Measure theory

Definition 1 (σ-algebra) *Let S be a set, and Σ a family of subsets of S, $\Sigma \subset \mathcal{P}(S)$. Σ is called a σ-algebra iff:*

(a) $\emptyset, S \in \Sigma$.
(b) $A \in \Sigma \Rightarrow A^C \in \Sigma$.
(c) $\{A_n\}_{n=1}^{\infty} \subseteq \Sigma \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \Sigma$.

Remark 2 *$\Sigma = \{\emptyset, S\}$ is a σ-algebra. $\Sigma = \mathcal{P}(S)$ is a σ-algebra.*

Proposition 3 *If Σ is a σ-algebra, then:*
(a) If $\{A_n\}_{n\geq 1} \in \Sigma$, then $\bigcap_{n=1}^{\infty} A_n \in \Sigma$.
(b) If $A, B \in \Sigma$, then $A \backslash B \in \Sigma$.

Proof. (a)

Note that $\bigcap_{n=1}^{\infty} A_n = \left[\bigcup_{n=1}^{\infty} A_n^C\right]^C$. But if A_n^C is in Σ, by property (c) of the definition of σ-algebra it should be the case that $\bigcup_{n=1}^{\infty} A_n^C$ is in Σ, and by property (b), $\left[\bigcup_{n=1}^{\infty} A_n^C\right]^C$. Then $\bigcap_{n=1}^{\infty} A_n \in \Sigma$.

(b)

Note that $A \backslash B = A \cap B^C$. Note further that $A \in \Sigma$, and if $B \in \Sigma$, then by property (b), $B^C \in \Sigma$. It follows that $A \cap B^C = A \backslash B \in \Sigma$. ■

Definition 4 *Let S be a set and $\mathcal{A}$ any collection of subsets of S. Then*

$$\sigma(\mathcal{A}) = \bigcap_{\substack{\beta \subseteq \mathcal{P}(S) \\ \beta \supseteq \mathcal{A} \\ \beta \text{ is a } \sigma\text{-algebra}}} \beta$$

is the σ-algebra generated by $\mathcal{A}$.

Proposition 5 *$\sigma(\mathcal{A})$ is a σ-algebra.*

Proof. First note that since $\mathcal{P}(S)$ is in the family of β we intersect, the operation is not an intersection over an empty family. To check the properties of a σ-algebra, we take 3 steps.

(a) Note that $\emptyset \in \beta\ \forall \beta$ such that it is a σ-algebra. Then

$$\emptyset \in \bigcap_{\substack{\beta \subseteq \mathcal{P}(S) \\ \beta \supseteq \mathcal{A} \\ \beta \text{ is a } \sigma\text{-algebra}}} \beta = \sigma(\mathcal{A}).$$

(b) Consider $K \in \sigma(\mathcal{A})$. Then

$$K \in \bigcap_{\substack{\beta \subseteq \mathcal{P}(S) \\ \beta \supseteq \mathcal{A} \\ \beta \text{ is a } \sigma\text{-algebra}}} \beta$$

and $K \in \beta\ \forall \beta \subseteq \mathcal{P}(S)$, $\beta \supseteq \mathcal{A}$, β is a σ-algebra. It follows that $K^C \in \beta\ \forall \beta \subseteq \mathcal{P}(S)$, $\beta \supseteq \mathcal{A}$, β is a σ-algebra, then

$$K^C \in \bigcap_{\substack{\beta \subseteq \mathcal{P}(S) \\ \beta \supseteq \mathcal{A} \\ \beta \text{ is a } \sigma\text{-algebra}}} \beta = \sigma(\mathcal{A}).$$

