Finite, countable and uncountable sets

Definition 1 (Injective function) *Let* $f : S \to T$ *be a function. We say that* f *is injective if* $f(x) = f(y) \Rightarrow x = y$.

Remark 2 f *is injective iff* $x \neq y \Rightarrow f(x) \neq f(y)$.

Definition 3 (Onto function) *Let* $f : S \to T$ *be a function. We say that* f *is onto iff* $\forall y \in T$*, there exists some* $x \in S$ *such that* $f(x) = y$.

Definition 4 (Bijection) *Let* $f : S \to T$ *be a function. We say that* f *is a bijection iff* f *is injective and onto.*

Definition 5 (Cardinality of a set) *Let* S *be a finite set,* $S = \{s_1, s_2, ..., s_n\}$*. Then the cardinality of* S *is* $|S| = n$.

Proposition 6 *Suppose that* S *and* T *are finite. Then* $|S| = |T|$ *iff there exists a bijection* $f : S \to T$.

Proof. "$\Rightarrow$"

Suppose that $|S| = |T| = n$, S, T are finite sets. Then S should be of the type $S = \{s_1, s_2, ..., s_n\}$ and $T = \{t_1, t_2, ..., t_n\}$. Consider the function $f : S \to T$, $s_i \to t_i$. This function is a bijection.

"$\Leftarrow$"

Suppose that $f : S \to T$ is a bijection. We need to show that $|S| = |T|$. Suppose not. Then we have 2 alternatives:

(a) $|S| > |T|$, where $S = \{s_1, s_2, ..., s_n, s_{n+1}, ..., s_m\}$ and $T = \{t_1, t_2, ..., t_n\}$. Then $f(s_1) \neq f(s_2) \neq ... \neq f(s_n)$, since f is injective. But $f(s_{n+1}) = t_j = f(s_i)$ for some $i \in \{1, ..., n\}$, which is a contradiction with the injectivity of f.

(b) $|S| < |T|$, where $S = \{s_1, s_2, ..., s_m\}$ and $T = \{t_1, t_2, ..., t_m, t_{m+1}, ..., t_n\}$. Then $\forall i \in \{1, ..., m\}$, $t_i = f(s_j)$ ($\forall j \in \{1, ..., m\}$), else the function is not onto. But then $t_{m+1} \neq f(s_j)$ for any $j \in \{1, ..., m\}$. Then f is not onto, which is a contradiction. ■

Example 7 *Consider a set* $A = \{a_1, a_2, ..., a_n\}$*. Then the power set of* A *is defined as*

$$\mathcal{P}(A) = \{B | B \textit{ is a subset of } A\}.$$

Then, $|\mathcal{P}(A)| = 2^n$.

Proof. Consider the set $X = \{0, 1\}^n$, $X = \{(x_1, ..., x_n) | x_i \in \{0, 1\}\}$. (A typical set has the form $X = \{0, 1, 1, 0, ..., 1\}$). Then $|X| = 2^n$. Now we find a bijection between $\mathcal{P}(A)$ and X, $f : \mathcal{P}(A) \to X$, $B \to f(B) = (x_1, x_2, ..., x_n)$ with $x_i = \begin{cases} 1 & \text{if } a_i \in B \\ 0 & \text{if } a_i \notin B \end{cases}$.

(For example, consider $A = \{1, 2, 3, 4\}$. If $B = \{1, 2, 3\}$ then $f(B) = \{1, 1, 1, 0\}$; if $B = \emptyset$, then $f(B) = \{0, 0, 0, 0\}$.)

(a) f is injective.

Consider $B_1, B_2 \in \mathcal{P}(A)$. Suppose that $f(B_1) = f(B_2)$. We need to show that $B_1 = B_2$. Note that $a_i \in B_1 \Leftrightarrow [f(B_1)]_i = 1 \Leftrightarrow [f(B_2)]_i = 1$ (since $f(B_1) = f(B_2)$) $\Leftrightarrow a_i \in B_2$. Then $B_1 = B_2$.

(b) f is onto.

We need to show that $\forall x \in X$, there exists some $B \in \mathcal{P}(A)$ such that $f(B) = x$. Consider $x = (x_1, x_2, ..., x_n) \in X$, and $B = \{a_i | x_i = 1\}$. Then $f(B) = x$.

