Sets

Definition 1 (Union and intersection of sets) *Let $\mathcal{F}$ be a family of sets, a typical member being denoted by A. Then the operations of union and intersection are defined as follows:*

$$\bigcup\nolimits_{A\in\mathcal{F}} A = \{p | p \text{ is in at least one } A \in \mathcal{F}\}$$

$$\bigcap\nolimits_{A\in\mathcal{F}} A = \{p | p \text{ is in every } A \in \mathcal{F}\}$$

Alternatively, the union can be defined by

$$\bigcup\nolimits_{A\in\mathcal{F}} A = \{p | \text{there exists some } A \in \mathcal{F} \text{ with } p \in A\}$$

and the intersection by

$$\bigcap\nolimits_{A\in\mathcal{F}} A = \{p | \forall A \in \mathcal{F},\ p \in A\}.$$

Example 2 *Let $\mathcal{F} = \{A_1, A_2, A_3, \ldots\}$ with $A_i = \left[0, \frac{1}{i}\right] \subseteq \mathbb{R}$. Then $\bigcup_{A\in\mathcal{F}} A = [0,1]$ and $\bigcap_{A\in\mathcal{F}} A = \{0\}$.*

Remark 3 *Sometimes, if the sets in the family are indexed by the naturals, then the following notation can be applied: $\bigcup_{A\in\mathcal{F}} A \equiv \bigcup_{i\in\mathbb{N}} A_i$, and $\bigcap_{A\in\mathcal{F}} A \equiv \bigcap_{i\in\mathbb{N}} A_i$.*

Definition 4 (Complement of a set) *Let A and B be sets in S. Then:*

$$A - B = \{p | p \in A, p \notin B\}.$$

The complement of set A, denoted A^C, is defined as $A^C = S - A$.

Theorem 5 (De Morgan) *Let (S, d) be a metric space and $\mathcal{F}$ be a family of sets in S. Then:*

(a) $\left(\bigcup_{A\in\mathcal{F}} A\right)^C = \bigcap_{A\in\mathcal{F}} A^C$.

(b) $\left(\bigcap_{A\in\mathcal{F}} A\right)^C = \bigcup_{A\in\mathcal{F}} A^C$.

Proof. (a)

"$\subseteq$"

Consider $p \in \left(\bigcup_{A\in\mathcal{F}} A\right)^C$. We need to show that $p \in \bigcap_{A\in\mathcal{F}} A^C$. Notice that if $p \in \left(\bigcup_{A\in\mathcal{F}} A\right)^C$ then $p \notin \bigcup_{A\in\mathcal{F}} A$. This means that $\forall A \in \mathcal{F}$, $p \notin A$. Then $\forall A \in \mathcal{F}$, $p \in A^C$ and $p \in \bigcap_{A\in\mathcal{F}} A^C$.

"$\supseteq$"

Consider $p \in \bigcap_{A\in\mathcal{F}} A^C$. We need to show that $p \in \left(\bigcup_{A\in\mathcal{F}} A\right)^C$. Notice that if $p \in \bigcap_{A\in\mathcal{F}} A^C$, then $\forall A \in \mathcal{F}$, $p \notin A$. Then there is no $A \in \mathcal{F}$ such that $p \in A$. Then $p \notin \bigcup_{A\in\mathcal{F}} A$. It follows that $p \in \left(\bigcup_{A\in\mathcal{F}} A\right)^C$.

(b) Assigned for homework. ■

Open and closed sets

Definition 6 (Open and closed balls) *Let (S, d) be a metric space, $p_0 \in S$, $r > 0$. Then the open ball $B(p_0, r)$ and the closed ball $\bar{B}(p_0, r)$ with center p_0 and radius r are defined by:*

$$B(p_0, r) = \{p | d(p, p_0) < r\}$$

$$\bar{B}(p_0, r) = \{p | d(p, p_0) \leq r\}$$

Remark 7 *It follows that $B(p_0, r) \neq \emptyset$, $\bar{B}(p_0, r) \neq \emptyset$, since p_0 is always an element of B and $\bar{B}$.*

