Convex sets and convex functions

Definition 1 *In $\mathbb{R}^n$, if $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, then*

$$x + y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n),$$

$$\lambda x = (\lambda x_1, \lambda x_2, ..., \lambda x_n).$$

Definition 2 (Convex set) *A set $S \subseteq \mathbb{R}^n$ is convex iff $\forall x, y \in S$ and $\lambda \in [0,1]$:*

$$\lambda x + (1-\lambda)y \in S.$$

Remark 3 *Any finite set $S \in \mathbb{R}^n$ with 2 or more elements is not convex. However, $\emptyset$ is convex, as is a singleton $\{x\}$.*

Proposition 4 *Consider $\{S_\alpha\}_{\alpha\in A}$ a family of convex sets. Then $\bigcap_{\alpha\in A} S_\alpha$ is convex.*

Proof. Consider $x, y \in \bigcap_{\alpha\in A} S_\alpha$. Since $\forall \alpha \in A$, $x, y \in S_\alpha$, it follows that $\lambda x + (1-\lambda)y \in S_\alpha$, $\forall \alpha \in A$, $\forall \lambda \in [0,1]$. Then $\lambda x + (1-\lambda)y \in \bigcap_{\alpha\in A} S_\alpha$, $\forall \alpha \in A$, $\forall \lambda \in [0,1]$, and therefore $\bigcap_{\alpha\in A} S_\alpha$ is convex. ■

Remark 5 *The union of convex sets is not necessarily convex.*

Example 6 *Consider $S_1 = \{1\}$, $S_2 = \{2\}$ in $\mathbb{R}$. Each element is convex, but their union is not (see Remark 3).*

Proposition 7 *Let $S \subseteq \mathbb{R}^L$ be a convex set. Consider $\{x_1, x_2, ..., x_L\} \subseteq S$. If $\{\lambda_i\}_{i=1,...,n}$, $\lambda_i \geq 0$ $\forall i \in \{1, ..., n\}$ and $\sum_{i=1}^n \lambda_i = 1$, then $\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n \in S$.*

Proof. The proof is by induction.

For $n = 1$, consider $x_1 \in S$, $\lambda_1 = 1$. Then $\lambda_1 x_1 = x_1 \in S$, which is convex.

For $n = n+1$, we assume that the property is true for families of n elements. Consider $\{x_1, ..., x_{n+1}\} \subseteq S$ and $\{\lambda_i\}_{i=1,...,n+1}$ where $\lambda_i \geq 0$ $\forall i \in \{1, ..., n\}$ and $\sum_{i=1}^n \lambda_i = 1$. We now set up 2 cases, depending on the value of λ_{n+1}.

(a) $\lambda_{n+1} = 1$. Then it follows that

$$\lambda_1 x_1 + ... + \lambda_n x_n + \lambda_{n+1} x_{n+1} = \lambda_{n+1} x_{n+1} = x_{n+1} \in S.$$

(b) $\lambda_{n+1} \neq 1$. Note that in this case,

$$\lambda_1 x_1 + ... + \lambda_{n+1} x_{n+1} = (1-\lambda_{n+1}) \left[\frac{\lambda_1}{1-\lambda_{n+1}} x_1 + ... + \frac{\lambda_n}{1-\lambda_{n+1}} x_n \right] + \lambda_{n+1} x_{n+1}$$

and the resulting structure resembles an expression of the type $(1-\lambda)a + \lambda b$, where $a = \frac{\lambda_1}{1-\lambda_{n+1}} x_1 + ... + \frac{\lambda_n}{1-\lambda_{n+1}} x_n$ and $b = x_{n+1}$. We need to check that the above is an element of S.