Since f is injective and onto, then f is bijective, and therefore $|\mathcal{P}(A)| = |X| = 2^n$. ■

Definition 8 *For infinite sets, we say that* $|A| = |B|$ *iff there exists a bijection* $f : A \to B$.

Definition 9 (Denumerable and countable sets) *If* $|A| = |\mathbb{N}|$ *then we say that* A *is denumerable. If* $|A| = |\mathbb{N}|$ *or* A *is finite, then* A *is countable.*

Proposition 10 $|\mathbb{N}| = |\mathbb{Z}|$.

Proof. Define a function $f : \mathbb{N} \to \mathbb{Z}$, $n \to \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{-(n+1)}{2} & \text{if } n \text{ is odd} \end{cases}$.

(a) f is injective.

Suppose that $f(n_1) = f(n_2)$.

(a.1) Let $f(n_1) = f(n_2) \geq 0$. Then $\frac{n_1}{2} = \frac{n_2}{2} \Rightarrow n_1 = n_2$.

(a.2) Let $f(n_1) = f(n_2) \leq 0$. Then $\frac{-(n+1)}{2} = \frac{-(n+1)}{2} \Rightarrow n_1 = n_2$.

(b) f is onto.

Consider any $z \in \mathbb{Z}$.

(b.1) Let $z \geq 0$, and consider $n = 2z$, which is an even natural number. Then $f(n) = \frac{n}{2} = \frac{(2z)}{2} = z$.

(b.2) Let $z < 0$, and consider $n = -(2z+1)$, an odd natural number. Then $f(n) = \frac{-(n+1)}{2} = \frac{-(-(2z+1)+1)}{2} = z$.

Then $|\mathbb{N}| = |\mathbb{Z}|$. ■

Theorem 11 *Any subset of a countable set is also countable.*

Proof. Assigned for homework. ■

Theorem 12 *Let $S_1, S_2, \ldots$ be a countable family of countable sets. Then $\bigcup_{i\geq 1} S_i$ is countable.*

Proof. S_m is countable for $m \geq 1$. Then it can be put into a bijection with a (possibly proper) subset of $\mathbb{N}$

$$S_m = \{(m,1), (m,2), (m,3), (m,4) \ldots\}.$$

Then:

$$\begin{array}{rcccc}
S_1 = & (1,1), & (1,2), & (1,3), & \ldots \\
S_2 = & (2,1), & (2,2), & (2,3), & \ldots \\
S_3 = & (3,1), & (3,2), & (3,3), & \ldots \\
\vdots & & & & \\
S_m = & (m,1) & (m,2) & (m,3) & \ldots \\
\vdots & & & &
\end{array}$$

And $(1,1)$ can be assigned to 1, $(2,1)$ to 2, $(1,2)$ to 3, $(3,1)$ to 4, $(2,2)$ to 5, and so on, so all elements can be covered without repeating.

Now consider $f : \{(m,n) \,|m,n \in \mathbb{N}^*\} \to \mathbb{N}^*$, $(m,n) \to \frac{(m+n-1)(m+n-2)}{2} + n$. Then f is an injection, but not necessarily onto (since some S_i's can be finite and some elements of some S_i's can be dropped because of repetition). However, for $S = \{(m,n) \,|m,n \in \mathbb{N}^*\}$ then $f : \{(m,n) \,|m,n \in \mathbb{N}^*\} \to f[S]$ is onto. Then, there is a bijection between $\bigcup_{i\geq 1} S_i$ and $f[S]$. Since $f[S] \subseteq \mathbb{N}$, then $f[S]$ is countable, because of Theorem 11. Then $\bigcup_{i\geq 1} S_i$ is countable. ■

Corollary 13 $|\mathbb{Q}| = |\mathbb{N}|$.

Proof. Note that $\mathbb{Q}$ can be expressed as

$$\mathbb{Q} = \bigcup\nolimits_{Q \in \mathbb{Z}\backslash\{0\}} \left\{ \frac{p}{q} | p \in \mathbb{Z} \right\}.$$

Call the set $\left\{ \frac{p}{q} | p \in \mathbb{Z} \right\} = B_Q$, the set of rational numbers with denominator q. Note that $|B_Q| = |\mathbb{Z}|$. ($f : B_Q \to \mathbb{Z}$, $\frac{p}{q} \to p$ is a bijection). Then B_Q is countable.