(c) Consider $\{A_n\}_{n=1}^{\infty} \subseteq \sigma(\mathcal{A})$. Then $\{A_n\}_{n=1}^{\infty} \subseteq \beta\ \forall \beta \subseteq \mathcal{P}(S)$, $\beta \supseteq \mathcal{A}$, β is a σ-algebra. It follows that $\bigcup_{n=1}^{\infty} A_n \in \beta\ \forall \beta \subseteq \mathcal{P}(S)$, $\beta \supseteq \mathcal{A}$, β is a σ-algebra, and then

$$\bigcup_{n=1}^{\infty} A_n \in \bigcap_{\substack{\beta \subseteq \mathcal{P}(S) \\ \beta \supseteq \mathcal{A} \\ \beta \text{ is a } \sigma\text{-algebra}}} \beta = \sigma(\mathcal{A}).$$

So $\sigma(\mathcal{A})$ is the smallest σ-algebra that contains the collection $\mathcal{A}$. ■

Definition 6 (Borel σ-algebra) *Set $S = \mathbb{R}$. Let $\mathcal{A} = \{(-\infty, b)\,|b \in \mathbb{R}\} \cup \{(a, \infty)\,|a \in \mathbb{R}\} \cup \{(a, b)\,|a, b \in \mathbb{R}\} \cup \{(-\infty, \infty)\}$. Then, $\sigma(\mathcal{A})$ is called the Borel σ-algebra of $\mathbb{R}$, and is denoted by $\mathcal{B}^1$. Any set in $\mathcal{B}^1$ is called a Borel set.*

Definition 7 *For $S = \mathbb{R}^l$ or any other metric space,*

$$\mathcal{A} = \{B(s, \varepsilon)\,|s \in S, \varepsilon > 0\}$$

$\sigma(\mathcal{A})$ is called the Borel σ-algebra.

Definition 8 (Measurable space) *If Σ is a σ-algebra in S, then we say that (S, Σ) is a measurable space.*

Definition 9 (Measure) *Let (S, Σ) be a measurable space. A measure is an extended real-valued function $\mu : \Sigma \to \bar{\mathbb{R}}$ such that:*
(a) $\mu(\emptyset) = 0$.
(b) $\mu(A) \geq 0\ \forall A \in \Sigma$.
(c) If $\{A_n\}_{n=1}^{\infty}$ is a countable, pairwise disjoint sequence of subsets in Σ, then $\mu(\{A_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} \mu(A_n)$.

Example 10 *Let $S = \{1, 2, 3, 4, 5, 6\}$, $\Sigma = \{\emptyset, \{1, 2, 3\}, \{4, 5, 6\}, \{1, 2, 3, 4, 5, 6\}\}$ and $\mu : \Sigma \to \bar{\mathbb{R}}$. If we imagine that the measure is a sort of probability, then we can assign the following values:*
$\mu(\emptyset) = 0$
$\mu(\{1, 2, 3\}) = \frac{1}{5}$
$\mu(\{4, 5, 6\}) = \frac{4}{5}$
$\mu(\{1, 2, 3, 4, 5, 6\}) = \mu(\{1, 2, 3\}) + \mu(\{4, 5, 6\}) = 1$
where the last step follows from property (c) of Definition 9.

Example 11 *Let $S = \{1, 2, 3, 4, 5, 6\}$, $\Sigma = \mathcal{P}(S)$ and $\mu : \Sigma \to \bar{\mathbb{R}}$. A naive way of defining a measure is to assign a value $\pi_i \geq 0$ to each of the singletons, so that*
$\mu(\emptyset) = 0$
$\mu(\{1\}) = \pi_1$
$\mu(\{2\}) = \pi_2$
$\mu(\{3\}) = \pi_3$
$\mu(\{4\}) = \pi_4$
$\mu(\{5\}) = \pi_5$
$\mu(\{6\}) = \pi_6$
and from property (c) of Definition 9, $\mu(A) = \sum_{i \in A} \pi_i\ \forall A \in \Sigma$.