Example 8 *Suppose $(S, d) = (\mathbb{R}, |\cdot|)$. Then $B(p_0, r) = (p_0 - r, p_0 + r)$ and $\bar{B}(p_0, r) = [p_0 - r, p_0 + r]$.*

Definition 9 (Isolated point) *Let (S, d) be a metric space. A point $p_0 \in S$ is an isolated point of A iff there exists $r > 0$ such that $B(p_0, r) \cap A = \{p_0\}$.*

Example 10 *Suppose $(S, d) = (\mathbb{R}, |\cdot|)$. Consider $A = [1, 2] \cup \{3\}$. (a) Prove that $\{3\}$ is an isolated point of A. (b) Prove that the elements in $[1, 2]$ are not isolated points.*

Proof. For (a), consider $B(3, \frac{1}{2}) \cap A = \{3\}$. Then $\{3\}$ is an isolated point. For (b), Consider $p \in [1,2]$. Then for any $r > 0$, $B(p,r) \cap A \neq \{p\}$. ■

Definition 11 (Limit point) *Let (S,d) be a metric space. A point $p_0 \in S$ is a limit point of $A \subseteq S$ if every open ball $B(p_0, r)$ contains a point $p \in A$, with $p \neq p_0$.*

Example 12 *Suppose $(S,d) = (\mathbb{R}, |\cdot|)$. Consider $A = [1,2] \cup \{3\}$. It is easy to see that $[1,2]$ are limit points of A, while $\{3\}$ is not a limit point of A. Now consider $A = (1,2)$. In this case 1 and 2 are not in the set but 1 and 2 are limit points of A.*

Definition 13 (Closed and open sets) *Let (S,d) be a metric space. A set A is closed iff it contains all of its limit points. A set A is open iff $\forall x \in A$, there exists some $r > 0$ such that $B(x,r) \subseteq A$.*

Example 14 *Consider $(S,d) = (\mathbb{R}, |\cdot|)$. Let $A = [1,2] \cup \{3\}$. Notice that $\{3\}$ is not a limit point (instead, it is an isolated point). The limit points of A are $[1,2] \subseteq A$, so A is closed. However, A is not open, since $\forall r > 0$, $B(1,r) \nsubseteq A$, $B(2,r) \nsubseteq A$, and $B(3,r) \nsubseteq A$.*

Example 15 *Consider $(S,d) = (\mathbb{R}, |\cdot|)$. Let $A = (1,2)$. The limit points of A are $[1,2] \nsubseteq A$. Then A is not closed. However, $\forall x \in A$, there exists $r > 0$ such that $B(x,r) \subseteq A$. Then A is open.*

Example 16 *Consider $(S,d) = (\mathbb{R}, |\cdot|)$. Let $A = [1,2)$. The limit points of A are $[1,2]$ but $2 \notin A$. Then A is not closed. Moreover, $\forall r > 0$, $B(1,r) \nsubseteq A$. Then A is not open.*

Remark 17 *If (S,d) is a metric space, then $A = \emptyset$ is open and closed. $A = S$ is also open and closed.*

Example 18 *Consider $(S,d) = (\mathbb{R}, |\cdot|)$. Let $A = \{3\}$. Notice that $\{3\}$ is an isolated point. Since it does not have any limit points, it contains all of them, so A is closed. However, $B(3,r) \nsubseteq A$ $\forall r > 0$, so A is not open.*

Theorem 19 *Let (S,d) be a metric space. A point p_0 is a limit point of a set A iff there exists a sequence $\{p_n\}_{n\in\mathbb{N}} \subseteq A$ with $p_n \neq p_0$ and $p_n \to p_0$.*

Proof. "$\Rightarrow$"

Assume that p_0 is a limit point of A. Consider $r = 1$. Then there exists $p \in B(p_0, 1) \cap A$ such that $p \neq p_0$. Call this point p_1. Consider $r = \frac{1}{2}$. Then there exists $p \in B(p_0, \frac{1}{2}) \cap A$ such that $p \neq p_0$. Call this point p_2. This process can be repeated up to $r = \frac{1}{n}$. Then there exists $p \in B(p_0, \frac{1}{n}) \cap A$ such that $p \neq p_0$. Call this point p_n.Then $\{p_n\}_{n\in\mathbb{N}} \subseteq A$, with $p_n \neq p_0$ and $p_n \to p_0$ since $d(p_n, p_0) < \frac{1}{n}$.