(b.1)

We claim that $\frac{\lambda_1}{1-\lambda_{n+1}} x_1 + ... + \frac{\lambda_n}{1-\lambda_{n+1}} x_n \in S$. Note that since $\{x_1, ..., x_n\} \subseteq S$ and $\frac{\lambda_i}{1-\lambda_{n+1}} \geq 0$, then

$$\sum\nolimits_{i=1}^{n} \frac{\lambda_i}{1-\lambda_{n+1}} = \frac{1}{1-\lambda_{n+1}} \sum\nolimits_{i=1}^{n} \lambda_i = \frac{1}{1-\lambda_{n+1}} (1-\lambda_{n+1}) = 1.$$

Then, by induction hypothesis, $\frac{\lambda_1}{1-\lambda_{n+1}} x_1 + ... + \frac{\lambda_n}{1-\lambda_{n+1}} x_n \in S$.

(b.2)

Since $\frac{\lambda_1}{1-\lambda_{n+1}} x_1 + ... + \frac{\lambda_n}{1-\lambda_{n+1}} x_n \in S$ and $x_{n+1} \in S$ by hypothesis, it follows that

$$(1-\lambda_{n+1}) \left[\frac{\lambda_1}{1-\lambda_{n+1}} x_1 + ... + \frac{\lambda_n}{1-\lambda_{n+1}} x_n \right] + \lambda_{n+1} x_{n+1} \in S.$$

■

Proposition 8 *Let $S \subseteq \mathbb{R}^n$. If S is convex, $\bar{S}$ is convex.*

Proof. Consider $x, y \in \bar{S}$. If $x \in \bar{S}$, then either $x \in S$ or $x \in S'$. In either case, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \to x$, and a sequence $\{y_n\}_{n\in\mathbb{N}}$ such that $y_n \to y$. Then $\lambda x_n + (1-\lambda)y_n \to \lambda x + (1-\lambda)y$. But since S is convex, we know that $\{\lambda x_n + (1-\lambda)y_n\}_{n\in\mathbb{N}} \subseteq S$. It follows that $\lambda x + (1-\lambda)y \in \bar{S}$. ■

Definition 9 *Let $A, B \subseteq \mathbb{R}^n$. Then $A + B$ is defined as*

$$A + B = \{x | x = a + b \text{ for some } a \in A,\ b \in B\}.$$

Example 10 *Let $A = [0, 1]$, $B = [2, 3] \cup (4, 5)$. Then $A + B = [2, 6)$.*

Proposition 11 *Suppose that the sets A and B are convex. Then $A + B$ is also convex.*

Proof. Consider $x, y \in A + B$. Since $x \in A + B$ then there exists $a_1 \in A$, $b_1 \in B$ such that $a_1 + b_1 = x$. Since $y \in A + B$ then there exists $a_2 \in A$, $b_2 \in B$ such that $a_2 + b_2 = y$. Then:

$$\lambda x + (1-\lambda)y = \lambda[a_1 + b_1] + (1-\lambda)[a_2 + b_2]$$

$$= [\lambda a_1 + (1-\lambda)a_2] + [\lambda b_1 + (1-\lambda)b_2]$$
$$= \bar{a} + \bar{b},$$

and since $[\lambda a_1 + (1-\lambda)a_2] = \bar{a} \in A$, and $[\lambda b_1 + (1-\lambda)b_2] = \bar{b} \in B$, then $\lambda x + (1-\lambda)y \in A + B$. ■

Definition 12 (Convex function) *Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$, and S be a convex set. The function f is convex iff $\forall x, y \in S$ and $\forall \lambda \in (0, 1)$:*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

Definition 13 (Concave function) *Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$, and S be a convex set. The function f is concave iff $-f$ is convex.*

Definition 14 (Strictly convex function) *Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$, and S be a convex set. The function f is strictly convex iff $\forall x, y \in S$ and $\forall \lambda \in (0, 1)$:*

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y).$$

Definition 15 (Strictly concave function) *Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$, and S be a convex set. The function f is strictly concave iff $-f$ is strictly convex.*

Definition 16 (Epigraph of a function) *Consider $S \subseteq \mathbb{R}^n$, and let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$. The epigraph of f is defined as:*

$$epi(f) = \{(x_1, ..., x_n, y) | y \geq f(x_1, ..., x_n), (x_1, ..., x_n) \in S\}.$$

Proposition 17 *Let $S \subseteq \mathbb{R}^n$, S a convex set. $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is convex iff $epi(f)$ is convex.*