Definition 12 (Measure space) *A measure space is a triple (S, Σ, μ) where S is a set, Σ is a σ-algebra in S and $\mu : \Sigma \to \bar{\mathbb{R}}$ is a measure.*

Proposition 13 *Let (S, Σ, μ) be a measure space. Then:*
(a) $A \subseteq B$, $A, B \in \Sigma \Rightarrow \mu(A) \leq \mu(B)$.
(b) $A \subseteq B$, $A, B \in \Sigma$, $\mu(A) < \infty \Rightarrow \mu(B \backslash A) = \mu(B) - \mu(A)$.

Proof. (a)
Note that $B = [B \cap A^C] \cup A$, where $[B \cap A^C]$, A are disjoint, and $[B \cap A^C], A \in \Sigma$. Then $\mu(B) = \mu(B \cap A^C) + \mu(A)$. However, $\mu(B \cap A^C) \geq 0$. It follows that $\mu(B) \geq \mu(A)$.
(b)
Note that $B = (B \backslash A) \cup A$, which are disjoint sets. Therefore $\mu(B) = \mu(B \backslash A) + \mu(A)$. If $\mu(A) < \infty$, we can subtract on both sides, so that $\mu(B \backslash A) = \mu(B) - \mu(A)$. ■

Remark 14 *The fact that $A \subseteq B$ is a key assumption for Proposition 13 to hold.*

Example 15 *Let $S = \{1, 2, 3, 4, 5, 6\}$, $\Sigma = \mathcal{P}(S)$ and $\mu : \Sigma \to \bar{\mathbb{R}}$ defined as in Example 11. If $A = \{1\}$ and $B = \{2\}$ then $B \backslash A = \{2\}$ and $\mu(B \backslash A) = \pi_2$, but $\mu(B) - \mu(A) = \pi_2 - \pi_1$, which is not necessarily π_2.*

Proposition 16 *Let (S, Σ, μ) be a measure space. Then:*
(a) If $\{A_n\}_{n=1}^{\infty}$ is such that $A_n \subseteq A_{n+1}$ then $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$.
(b) If $\{B_n\}_{n=1}^{\infty}$ is such that $B_n \supseteq B_{n+1}$ and $\mu(B_n) < \infty$ for some n, then $\mu\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \mu(B_n)$.

Proof. (a)

We have two cases:

(a.1) $\mu(A_n) < \infty\ \forall n$.

Notice that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} (A_n \backslash A_{n-1})\right).$$

Set $A_0 = \emptyset$. Then:

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} (A_n \backslash A_{n-1})\right) &= \sum_{n=1}^{\infty} \mu(A_n \backslash A_{n-1}) \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \mu(A_n \backslash A_{n-1}) \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} [\mu(A_n) - \mu(A_{n-1})] \\
&= \lim_{N\to\infty} \mu(A_N) - \mu(A_0) \\
&= \lim_{N\to\infty} \mu(A_N).
\end{aligned}$$

(a.2) There exists some $\bar{N}$ such that $\mu(A_{\bar{N}}) = \infty$.

Then $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \infty$ but $\lim_{n\to\infty} \mu(A_n) = \infty$, and the equality holds.

(b)

Without loss of generality, set $\mu(B_1) < \infty$. Consider the family $\{B_1 \backslash B_n\}_{n=1}^{\infty}$, and note that $B_1 \backslash B_n \subseteq B_1 \backslash B_{n+1}$ and that by part (a):

$$\mu\left(\bigcup_{n=1}^{\infty} [B_1 \backslash B_n]\right) = \lim_{n\to\infty} \mu(B_1 \backslash B_n). \tag{1}$$

However, note that

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} [B_1 \backslash B_n]\right) &= \mu\left(\bigcup_{n=1}^{\infty} B_1 \cap B_n^C\right) \\
&= \mu\left(B_1 \cap \left(\bigcup_{n=1}^{\infty} B_n^C\right)\right) \\
&= \mu\left(B_1 \cap \left(\bigcap_{n=1}^{\infty} B_n\right)^C\right).
\end{aligned} \tag{2}$$

