Sequences and convergence

Definition 1 (Sequence) *Let* $p_1, p_2, ..., p_n, ...$ *denote a sequence of elements of a metric space* (S, d)*. We use* $\{p_n\}_{n\in\mathbb{N}}$ *to denote such a sequence.*

Example 2 *Let* $S = \mathbb{R}^N$ *and* $d(x, y) = \sum_{i=1}^{N} |x_i - y_i|$*. Then, the first elements of the sequence* $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ *are* $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$*.*

Definition 3 (Convergence) *Let* (S, d) *be a metric space and* $p_0 \in S$*. Then* $\{p_n\}_{n\in\mathbb{N}}$ *converges to* p_0 *(denoted by* $p_n \to p_0$*,* $\lim_{n\to\infty} p_n = p_0$*) iff* $\forall\varepsilon > 0$*, there exists some* $N \in \mathbb{N}$ *such that* $\forall n > N$*,* $d(p_n, p_0) < \varepsilon$*.*

Example 4 *Let* $S = \mathbb{R}$*,* $d(x, y) = |x - y|$ *and* $\{p_n\}_{n\in\mathbb{N}} = \frac{1}{n}$*. Prove that* $\{p_n\}_{n\in\mathbb{N}}$ *converges to zero.*

Proof. Consider $\varepsilon > 0$. Note that:

$$d(p_n, 0) = \left|\frac{1}{n} - 0\right| = \frac{1}{n}.$$

Therefore, we need that $d(p_n, 0) = \frac{1}{n} < \varepsilon$. Consider $N = \frac{\lceil 1 \rceil}{\varepsilon}$, the "first" natural number above $\frac{1}{\varepsilon}$. Then $\forall n > N$, $\frac{1}{n} < \frac{1}{N} < \frac{1}{\varepsilon}$, and $d(p_n, 0) < \varepsilon$. ■

Proposition 5 (Uniqueness of the limit) *Let* (S, d) *be a metric space. If* $p_n \to p_0$ *and* $p_n \to p_0'$*, then* $p_0 = p_0'$*.*

Proof. Consider $\varepsilon > 0$. If $p_n \to p_0$ then there exists some $N \in \mathbb{N}$ such that

$$d(p_n, p_0) < \varepsilon \ \forall n > N.$$

If p_n is also converging to p_0', then there exists some $N' \in \mathbb{N}$ such that

$$d(p_n, p_0) < \varepsilon \ \forall n > N'.$$

Now, consider $n > \max\{N, N'\}$, and note that:

$$d(p_0, p_0') \leq d(p_0, p_n) + d(p_n, p_0')$$

but, by definition, $d(p_0, p_n) < \varepsilon$ and $d(p_n, p_0') < \varepsilon$. It follows that

$$d(p_0, p_0') < 2\varepsilon$$

so the only possibility is that $d(p_0, p_0') = 0$ and $p_0 = p_0'$. ■

Proposition 6 *Let* (S, d_1) *be a metric space. Suppose that* $p_n \xrightarrow{d_1} p_0$*. If* d_1 *is equivalent to* d_2*, then* $p_n \xrightarrow{d_2} p_0$*.*

Proof. By definition, $\forall\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $\forall n > N$, $d_1(p_n, p_0) < \varepsilon$. Consider $\bar{\varepsilon} > 0$. Then there exists some $\bar{N} \in \mathbb{N}$ such that $\forall n > \bar{N}$,

$$d_1(p_n, p_0) < \frac{\bar{\varepsilon}}{C}$$

where C is such that $d_2(p_n, p_0) \leq C d_1(p_n, p_0)$, which exists because d_1 and d_2 are equivalent. Then consider $N = \bar{N}$, and $\forall n > \bar{N}$,

$$\begin{aligned} d_2(p_n, p_0) &\leq C d_1(p_n, p_0) \\ &< C \cdot \frac{\bar{\varepsilon}}{C} \\ &= \bar{\varepsilon} \end{aligned}$$

and so $d_2(p_n, p_0) < \varepsilon$. ■

Some results in $\mathbb{R}$

The following results are stated for $S = \mathbb{R}$ and $d(x,y) = |x-y|$.

Theorem 7 *Let $S = \mathbb{R}$ and $d(x,y) = |x-y|$. Consider the sequences $\{p_n\}_{n\in\mathbb{N}}$, $\{q_n\}_{n\in\mathbb{N}}$. Then:*
(a) If $p_n \to p_0$ and $q_n \to q_0$, then $p_n + q_n \to p_0 + q_0$.
(b) If $p_n \to p_0$, then $cp_n \to cp_0$ $\forall c \in \mathbb{R}$.
(c) If $p_n \to p_0$ and $q_n \to q_0$, then $p_n q_n \to p_0 q_0$.
(d) If $p_n \to p_0$ and $q_n \to q_0 \neq 0$, then $\frac{p_n}{q_n} \to \frac{p_0}{q_0}$.

Proof. (a) Consider $\varepsilon > 0$. Now, by the triangular inequality,

$$|(p_n + q_n) - (p_0 + q_0)| \le |p_n - p_0| + |q_n - q_0|$$

and note that if $|p_n - p_0| < \frac{\varepsilon}{2}$ and $|q_n - q_0| < \frac{\varepsilon}{2}$ the proof would be done. Consider $N_1 \in \mathbb{N}$ such that $\forall n > N_1$, $|p_n - p_0| < \frac{\varepsilon}{2}$. Also, consider $N_2 \in \mathbb{N}$ such that $\forall n > N_2$, $|q_n - q_0| < \frac{\varepsilon}{2}$. (This follows from the fact that ε can be made as small as desired.) Then, set $N = \max\{N_1, N_2\}$, and $\forall n > N$,

$$|p_n - p_0| < \frac{\varepsilon}{2},\ |q_n - q_0| < \frac{\varepsilon}{2}.$$

Therefore,

$$\begin{aligned} |(p_n + q_n) - (p_0 + q_0)| &\le |p_n - p_0| + |q_n - q_0| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

(b) Assigned for homework.
(c) Note that:

$$|(p_n q_n) - (p_0 q_0)| \le |(p_n - p_0)(q_n - q_0) + p_0 (q_n - q_0) + q_0 (p_n - p_0)|.$$

Using the triangular inequality and the fact that $|ab| = |a| \cdot |b|$, it follows that:

$$|(p_n - p_0)(q_n - q_0) + p_0 (q_n - q_0) + q_0 (p_n - p_0)| \le |p_n - p_0| \cdot |q_n - q_0| + |p_0| \cdot |q_n - q_0| + |q_0| \cdot |p_n - p_0|$$

Note further that:
(c.1) There exists $N_1 \in \mathbb{N}$ such that $\forall n > N_1$, $|p_n - p_0| < \frac{\sqrt{\varepsilon}}{\sqrt{3}}$.
(c.2) There exists $N_2 \in \mathbb{N}$ such that $\forall n > N_2$, $|q_n - q_0| < \frac{\sqrt{\varepsilon}}{\sqrt{3}}$.
(c.3) There exists $N_3 \in \mathbb{N}$ such that $\forall n > N_3$, $|q_n - q_0| < \frac{\varepsilon}{3|p_0|}$.
(c.4) There exists $N_4 \in \mathbb{N}$ such that $\forall n > N_4$, $|p_n - p_0| < \frac{\varepsilon}{3|q_0|}$.
Consider $N = \max\{N_1, N_2, N_3, N_4\}$. Then $\forall n > N$,

$$\begin{aligned} |(p_n q_n) - (p_0 q_0)| &\le |p_n - p_0| \cdot |q_n - q_0| + |p_0| \cdot |q_n - q_0| + |q_0| \cdot |p_n - p_0| \\ &< \left(\frac{\sqrt{\varepsilon}}{\sqrt{3}}\right)\left(\frac{\sqrt{\varepsilon}}{\sqrt{3}}\right) + |p_0| \left(\frac{\varepsilon}{3\,|p_0|}\right) + |q_0| \left(\frac{\varepsilon}{3\,|q_0|}\right) \\ &= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