"$\Leftarrow$"

Assume that there exists a sequence $\{p_n\}_{n\in\mathbb{N}} \subseteq A$, with $p_n \neq p_0$ and $p_n \to p_0$. Consider $r > 0$. Since $p_n \to p_0$, there exists some n such that $d(p_n, p_0) < r$. Then $p_n \in B(p_0, r)$. By properties of the sequence, $\{p_n\}_{n\in\mathbb{N}} \subseteq A$ and $p_n \neq p_0$. Then $p_n \in B(p_0, r) \cap A$, $p_n \neq p_0$ and p_0 is a limit point of A. ■

Corollary 20 *Let (S,d) be a metric space. A set X is closed iff for every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that $x_n \to x$, $x \in X$.*

Proof. Assigned for homework. ■

Theorem 21 *The following propositions are true:*

(a) An open ball is an open set.

(b) A closed ball is a closed set.

Proof. (a)

Consider the open ball $B(p_0, r) = \{p | d(p, p_0) < r\}$. Let $p \in B(p_0, r)$ and set $\bar{r} = r - d(p, p_0) > 0$. Take any $x \in B(p, \bar{r})$,then $d(p, x) < \bar{r}$. It follows that

$$d(x, p_0) \leq d(x, p) + d(p, p_0)$$

$$< r - d(p, p_0) + d(p, p_0)$$

$$= r.$$

So $d(x,p_0)<r$ and $x\in B(p_0,r)$.

(b)

Consider the closed ball $\bar{B}(p_0,r)=\{p|d(p,p_0)\leq r\}$. If p is a limit point of $\bar{B}(p_0,r)$, then there exists a sequence $\{p_n\}_{n\in\mathbb{N}}\subseteq\bar{B}(p_0,r)$ with $p_n\neq p$ and $p_n\to p$. By using the triangular inequality,

$$d(p,p_0)\leq d(p,p_n)+d(p_n,p_0)$$

where the term $d(p_n,p_0)\leq r$ since the sequence $\{p_n\}_{n\in\mathbb{N}}\subseteq\bar{B}(p_0,r)$. Consider $\varepsilon>0$. Then there exists some n_ε^* such that $d(p_{n_\varepsilon^*},p)<\varepsilon$ since $p_n\to p$. Then:

$$\begin{aligned}d(p,p_0)&\leq d(p,p_n)+d(p_n,p_0)\\&<r+\varepsilon.\end{aligned}$$

Since this is true $\forall\varepsilon>0$, then it has to be the case that

$$d(p,p_n)\leq r$$

and then $p\in\bar{B}(p_0,r)$. ■

Theorem 22 *Consider a family of open sets Θ. Then:*

(a) $\bigcup_{A\in\Theta}A$ is open.

(b) $\bigcap_{A\in\Theta}A$ is open if Θ is finite.

Proof. (a)

Consider $x\in\bigcup_{A\in\Theta}A$. Then there exists $A^*\in\Theta$ such that $x\in A^*$, where A^* is open. It follows that there exists $\varepsilon>0$ such that $B(x,\varepsilon)\subseteq A^*$. Then, by the definition of union, $B(x,\varepsilon)\subseteq\bigcup_{A\in\Theta}A$.