Proof. "⇒"

Consider $(x_1, ..., x_n, y) \in epi(f)$, $(w_1, ..., w_n, v) \in epi(f)$ and $\lambda \in [0, 1]$. Now:

$$\lambda(x_1, ..., x_n, y) + (1-\lambda)(w_1, ..., w_n, v)$$

$$= (\lambda x_1 + (1-\lambda)w_1 + ... + \lambda x_n + (1-\lambda)w_n + \lambda y + (1-\lambda)v).$$

But note that

$$f(\lambda x_1 + (1-\lambda)w_1 + ... + \lambda x_n + (1-\lambda)w_n)$$
$$\leq \lambda f(x_1, ..., x_n) + (1-\lambda) f(w_1, ..., w_n)$$
$$\leq \lambda y + (1-\lambda)v$$

where line two follows from the convexity of f, and line three because $(x_1, ..., x_n, y)$, $(w_1, ..., w_n, v) \in epi(f)$. Then

$$(\lambda x_1 + (1-\lambda)w_1 + ... + \lambda x_n + (1-\lambda)w_n + \lambda y + (1-\lambda)v) \in epi(f).$$

"⇐"

Assume that $epi(f)$ is convex. We know that $(x, f(x))$, $(y, f(y)) \in epi(f)$. Then,

$$\lambda (x, f(x)) + (1-\lambda)(y, f(y)) \in epi(f) \ \forall \lambda \in [0,1],$$

Then

$$(\lambda x + (1-\lambda) y, \lambda f(x) + (1-\lambda) f(y)) \in epi(f).$$

Then

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y),$$

and f is convex. ∎

Theorem 18 *Consider* $\{f_i\}_{i=1,\ldots,n}$, $f_i : S \subseteq \mathbb{R}^n \to \mathbb{R}$ *and* f_i *convex,* S *convex. Then* $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ *defined by*

$$f(x) = \max_{i=1,\ldots,n} f_i(x)$$

is also convex.

Proof. The proof is done in 2 steps.

(a) We claim that $epi(f) = \bigcap_{i=1,\ldots,n} epi(f_i)$. To see that this is so, note that $(x,y) \in epi(f) \Leftrightarrow f(x) \leq y \Leftrightarrow \max_{i=1,\ldots,n} f_i(x) \leq y \Leftrightarrow f_i(x) \leq y \ \forall i \in \{1,...,n\} \Leftrightarrow (x,y) \in epi(f_i) \ \forall i \in \{1,...,n\} \Leftrightarrow (x,y) \in \bigcap_{i=1,\ldots,n} epi(f_i)$.

(b) It follows that $epi(f)$ is convex, since it is the intersection of convex sets. Then, by Proposition 17, f is convex. ∎

Proposition 19 *Let* $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$, S *is a compact and convex set, and* f *is continuous and strictly concave. Then there exists a unique* $x^* \in S$ *such that* $f(x^*) \geq f(x) \ \forall x \in S$.

Proof. The proof is done in 2 steps.

(a) The existance of $x^* \in S$ is guaranteed by the fact that S is compact and f is continuous (Theorem 40 in Compact sets).

(b) To see that x^* is unique, consider $x^* \in S$ such that $f(x^*) \geq f(x) \ \forall x \in S$, and suppose there is a $y^* \in S$ is such that $f(y^*) \geq f(x) \ \forall x \in S$, and $x^* \neq y^*$. Then, $\forall \lambda \in (0,1)$ and using the strict concavity of f:

$$f(\lambda x^* + (1-\lambda) y^*) > \lambda f(x^*) + (1-\lambda) f(y^*).$$

But then, using that x^*, y^* are maximums:

$$\begin{aligned} f(\lambda x^* + (1-\lambda) y^*) &> \lambda f(x^*) + (1-\lambda) f(y^*) \\ &\geq \lambda f(\lambda x^* + (1-\lambda) y^*) + (1-\lambda)(\lambda x^* + (1-\lambda) y^*) \\ &= f(\lambda x^* + (1-\lambda) y^*), \end{aligned}$$

which is a contradiction. ∎