Some examples

Example 1 (Completion of a σ-algebra) *Consider (S, Σ, μ), a measure space. Define the set $\mathcal{C}$ as*

$$\mathcal{C} = \{C \subseteq S | C \subseteq A \text{ for some } A \in \Sigma \text{ with } \mu(A) = 0\},$$

where the sets in $\mathcal{C}$ are not necessarily in the σ-algebra. Now define the set Σ' as

$$\Sigma' = \left\{B' \in S | B' = (B \cup C_1) \cap C_2^C \text{ for some } B \in \Sigma, \, C_1, C_2 \in \mathcal{C}\right\}.$$

Prove that Σ' is a σ-algebra.

Proof. We need to prove that Σ' satisfies the properties of a σ-algebra.

(a) $\emptyset, S \in \Sigma'$ since $\emptyset, S \in \Sigma \Longrightarrow \emptyset, S \in \Sigma'$ (note that $\emptyset, S \in \Sigma$, and that for any $A \in \Sigma$, $(A \cup \emptyset) \cap \emptyset^C = A \in \Sigma'$).

(b) Suppose that $B' \in \Sigma'$. Then we can express B' as

$$B' = (B \cup C_1) \cap C_2^C$$

where $B \in \Sigma$, $C_1, C_2 \in \mathcal{C}$. However, note that

$$\begin{aligned}(B')^C &= (B \cup C_1)^C \cup \left(C_2^C\right)^C \\ &= \left(B^C \cap C_1^C\right) \cup C_2 \\ &= \left(B^C \cup C_2\right) \cap \left(C_1^C \cup C_2\right) \\ &= \left(B^C \cup C_2\right) \cap \left(C_1 \cap C_2^C\right).\end{aligned}$$

In this case, notice that $B^C \in \Sigma$ and $C_2 \in \mathcal{C}$, so it only remains to prove that $\left(C_1 \cap C_2^C\right) \in \mathcal{C}$. To see that this is so, note that since $C_1 \in \mathcal{C}$ then $C_1 \subseteq A \in \Sigma$ with $\mu(A) = 0$. It follows that $C_1 \cap C_2^C \subseteq A \in \Sigma$ with $\mu(A) = 0$, and then $C_1 \cap C_2^C \in \mathcal{C}$. Then $(B')^C \in \mathcal{C}$.

(c) Note that if $\{A_n\}_{n=1}^\infty \subseteq \Sigma'$, then

$$A_n = (B_n \cup C_{1,n}) \cap C_{2,n}^C,$$

where $B_n \in \Sigma$ and $C_{1,n}, C_{2,n} \in \mathcal{C}$. It follows that:

$$\begin{aligned}\bigcup_{n=1}^\infty A_n &= \bigcup_{n=1}^\infty [B_n \cup C_{1,n}] \cap \bigcup_{n=1}^\infty C_{2,n}^C \\ &= \left[\left(\bigcup_{n=1}^\infty B_n\right) \cup \left(\bigcup_{n=1}^\infty C_{1,n}\right)\right] \cap \left[\bigcap_{n=1}^\infty C_{2,n}\right]^C,\end{aligned}$$

where $\bigcup_{n=1}^\infty B_n \in \Sigma$ and $\bigcap_{n=1}^\infty C_{2,n} \in \mathcal{C}$. It only remains to prove that $\bigcup_{n=1}^\infty C_{1,n} \in \mathcal{C}$. To see that this is so, note that if $C_{1,n} \in \mathcal{C}$ then $C_{1,n} \subseteq A \in \Sigma$ with $\mu(A_n) = 0$. Then,

$$\bigcup_{n=1}^\infty C_{1,n} \subseteq \bigcup_{n=1}^\infty A_n \in \Sigma \text{ with } \mu\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \mu(A_n) = 0. \tag{1}$$

Then $\bigcup_{n=1}^\infty C_{1,n} \in \mathcal{C}$ and $\bigcup_{n=1}^\infty A_n \in \Sigma'$.