(b)

Consider $x\in\bigcap_{A\in\Theta}A$. This is equivalent to saying that $x\in A\ \forall A\in\Theta$. Since Θ is finite, consider $\Theta=\{A_1,A_2,...,A_N\}$. Since $x\in A_1$, which is open, then there exists $r_1>0$ such that $B(x,r)\subseteq A_1$. This can be repeated for every $A_n\in\Theta$. Now consider $r=\min\{r_1,r_2,...,r_N\}$. Then $B(x,r)\subseteq A_1\cap A_2\cap...\cap A_N$, and it follows that $B(x,r)\in\bigcap_{A\in\Theta}A$. ■

Remark 23 *The finiteness of Θ is important for the result in (b).*

Example 24 *Consider $A_n=\left(-\frac{1}{n},\frac{1}{n}\right)$, which is open. Then $\bigcap_{n\in\mathbb{N}}A_n=\{0\}$, which is a singleton, and is not open.*

Theorem 25 *Let (S,d) be a metric space. A set A is open iff A^C is closed.*

Proof. "⇒"

Suppose that A is open. Consider $\{x_n\}_{n\in\mathbb{N}}\subseteq A^C$, such that $x_n\to x$. We need to prove that $x\in A^C$. Suppose not, then $x\in A$. Since A is open, then there exists some r such that $B(x,r)\subseteq A$. Since $x_n\to x$, there exists some n^* such that $d(x_n,x)<r\Rightarrow x_n^*\in B(x,r)\subseteq A\Rightarrow x_n^*\in A$, which is a contradiction since $\{x_n\}_{n\in\mathbb{N}}\subseteq A^C$. Then $x\in A^C$.

"⇐"

Suppose that A^C is closed. Let $x\in A$, and consequently $x\notin A^C$. Since A^C is closed, it contains all of its limit points. Then x is not a limit point of A^C. This means that there exists some $r>0$ such that $B(x,r)\cap A^C=\emptyset$. Then $B(x,r)\subseteq A$. ■

Theorem 26 *Consider $\mathcal{F}$, a family of closed sets. Then:*

(a) The arbitrary intersection of sets $\bigcap_{F\in\mathcal{F}}\mathcal{F}$ is closed.

(b) $\bigcup_{F\in\mathcal{F}}F$ is closed if $\mathcal{F}$ is finite.

Proof. (a)

Note that

$$\bigcap_{F\in\mathcal{F}}F=\left(\bigcup_{F\in\mathcal{F}}F^C\right)^C$$

if the F's are closed sets, then the F^C's are open. But the union of open sets is open, and the complement of an open set is closed.

(b)

Note that

$$\bigcup_{F\in\mathcal{F}} F = \left(\bigcap_{F\in\mathcal{F}} F^C\right)^C$$

and that F^C is open. The intersection of a finite group of open sets is still open ($\bigcap_{F\in\mathcal{F}} F^C$), but its complement is closed. ■

Definition 27 *(Preimage) Let $f : S \to T$. For $A \subseteq T$ we define the* preimage *as:*

$$f^{-1}[A] = \{x \in S | f(x) \in A\}.$$

Theorem 28 *Consider $f : (S, d_S) \to (T, d_T)$. A function is continuous iff $\forall A \subseteq T$ with A open, we have that $f^{-1}[A]$ is open.*

Proof. "$\Rightarrow$"

Suppose that f is continuous. Consider $x \in f^{-1}[A]$. By definition, $f(x) \in A$. Since A is open, then there exists some $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subseteq A$. Then

$$\{z | d_T(z, f(x)) < \varepsilon\} \subseteq A. \tag{1}$$

Since f is continuous, there exists some $\delta > 0$ such that

$$d_S(y, x) < \delta \Rightarrow d_T(f(y), f(x)) < \varepsilon. \tag{2}$$

If $d_S(y, x) < \delta$ then by (1) and (2),

$$d_S(y, x) < \delta \Rightarrow f(y) \in A \Rightarrow y \in f^{-1}[A].$$

Then $B(x, \delta) \subseteq f^{-1}[A]$.