But note that the union is made in $Q \in \mathbb{Z}\backslash\{0\}$, which is countable. So we have that $\mathbb{Q}$ is a countable union of countable sets, so $\mathbb{Q}$ is countable. If a set is countable, it can be the case that either $|\mathbb{Q}| = |\mathbb{N}|$ or that $\mathbb{Q}$ is finite. Since $\mathbb{Q}$ is not finite, then $|\mathbb{Q}| = |\mathbb{N}|$. ■

Theorem 14 $\mathbb{R}$ *is not countable.*

Proof. It is enough to show that $[0,1]$ is not countable. Suppose that $[0,1]$ is countable. Then its elements can be put in a sequence:

$$\begin{aligned} x_1 &= 0.x_{11}x_{12}x_{13}\dots \\ x_2 &= 0.x_{21}x_{22}x_{23}\dots \\ x_3 &= 0.x_{31}x_{32}x_{33}\dots \\ x_4 &= 0.x_{41}x_{42}x_{43}\dots \\ &\vdots \end{aligned}$$

where x_{ij} are the digits of the number. Now we construct a number $\bar{x}$ such that it is not in the list. Consider $\bar{x} = 0.\bar{x}_1\bar{x}_2\bar{x}_3...$ defined as:

$$\bar{x}_i = \begin{cases} 5 & \text{if } x_{ii} = 4 \\ 4 & \text{otherwise} \end{cases}.$$

Then $\bar{x} \neq x_n \ \forall n \in \mathbb{N}$ since $\bar{x}_n \neq x_{nn}$. Therefore, no bijection between $\mathbb{N}$ and $[0,1]$ can be established, and $\mathbb{R}$ is not countable. ■

Theorem 15 *Any family of disjoint nonempty open intervals in* $\mathbb{R}$ *is countable.*

Proof. Let $\{(a_\lambda, b_\lambda)\}_{\lambda\in\Lambda}$ be a family of open sets. Then there exists x_λ in (a_λ, b_λ) such that $x_\lambda \in \mathbb{Q}$. Consider $f : \Lambda \to \mathbb{Q}$, $\lambda \to x_\lambda$. We now claim that f is injective. To see that this is so, suppose that $\lambda_1 \neq \lambda_2$, and without loss of generality, suppose that $b_{\lambda_1} < a_{\lambda_2}$ (and the intervals are disjoint). Then $x_{\lambda_1} < x_{\lambda_2}$ implies that $f(\lambda_1) \neq f(\lambda_2)$. Then $f : \Lambda \to f[\Lambda]$ is a bijection, and it follows that $|\Lambda| = |f[\Lambda]|$.

Since $f[\Lambda]$ is a subset of $\mathbb{Q}$, which is countable, we know that $f[\Lambda]$ is countable. Then Λ is countable. ■

Theorem 16 *Suppose* $A_1, ..., A_n$ *is a finite family of countable sets. Then the cartesian product* $A_1 \times A_2 \times ... \times A_N$ *is countable.*

Proof. The proof is by induction.

For $n = 1$, A_1 is countable by hypothesis.

For $n = n + 1$, consider

$$A_1 \times ... \times A_{n+1} = \{(x_1, ..., x_{n+1}) \, | x_i \in A_i\},$$

which can also be expressed as

$$A_1 \times ... \times A_{n+1} = \bigcup_{(x_1,...,x_n)\in A_1\times...\times A_n} \{(x_1, ...x_n, x_{n+1}) \, | x_{n+1} \in A_{n+1}\}.$$

(Example: $A_1 = \{a, b\}$, $A_2 = \{x, y\}$, $A_3 = \{u, v, w\}$. Then $A_1 \times A_2 \times A_3 = \{(a, x, u), (a, x, v), (a, x, w)\} \cup \{(a, y, u), (a, y, v), (a, y, w)\} \cup \{(b, x, u), ...\} \cup \{(b, y, u), ...\}$.)

Note that we are taking the union over $A_1 \times ... \times A_{n+1}$, which is countable. Furthermore, note that $\{(x_1, ...x_n, x_{n+1}) \, | x_{n+1} \in A_{n+1}\} = B_{x_1x_2...x_n}$. Then:

(a) $|B_{x_1x_2...x_n}| = |A_{n+1}|$. Consider $f : B_{x_1x_2...x_n} \to A_{n+1}$, $(x_1, ..., x_n) \to x_{n+1}$. Then:

(a.1) f is injective. Note that $f(x_1, ..., x_n, x_{n+1}) = f(x_1, ..., x_n, y_{n+1}) \Rightarrow x_{n+1} = y_{n+1} \Rightarrow (x_1, ..., x_{n+1}) = (y_1, ..., y_{n+1})$.