And then, by (1) and (2):

$$\begin{aligned}
\mu\left(B_1 \cap \left(\bigcap_{n=1}^{\infty} B_n\right)^C\right) &= \lim_{n\to\infty} \mu(B_1 \backslash B_n) \\
\mu\left(B_1 \backslash \left(\bigcap_{n=1}^{\infty} B_n\right)\right) &= \lim_{n\to\infty} \mu(B_1 \backslash B_n) \\
\mu(B_1) - \mu\left(\bigcap_{n=1}^{\infty} B_n\right) &= \lim_{n\to\infty} [\mu(B_1) - \mu(B_n)] \\
\mu(B_1) - \mu\left(\bigcap_{n=1}^{\infty} B_n\right) &= \mu(B_1) - \lim_{n\to\infty} \mu(B_n) \\
\mu\left(\bigcap_{n=1}^{\infty} B_n\right) &= \lim_{n\to\infty} \mu(B_n).
\end{aligned}$$

■

Definition 17 (Finite and probability measures) *In (S, Σ, μ), a measure space, μ is called a finite measure if $\mu(S) < \infty$, and μ is called a probability measure if $\mu(S) = 1$.*

Definition 18 (Probability space) *If μ is a probability measure then (S, Σ, μ) is called a probability space.*

Defining measures on $\mathbb{R}$

Consider $(\mathbb{R}, \mathcal{B}^1, \mu)$. We will try to define a proper μ.

Definition 19 (Algebra) *Let S be a set and $\mathcal{A}$ be a family of subsets of S. Then $\mathcal{A}$ is called an algebra if:*
(a) $\emptyset, S \in \mathcal{A}$.
(b) $A \in \mathcal{A} \Rightarrow A^C \in \mathcal{A}$.
(c) $A_1, ..., A_n \in \mathcal{A} \Rightarrow \bigcup_{i=1}^n A_i \in \mathcal{A}$.

Remark 20 *Every σ-algebra is an algebra, but the opposite is not necessarily true.*

Example 21 *In $\mathbb{R}$, consider the family of complements and* finite unions *of sets of the form $(-\infty, b]$, $(a, b]$, (a, ∞), $(-\infty, \infty)$. Then the family is an algebra but not a σ-algebra, since $\bigcup_{n\in\mathbb{N}} \left(-\infty, -\frac{1}{n}\right) = (-\infty, 0)$ is not in the set.*

Definition 22 (Measure) *Let S be a set and let $\mathcal{A}$ be an algebra of subsets of S. Then a measure on $\mathcal{A}$ is a function $\mu : \mathcal{A} \to \mathbb{R}$ such that:*
(a) $\mu(\emptyset) = 0$.
(b) $\mu(A) \geq 0 \ \forall A \in \mathcal{A}$.
(c) If $\{A_n\}_{n=1}^\infty$ is any disjoint sequence with $\bigcup_{i=1}^\infty A_i \in \mathcal{A}$, then $\mu\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \mu(A_i)$.

Example 23 *In the algebra defined in Example 21, it is easy to define a measure. For example:*
$\mu((-\infty, b]) = \infty$,
$\mu((a, b]) = b - a$,
$\mu((a, \infty)) = \infty$,
$\mu((-\infty, \infty)) = \infty$,
$\mu\left(\bigcup_{i=1}^n (a_i, b_i]\right) = \sum_{i=1}^n (b_i - a_i)$.