(d) Assigned for homework. ■

Definition 8 (Bounded sequence) *A sequence $\{p_n\}_{n\in\mathbb{N}}$ is bounded if there exists some M such that $|p_n| \le M$ $\forall n \in \mathbb{N}$.*

Example 9 *Let $\{p_n\}_{n\in\mathbb{N}} = (-1)^n$. The sequence is bounded since $|p_n| \le 1$.*

Example 10 *Let $\{p_n\}_{n\in\mathbb{N}} = n$. This sequence is not bounded.*

Theorem 11 *Every convergent sequence is bounded.*

Proof. If a sequence $\{p_n\}_{n\in\mathbb{N}}$ converges, then it has a limit p_0. Also, there exists a certain $N \in \mathbb{N}$ such that $\forall n > N$, $|p_n - p_0| < 1$ (since ε can take on any value). But $|p_n - p_0| < 1 \Leftrightarrow -1 < p_n - p_0 < 1 \Leftrightarrow p_0 - 1 < p_n < p_0 + 1 \Leftrightarrow |p_n| < |p_0| + 1 \Leftrightarrow |p_n| < M$. Note that the set $\{n | n \leq N\}$ has a finite number of elements. Then, since the infinitely many points in the range $n > N$ are bounded by $|p_0 + 1|$, it is safe to state that:

$$|p_n| \leq \max\{|p_0 + 1|, |p_1|, |p_2|, ..., |p_N|\} \ \forall n \in \mathbb{N}$$

and therefore, the sequence is bounded. ■

Remark 12 *The converse is not true. The sequence* $\{p_n\}_{n\in\mathbb{N}} = (-1)^n$ *is bounded but it does not converge.*

Theorem 13 *Let* $\{p_n\}_{n\in\mathbb{N}}$, $\{q_n\}_{n\in\mathbb{N}}$, $\{r_n\}_{n\in\mathbb{N}}$ *be sequences such that* $p_n \leq q_n \leq r_n \ \forall n > N$. *Suppose that* $\lim_{n\to\infty} p_n = \lim_{n\to\infty} r_n = L$. *Then* $\lim_{n\to\infty} q_n = L$.

Proof. Consider $\varepsilon > 0$. Since $p_n \to L$ then there exists $N_1 \in \mathbb{N}$ such that $\forall n > N_1$,

$$|p_n - L| < \varepsilon \Leftrightarrow -\varepsilon < p_n - L < \varepsilon \Leftrightarrow L - \varepsilon < p_n < L + \varepsilon.$$

Also, since $r_n \to L$, there exists some $N_2 \in \mathbb{N}$ such that $\forall n > N_2$,

$$|r_n - L| < \varepsilon \Leftrightarrow L - \varepsilon < r_n < L + \varepsilon.$$

It is the case that $\forall n > N$, $p_n \leq q_n \leq r_n$. Consider $N^* = \max\{N, N_1, N_2\}$. It follows that $\forall n > N^*$, the 3 conditions are met and

$$L - \varepsilon < p_n \leq q_n \leq r_n < L + \varepsilon,$$

which implies that

$$L - \varepsilon < q_n < L + \varepsilon \Leftrightarrow -\varepsilon < q_n - L < \varepsilon \Leftrightarrow |q_n - L| < \varepsilon \Leftrightarrow q_n \to L.$$

■

Example 14 *Prove that* $\lim_{n\to\infty} \sqrt[n]{p} = 1 \ \forall p > 0$.

Proof. There are 3 cases to consider.
(a) $p = 1$. Then $\sqrt[n]{p} = 1$ in every case and therefore $\sqrt[n]{p} \to 1$ (trivial case).
(b) $p > 1$. Define a sequence $\{x_n\}_{n\in\mathbb{N}} = \sqrt[n]{p} - 1$. Note that:
(b.1) $x_n \geq 0$.
(b.2) $(1 + x_n)^n = p$, which is equivalent to:

$$1 + nx_n + \begin{pmatrix} n \\ 2 \end{pmatrix} x_n^2 + \begin{pmatrix} n \\ 3 \end{pmatrix} x_n^3 + ... + \begin{pmatrix} n \\ n \end{pmatrix} x_n^n.$$

Also note that

$$1 + nx_n + \begin{pmatrix} n \\ 2 \end{pmatrix} x_n^2 + \begin{pmatrix} n \\ 3 \end{pmatrix} x_n^3 + ... + \begin{pmatrix} n \\ n \end{pmatrix} x_n^n \geq nx_n.$$

Then $p \geq nx_n$ and $x_n \leq \frac{p}{n}$. Using (b.1) and (b.2), we get that:

$$0 \leq x_n \leq \frac{p}{n}$$

and since $\{0\} \to 0$ and $\left\{\frac{p}{n}\right\} \to 0$, then $x_n \to 0$ and $\sqrt[n]{p} \to 1$.
(c) $p < 1$. Assigned for homework. ■

Definition 15 (Cauchy sequences) *Let* (S, d) *be a metric space. A sequence* $\{p_n\}_{n\in\mathbb{N}}$ *satisfies the Cauchy criterion (it is a Cauchy sequence) iff* $\forall \varepsilon > 0$ *there exists some* N *such that* $\forall m, n > N$,

$$d(x_m, x_n) < \varepsilon.$$

Proposition 16 *If a sequence converges, then it satisfies the Cauchy criterion.*

Proof. Suppose that $\{p_n\}_{n\in\mathbb{N}}$ converges. Consider $\varepsilon > 0$, and note that there exists an N such that $\forall n > N$, $d(p_n, L) < \frac{\varepsilon}{2}$ because $p_n \to L$. Then, $\forall m, n > N$, by the triangular inequality,

$$\begin{aligned} d(p_m, p_n) &\leq d(p_m, L) + d(L, p_n) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

■

Theorem 17 (Nested Interval Theorem) *Suppose that there is a sequence of intervals $I_n = [a_n, b_n]$, $n = 1, 2, ...,$ with $I_{n+1} \subseteq I_n$ and $\lim_{n\to\infty}(a_n - b_n) = 0$. Then there is one and only one number x_0 which is in every I_n.*

Remark 18 *The Nested Interval Theorem is only true if the intervals are closed.*

Definition 19 (Subsequences) *Suppose that $k_1, k_2, k_3, ..., k_n, ...$ is an increasing sequence of positive integers. Then $\{x_{k_n}\}_{n\in\mathbb{N}}$ is a subsequence of $\{x_n\}_{n\in\mathbb{N}}$.*

Example 20 *Suppose $\{x_1, x_2, x_3, x_4, ...\}$ is a sequence. Then:*
(a) $\{x_1, x_3, x_5, ...\}$ is a valid subsequence.
(b) $\{x_{10}, x_{25}, x_{31}, ...\}$ is a valid subsequence.
(c) $\{x_1, x_3, x_5\}$ is not a valid subsequence (the subsequence should go to infinity).
(d) $\{x_1, x_7, x_3, ...\}$ is not a valid subsequence (the elements should be in an increasing order).

Remark 21 *Any subsequence is in itself a subsequence.*

Theorem 22 (Bolzano-Weierstrass) *In $(\mathbb{R}, |\cdot|)$, every bounded sequence has a convergent subsequence.*

Proof. Since the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded, there exists some interval $[a_1, b_1]$ such that $x_n \in [a_1, b_1]$ $\forall n \in \mathbb{N}$. Let $\frac{(a_1+b_1)}{2}$ be the average of a_1 and b_1. Now consider the left and right hand intervals defined by:

$$\left[a_1, \frac{a_1 + b_1}{2}\right], \left[\frac{a_1 + b_1}{2}, b_1\right].$$