(In (1) we stated that $\mu\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \mu(A_n)$. Take any 3 non disjoint sets A_1, A_2, A_3. Then $\mu(A_1 \cup A_2 \cup A_3) = \mu(A_1 \cup (A_2 \backslash A_1) \cup (A_3 \backslash (A_1 \cup A_2))$. Since this is a union of disjoint sets, then $\mu(A_1 \cup A_2 \cup A_3) = \mu(A_1) + \mu(A_2 \backslash A_1) + \mu(A_3 \backslash (A_1 \cup A_2))$. But note that $\mu(A_2) \geq \mu(A_2 \backslash A_1)$ and $\mu(A_3) \geq \mu(A_3 \backslash (A_1 \cup A_2))$. Then it follows that $\mu(A_1 \cup A_2 \cup A_3) \leq \mu(A_1) + \mu(A_2) + \mu(A_3)$.) ∎

Example 2 *Let (S, d) be a metric space, and $E \subseteq S$. Define $\rho_E : S \to \mathbb{R}_+$, $x \to \inf_{z \in E} d(x, z)$. (a) Prove that ρ_E is continuous. (b) Prove that $\bar{E} = \{x | \rho_E(x) = 0\}$.*

Proof. (a) We need to show that $\forall \varepsilon > 0$, there exists some $\delta > 0$ such that $d(x,y) < \delta \Rightarrow |\rho_E(x) - \rho_E(y)| < \varepsilon$.

Consider $\varepsilon > 0$, and suppose $d(x,y) < \delta$. Then:

$$\rho_E(x) = \inf_{z \in E} d(x,z)$$

$$\leq \inf_{z \in E} \{d(x,y) + d(y,z)\}$$

$$< \delta + \inf_{z \in E} d(y,z)$$

$$= \delta + \rho_E(y),$$

and it follows that $\rho_E(x) - \rho_E(y) = \delta$. Reverting the roles of x and y we get that $\rho_E(y) - \rho_E(x) = \delta$, and then $|\rho_E(x) - \rho_E(y)| < \delta$. Taking $\delta = \varepsilon$, we prove the continuity.

(b) We need 2 steps.

"$\subseteq$"

Suppose that $x \in \bar{E}$. Then there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq E$ such that $x_n \to x$. Consider $\varepsilon > 0$. Then there exists some $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$. Then:

$$\rho_E(x) = \inf_{z \in E} d(x,z) < \varepsilon.$$

Since the operation is done $\forall \varepsilon$, we know that $\rho_E(x) = 0$.

"$\supseteq$"

Suppose that $\rho_E(x) = 0$. Then

$$\inf_{z \in E} d(x,z) = 0$$

and it follows that $\{z_n\}_{n \in \mathbb{N}} \subseteq E$ and $d(x, z_n) < \frac{1}{n}$, therefore $d(x, z_n) \to 0$ and $z_n \to x$, then $x \in \bar{E}$. ∎

Example 3 *Let (X,d) be a metric space, $A, B \subseteq X$, A closed, B compact. Then the distance between A, B is*

$$\rho(A,B) = \inf_{x \in A} \inf_{y \in B} d(x,y)$$

Then $A \cap B \neq \emptyset \Leftrightarrow \rho(A,B) = 0$.

Proof. "$\Rightarrow$"

Suppose that $A \cap B \neq \emptyset$. Then there exists one point that is in both, $z \in A \cap B$. Note that since $z \in A$ and $z \in B$,

$$\rho(A,B) = \inf_{x \in A} \inf_{y \in B} d(x,y) \leq d(z,z) = 0$$

Then $\rho(A,B) = 0$.

"$\Leftarrow$"

Suppose $A \cap B = \emptyset$, and define $\rho_A : B \to \mathbb{R}$, $x \to \inf_{y \in A} d(x,y)$. Since ρ_A is continuous, and it is defined over a compact set, then it attains its minimum at some point x^*. We need to show that $\rho_A(x^*) > 0$.

Recall that if $x^* \in B \Rightarrow x^* \notin A$, $x^* \notin \bar{A}$, $\rho_A(x^*) > 0$. Then

$$\rho(A,B) = \inf_{x \in B} \left[\inf_{y \in A} d(x,y) \right]$$

$$= \inf_{x \in B} \rho_A(x)$$

$$= \min_{x \in B} \rho_A(x) = \rho_A(x^*) > 0.$$

And since $\rho(A,B) > 0$, it is a contradiction since $\rho(A,B) = 0$. Then $A \cap B \neq \emptyset$. ∎

Remark 4 *Closedness is important. For example, consider $A = (0,1)$ and $B = (2,1)$. Then $\rho(A,B) = 0$ but $A \cap B = \emptyset$. Also, compactness of one of the sets is important. Consider $A = \left\{(x,y) \,|y \geq \frac{1}{x}, x > 0\right\}$ and $B = \left\{(x,y) \,|y \geq -\frac{1}{x}, x > 0\right\}$. Then $\rho(A,B) = 0$ but $A \cap B = \emptyset$.*

Example 5 *Consider (X,d) a metric space, $A,B \subseteq X$, A compact, B closed. