:*

$$cd_1(x, y) \leq d_2(x, y) \leq Cd_1(x, y)\ \forall x, y \in S,\ c \in \mathbb{R}^+,\ C \in \mathbb{R}.$$

Note that this can also be expressed as:

$$\frac{1}{C} d_2(x, y) \leq d_1(x, y) \leq \frac{1}{c} d_2(x, y).$$

Example 8 *Let $S = \mathbb{R}^N$. Consider the metrics $d_1(x, y) = \sum_{i=1}^N |x_i - y_i|$ and $d_3(x, y) = \max_{i=1,\ldots,N} |x_i - y_i|$. Prove that d_1 and d_3 are equivalent.*

Proof. Note first that the maximum of an absolute difference is less than the sum of all absolute differences. Then:

$$d_3(x, y) = \max_{i=1,\ldots,N} |x_i - y_i| \leq \sum_{i=1}^N |x_i - y_i| = d_1(x, y).$$

Also, since the maximum of a set of numbers is always more than its average, we have that:

$$d_3(x, y) = \max_{i=1,\ldots,N} |x_i - y_i| \geq \frac{1}{N} \sum_{i=1}^N |x_i - y_i| = \frac{1}{N} d_1(x, y).$$

So, $c = \frac{1}{N}$ and $C = 1$, therefore d_1 and d_3 are equivalent. ∎

Example 9 *Let $S = \mathbb{R}^N$. Consider the metrics $d_1(x,y) = \sum_{i=1}^{N} |x_i - y_i|$, $d_2(x,y) = \sqrt{\sum_{i=1}^{N} (x_i - y_i)^2}$ and $d_3(x,y) = \max_{i=1,...,N} |x_i - y_i|$. Prove that: (a) d_1 is equivalent to d_2. (b) d_2 is equivalent to d_3.*

Proof. Assigned for homework. ■

Example 10 *Let $S = \mathbb{R}^N$. Consider $d_3(x,y) = \max_{i=1,...,N} |x_i - y_i|$ and $d_4(x,y) = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases}$. Prove that d_3 and d_4 are not equivalent.*

Proof. Suppose that they are equivalent. Then there exists a constant C such that

$$d_3(x,y) \leq C d_4(x,y) \; \forall x,y \in S.$$

Consider $x = (0,0,0,...,0)$ and $y = (C+1,0,0,...,0)$. Then $d_3 = C+1$ and $d_4 = 1$. Using this result,

$$d_3(x,y) > C d_4(x,y)$$

which is a contradiction. So the metrics are not equivalent. ■

Remark 11 *Equivalence among metrics is established with respect to a fixed S, which in the previous example was $\mathbb{R}^N$. However, the metrics in the previous example might be equivalent for some other set T.*