In either one of them, there are infinitely many terms of $\{x_n\}_{n\in\mathbb{N}}$. Without loss of generality, set $\left[a_1, \frac{a_1+b_1}{2}\right] = [a_2, b_2]$. Let $\frac{(a_2+b_2)}{2}$ be the average of a_2 and b_2. It is possible to define again some new intervals,

$$\left[a_2, \frac{a_2 + b_2}{2}\right], \left[\frac{a_2 + b_2}{2}, b_2\right],$$

and to repeat this procedure until the interval $[a_{n+1}, b_{n+1}]$ has been constructed. Define $I_n = [a_n, b_n]$, and note that:
(a) $I_{n+1} \subseteq I_n$
(b) $b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n) = \frac{1}{2}\left(\frac{1}{2}(b_{n-1} - a_{n-1})\right) = ... = \left(\frac{1}{2}\right)^n (b_1 - a_1) \to 0.$

Then it is possible to apply Theorem 17 (the Nested Interval Theorem) and state that there exists an x_0 that is contained in all the intervals.

The last step is to construct a sequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ that converges to x_0. Choose $x_{k_1} \in [a_1, b_1]$, $x_{k_2} \in [a_2, b_2]$ such that $k_2 > k_1$, $x_{k_3} \in [a_3, b_3]$ such that $k_3 > k_2$, and up to $x_{k_n} \in [a_n, b_n]$. (This is not a problem since there are infinitely many points in all the intervals.) By construction, $a_n \leq x_{k_n} \leq b_n$, and $a_n \to x_0$, $b_n \to x_0$. Therefoe, $x_{k_n} \to x_0$. ■

Example 23 *Let $\{x_n\}_{n\in\mathbb{N}} = (-1)^n$. This sequence does not converge, but the subsequences $\{x_1, x_3, x_5, ...\}$ and $\{x_2, x_4, x_6, ...\}$ do converge. Now consider $\{x_n\}_{n\in\mathbb{N}} = n$. This sequence does not converge; however, since it is not bounded, it has no convergent subsequences.*

Remark 24 *Theorem 22 (Bolzano-Weierstrass) does not tell that if a sequence is not bounded then it cannot converge.*

Example 25 *Let* $\{x_n\}_{n\in\mathbb{N}} = \begin{cases} n & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$. *This sequence is not bounded, but it has a convergent subsequence.*

Theorem 26 *In* $(\mathbb{R}, |\cdot|)$, *any Cauchy sequence converges.*

Proof. The proof requires 3 steps.

(a) Since $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy, there exists some $N \in \mathbb{N}$ such that $\forall m, n > N$, $|x_m - x_n| < 1$. In particular, $\forall n > N$, $|x_n - x_{N+1}| < 1$ (where we fix $m = N + 1$). Then:

$$-1 < x_n - x_{N+1} < 1 \Leftrightarrow |x_n| < |x_{N+1}| \;\; \forall n > N$$

and

$$|x_n| \leq \max\{|x_1|, |x_2|, ..., |x_N|, 1 + |x_{N+1}|\},$$

so the sequence is bounded.

(b) By Theorem 22 (Bolzano-Weierstrass), there exists some subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ such that $x_{k_n} \to L$.

(c) Consider $\varepsilon > 0$. Then, by the triangular inequality,

$$|x_n - L| \leq |x_n - x_{k_m}| + |x_{k_m} - L|.$$

Note that because of (b), there exists some $N_1 \in \mathbb{N}$ such that $\forall m > N_1$,

$$|x_{k_m} - L| < \frac{\varepsilon}{2}.$$

Also, there exists some $N_2 \in \mathbb{N}$ such that $\forall m, n > N_2$,

$$|x_n - x_m| < \frac{\varepsilon}{2}$$

since $k_m \geq m$. Then $|x_n - x_{k_m}| < \frac{\varepsilon}{2}$ because $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Finally, $\forall n > N_2$, consider x_{k_m} such that $m > \max\{N_1, N_2\}$. Then:

$$\begin{aligned} |x_n - L| &\leq |x_n - x_{k_m}| + |x_{k_m} - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