"$\Leftarrow$"

Suppose that $f^{-1}[A]$ is open $\forall A \subseteq T$. Consider $B(f(x), \varepsilon)$. If $f(y) \in B(f(x), \varepsilon)$ then we are done. Note first that $B(f(x), \varepsilon)$ is an open set. Then:

$$f^{-1}[B(f(x), \varepsilon)] \text{ is open} \Rightarrow \{z | f(z) \in B(f(x), \varepsilon)\} \text{ is open} \Rightarrow \{z | d_T(f(z), f(x)) < \delta\} \text{ is open.}$$

Note that

$$x \in \{z | d_T(f(z), f(x)) < \varepsilon\}$$

is open. Then, there exists some $\delta > 0$ such that

$$B(x, \delta) \subseteq \{z | d_T(f(z), f(x)) < \varepsilon\}.$$

Then:

$$y \in B(x, \delta) \Rightarrow y \in \{z | d_T(f(z), f(x)) < \varepsilon\}$$

which means that

$$d_S(y, x) < \delta \Rightarrow d_T(f(y), f(x)) < \varepsilon.$$

■

Definition 29 (Closure of a set) *Let (S, d) be a metric space, and $E \subseteq S$. Define E' as*

$$E' = \{x | x \text{ is a limit point of } E\}.$$

The closure of E, denoted by $\bar{E}$, is defined by

$$\bar{E} = E \cup E'.$$

Example 30 *Consider $E = [0, 1)$. For E, $E' = [0, 1]$ and $\bar{E} = [0, 1]$. Now consider $E = [0, 1) \cup \{3\}$. Then $E' = [0, 1]$ and $\bar{E} = [0, 1] \cup \{3\}$.*

Theorem 31 *The following propositions are true:*
(a) $\bar{E}$ is closed.
(b) $\bar{E} = E$ iff E is closed.
(c) If $F \supseteq E$ and F is closed, then $F \supseteq \bar{E}$.

Proof. (b)

"$\Rightarrow$"

Assume that $\bar{E} = E$. By (a), $\bar{E}$ is closed. Then E is closed.

"$\Leftarrow$"

Assume that E is closed. Then $E' \subseteq E$. Then $E \cup E' = E$. It follows that $\bar{E} = E$.

(c)

Consider $F \supseteq E$, with F closed and $x \in E \cup E' = \bar{E}$. Then there are 2 possibilities:

(c.1) $x \in E$, and we are done since $E \subseteq F$ and then $x \in F$.

(c.2) $x \in E'$, then x is a limit point of E. Theorem 19 stated that there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ such that $x_n \neq x$ and $x_n \to x$. If $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ then

$$\{x_n\}_{n\in\mathbb{N}} \subseteq F,\ F \text{ closed}$$

and it follows that $x \in F$.

(a)

Consider $x \in \bar{E}^C = (E \cup E')^C \Rightarrow x \in E^C \cap E'^C$ (this is because of De Morgan). Then $x \notin E$, $x \notin E'$. Since $x \notin E'$, x is not a limit point. Then there exists some $r > 0$ such that $B(x, r) \cap E$ is either x (an isolated point) or $\emptyset$. Since $x \notin E$ by definition, then $B(x, r) \cap E = \emptyset$.

Now consider $B(x, \frac{r}{2}) \cap \bar{E} = \emptyset$, and note that if $B(x, \frac{r}{2}) \cap \bar{E} = \emptyset$ then $B(x, \frac{r}{2}) \subseteq \bar{E}^c$; it follows that $\bar{E}^c$ is open and, consequently, $\bar{E}$ is closed.

To prove that $B(x, r) \cap E = \emptyset$ implies that $B(x, \frac{r}{2}) \cap \bar{E} = \emptyset$, suppose that $B(x, \frac{r}{2}) \cap \bar{E} \neq \emptyset$. Then there is some $y \in B(x, \frac{r}{2}) \cap \bar{E}$, where $\bar{E} = E \cup E'$. Notice that $y \notin E$ since $B(x, r) \cap E = \emptyset$. Then it must be that $y \in E'$, a limit point. It follows that there exists some sequence $\{y_n\}_{n\in\mathbb{N}} \subseteq E$, $y_n \neq y$ and $y_n \to y$. Now, note that:

$$\begin{aligned} d(y_n, x) &\leq d(y_n, y) + d(y, x) \\ &< \frac{r}{2} + \frac{r}{2} \\ &= r \end{aligned}$$

where the middle line follows from (i) the fact that by choosing a large n, y_n and y are converging, and (ii) the fact that $y \in B(x, \frac{r}{2})$ so that $d(y, x) < \frac{r}{2}$. But this means that $y \in B(x, r)$, so $y \notin E$, but our assumption was that $\{y_n\} \subseteq E$, so we reach a contradiction.