(a.2) f is onto. Consider any element $a \in A_{n+1}$. Then $f(x_1, ..., x_n, a) = a$.

(b) $A_1 \times ... \times A_n$ is countable by induction hypothesis.

Then $A_1 \times ... \times A_{n+1}$ is a countable union of countable sets, and therefore it is countable. ■

Remark 17 *If the finiteness of the family is dropped, then the previous result is no longer true.*

Example 18 *Consider* $A = \{0,1\}^{\mathbb{N}} = \{0,1\}\times \{0,1\} \times \{0,1\} \times \dots$ *Then* A *is not countable. Note that* A *is a set of infinite sequences of* 0*s and* 1*s:*

$$A = \{(x_1, x_2, ...) \, | x_i \in \{0,1\}\}.$$

Suppose that A *is countable. Then:*

$$\begin{aligned} x_1 &= (x_{11}, x_{12}, x_{13}, ...) \\ x_2 &= (x_{21}, x_{22}, x_{23}, ...) \\ x_3 &= (x_{31}, x_{32}, x_{33}, ...) \\ &\vdots \end{aligned}$$

Consider $\bar{x} \in \{0,1\}^{\mathbb{N}}$ *defined by* $\bar{x} = (\bar{x}_1, \bar{x}_2, ...)$, $\bar{x}_i = \begin{cases} 1 & \textit{if } x_{ii} = 0 \\ 0 & \textit{if } x_{ii} = 1 \end{cases}$. *Then* $\bar{x} \neq x_n$ $\forall n \in \mathbb{N}$ *since* $\bar{x}_n \neq x_{nn}$. *It follows that* A *is not countable.*

Remark 19 *There is a bijection between* $\mathcal{P}(\{a_1, ..., a_n\})$ *and* $\{0,1\}^{\mathbb{N}}$. *In addition, there is a bijection between* $\mathcal{P}(\mathbb{N})$ *and* $\{0,1\}^{\mathbb{N}}$. *Then* $\mathcal{P}(\mathbb{N})$ *is not countable.*

Theorem 20 *If* $f : \mathbb{R} \to \mathbb{R}$ *is a nondecreasing function, then the set of points where* f *is discontinuous is at most countable.*

Proof. Consider the family $\{x_\lambda\}_{\lambda \in \Lambda}$, the set of points where f is discontinuous. We need to show that Λ is countable. Note that if x is discontinuous at x_λ, then

$$\lim_{x \to x_\lambda^-} f(x) \neq \lim_{x \to x_\lambda^+} f(x)$$

Since f is nondecreasing,

$$\lim_{x \to x_\lambda^-} f(x) < \lim_{x \to x_\lambda^+} f(x)$$

It is possible to find $Q_\lambda \in \mathbb{Q}$ such that

$$\lim_{x \to x_\lambda^-} f(x) < Q_\lambda < \lim_{x \to x_\lambda^+} f(x)$$

Now consider $g : \Lambda \to \mathbb{Q}$, $\lambda \to Q_\lambda$. Then g is injective. (Note that if $\lambda_1 \neq \lambda_2$, then either $x_{\lambda_1} > x_{\lambda_2}$ or $x_{\lambda_1} < x_{\lambda_2}$. In the first case, $x_{\lambda_1} > x_{\lambda_2} \Rightarrow Q_{\lambda_1} > Q_{\lambda_2} \Rightarrow g(\lambda_1) > g(\lambda_2)$. In the second case, $x_{\lambda_1} < x_{\lambda_2} \Rightarrow Q_{\lambda_1} < Q_{\lambda_2} \Rightarrow g(\lambda_1) < g(\lambda_2)$ and it is true that $g(\lambda_1) \neq g(\lambda_2)$.)

It follows that $g : \Lambda \to g[\Lambda]$ is a bijection, so $|\Lambda| = |g[\Lambda]|$. Since $g[\Lambda] \subseteq \mathbb{Q}$, then $g[\Lambda]$ is countable. Therefore, Λ is countable. ■