■

Definition 27 *A metric space* (S, d) *is complete iff every Cauchy sequence converges.*

Remark 28 $(\mathbb{R}, |\cdot|)$ *is complete.* $(\mathbb{Q}, |\cdot|)$ *is not complete.*

Example 29 *Consider the metric space* $(\mathbb{Q}, |\cdot|)$. *Let* $x_1 = 3$, $x_2 = 3.1$, $x_3 = 3.14$, $x_4 = 3.141$, $x_5 = 3.1415$, ... *Then* $\{x_n\}_{n\in\mathbb{N}}$ *is a Cauchy sequence that converges to* π. *Note that* $\forall m, n > N$, $|x_n - x_m| < 10^{-N}$, *and by choosing a sufficiently large* N, $10^{-N} < \varepsilon$. *But although the sequence goes to* π, $\{x_n\}_{n\in\mathbb{N}}$ *does not converge to any limit in* $\mathbb{Q}$ *(since* $\pi \notin \mathbb{Q}$*).*

Generalization to $\mathbb{R}^K$

Consider $d_3(x,y) = \max_{i=1,...,K} \left| x^i - y^i \right|$.

Theorem 30 *Set* $(S,d) = (\mathbb{R}^K, d_3)$*. A sequence* $\{x_n\}_{n\in\mathbb{N}}$ *converges to* x *iff* $\forall i \in \{1,...,K\}$*,* $x_n^i \to x^i$*.*

Proof. "⇒"

Suppose that the vector $x_n \to x$ and consider $\varepsilon > 0$. Note that:

$$\left| x_n^i - x^i \right| \le \max_{j=1,...,K} \left| x_n^j - x^j \right| = d(x_n, x).$$

Since the vector $x_n \to x$, there is some $N \in \mathbb{N}$ such that $\forall n > N$, $d(x_n, x) < \varepsilon$. Then, $\forall n > N$,

$$\left| x_n^i - x^i \right| \le d(x_n, x) < \varepsilon.$$

"⇐"

Suppose that $x_n^i \to x^i$ $\forall i \in \{1,...,K\}$, and consider $\varepsilon > 0$. By definition, the following is true:

$$d(x_n, x) = \max_{i=1,...,K} \left| x_n^i - x^i \right|.$$

Note that for a maximum to be less than ε, then every one of its components should be less than ε. By definition, there exists some $N_1 \in \mathbb{N}$ such that $\forall n > N_1, \left| x_n^1 - x^1 \right| < \varepsilon$. Also, there is some $N_2 \in \mathbb{N}$ such that $\forall n > N_2$, $\left| x_n^2 - x^2 \right| < \varepsilon$. This reasoning can be extended up to N_K. Then, consider $N^* = \max\{N_1, N_2, ..., N_K\}$, such that $\forall n > N^*$,

$$\max_{i=1,...,K} \left| x_n^i - x^i \right| < \varepsilon.$$

■

Theorem 31 *The metric space* $(\mathbb{R}^K, d_3)$ *is complete.*

Proof. It is necessary to prove that a sequence converges iff it satisfies the Cauchy property.

"⇒"

This is true in every metric space.

"⇐"

Suppose that a sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfies the Cauchy property. Then, there exists some $N \in \mathbb{N}$ such that $\forall m, n > N$, $d_3(x,y) = \max_{i=1,...,K} \left| x_n^i - x^i \right| < \varepsilon$, or equivalently,

$$\left| x_n^i - x^i \right| < \varepsilon \ \forall i \in \{1,...,K\}.$$

Then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence $\forall i \in \{1,...,K\}$. Since $(\mathbb{R}, |\cdot|)$ is complete, $x_n^i \to x^i$ $\forall i \in \{1,...,K\}$. Using Theorem 30, the whole sequence converges and $x_n \to x$. ■