Therefore, $B(x, r) \cap E = \emptyset \Rightarrow B(x, \frac{r}{2}) \cap \bar{E} = \emptyset \Rightarrow B(x, \frac{r}{2}) \subseteq \bar{E}^C \Rightarrow \bar{E}^C$ is open. Consequently, $\bar{E}$ is closed. ■

Definition 32 (Interior point) *Let (S, d) be a metric space. Consider $A \subseteq S$. A point $x \in A$ is an interior point of A iff there exists a ball $B(x, r) \subseteq A$.*

Example 33 *Let $A = [1, 2)$. $x = \frac{3}{2}$ is an interior point of A. $x = 1$ is not an interior point of A.*

Remark 34 *To be an interior point, it is required that $x \in A$.*

Definition 35 (Interior of a set) *Let (S, d) be a metric space. Consider $A \subseteq S$. The interior of a set, denoted $\mathring{A}$, is defined as $\mathring{A} = \{x | x$ is an interior point of $A\}$.*

Example 36 *Let $A = [1, 2] \cup \{3\}$. Then $\mathring{A} = (1, 2)$.*

Theorem 37 *The following propositions are true:*
(a) $\mathring{A}$ is open.
(b) $A = \mathring{A}$ iff A is open.
(c) If B is open, $B \subseteq A$, then $B \subseteq \mathring{A}$.

Proof. (a)

Consider $x \in \mathring{A}$. We need to show that there exists some $r > 0$ such that $B(x, r) \subseteq \mathring{A}$. Since $x \in \mathring{A}$, there exists some $\bar{r} > 0$ such that $B(x, \bar{r}) \subseteq A$. We need to show that $B(x, \frac{\bar{r}}{2}) \subseteq \mathring{A}$. Consider $y \in B(x, \frac{\bar{r}}{2})$, and $B(y, \frac{\bar{r}}{2})$. If $z \in B(y, \frac{\bar{r}}{2})$,

$$\begin{aligned} d(z,x) &\leq d(z,y) + d(y,z) \\ &< \frac{\bar{r}}{2} + \frac{\bar{r}}{2} \\ &= \bar{r} \end{aligned}$$

So $z \in B(y, \frac{\bar{r}}{2}) \subseteq A \Rightarrow B(y, \frac{\bar{r}}{2}) \subseteq A \Rightarrow y \in \mathring{A} \Rightarrow B(x, \frac{\bar{r}}{2}) \subseteq \mathring{A}$, and $\mathring{A}$ is open.

(b)

We need to prove the statement both ways.

"$\Rightarrow$"

If $A = \mathring{A}$ and $\mathring{A}$ is open (which was proven in (a)), then A is open.

"$\Leftarrow$"

Suppose that A is open. Then $\forall x \in A$, there exists some $r > 0$ such that $B(x, r) \subseteq A$. Then every $x \in A$ is an interior point, so $A \subseteq \mathring{A}$. Since the other inclusion ($A \supseteq \mathring{A}$) is always true, then $A = \mathring{A}$.

(c)

Consider $x \in B$. Then there exists some $r > 0$ such that $B(x, r) \subseteq B$. But since $B \subseteq A$ by assumption,

$$B(x, r) \subseteq B \subseteq A$$

and then $x \in \mathring{A}$. ■

Remark 38 *Parts (a) and (c) of the previous theorem imply that* $\mathring{A}$ *is the largest open set contained in* A.