Theorem 24 (Caratheodory) *Let S be a set, and $\mathcal{A}$ an algebra of its subsets. Consider $\Sigma = \sigma(\mathcal{A})$. Then there exists a measure $\mu^* : \Sigma \to \mathbb{R}$ such that $\mu^*(A) = \mu(A) \ \forall A \in \mathcal{A}$.*

Definition 25 (σ-finite measure) *Let S be a set, $\mathcal{A}$ an algebra of its subsets and $\mu : \mathcal{A} \to \mathbb{R}$ a measure. μ is said to be σ-finite if there exists a sequence $\{A_n\}_{n=1}^\infty \subseteq \mathcal{A}$ such that $\bigcup_{i=1}^n A_i = S$ and $\mu(A_i) < \infty \ \forall i \in \mathbb{N}$.*

Theorem 26 (Hahn) *If $\mu : \mathcal{A} \to \mathbb{R}$ is σ-finite, then there exists a unique extension $\mu^*(A) = \mu(A) \ \forall A \in \mathcal{A}$ defined by $\mu^* : \sigma(\mathcal{A}) \to \mathbb{R}$ that is a measure.*

Example 27 *In Example 23, $\sigma(\mathcal{A}) = \mathcal{B}^1$.*

Measurable functions

Definition 28 (Σ-measurable function) *Given a measurable space (S, Σ, μ), a real valued function $f : S \to \mathbb{R}$ is measurable with respect to Σ (or Σ-measurable) if*

$$\{s \in S | f(s) \leq a\} \in \Sigma, \ \forall a \in \mathbb{R}$$

where $\{s \in S | f(s) \leq a\} = f^{-1}[(-\infty, a]]$.

Definition 29 (Random variable) *If (S, Σ, μ) is a probability space, then measurable functions are called random variables.*

Remark 30 *If $\Sigma = \mathcal{P}(S)$, then every $f : S \to \mathbb{R}$ is Σ-measurable.*

Remark 31 *If $\Sigma = \{\emptyset, S\}$, then only constant functions are Σ-measurable.*

Proof. Consider $f : S \to \mathbb{R}$ that is not constant. Then there exist $u, v \in S$ such that $f(u) = x$, $f(v) = y$ and $x \neq y$. Without loss of generality, suppose $x < a < y$. Then $f^{-1}[(-\infty, a]] \neq \emptyset$ (since $u \in f^{-1}[(-\infty, a]]$) and $f^{-1}[(-\infty, a]] \neq S$ (since $u \notin f^{-1}[(-\infty, a]]$). Then $f^{-1}[(-\infty, a]] \notin \Sigma$ and f is not Σ-measurable.

Now note that if $f : S \to \mathbb{R}$ is constant, then $f(s) \equiv x \ \forall s \in S$ and $f^{-1}[(-\infty, a]] = \begin{cases} \emptyset & \text{if } a < x \\ S & \text{if } a \geq x \end{cases}$. It follows that $f^{-1}[(-\infty, a]] \in \Sigma$ and f is Σ-measurable. ■

Example 32 *Let* $S = \{1,2,3,4,5,6\}$*,* $\Sigma = \{\emptyset, \{1,2,3\}, \{4,5,6\}, \{1,2,3,4,5,6\}\}$ *and* $\mu : \Sigma \to \bar{\mathbb{R}}$*. Which* $f : S \to \mathbb{R}$ *are measurable? In this case, we claim that functions of the type* $f(1) = f(2) = f(3) = x$ *and* $f(4) = f(5) = f(6) = y$ *are acceptable.*

Proof. Without loss of generality, let $f(1) = f(2) = f(3) = x$, and $f(4) = f(5) = f(6) = y$, and assume $x < y$. If $a < x$, then $f^{-1}[(-\infty, a]] = \emptyset \in \Sigma$. If $x \leq a < y$, then $f^{-1}[(-\infty, a]] = \{1,2,3\} \in \Sigma$. Finally, if $a \geq y$, then $f^{-1}[(-\infty, a]] = \{1,2,3,4,5,6\} \in \Sigma$. Then f is Σ-measurable.

Note that if $f(1) \neq f(2)$, for example if $f(1) = x$ and $f(2) = y$, $x < y$, then for $x < a < y$, $1 \in f^{-1}[(-\infty, a]]$ but $2 \notin f^{-1}[(-\infty, a]]$. It follows that $f^{-1}[(-\infty, a]] \notin \Sigma$, and f is not Σ-measurable. ■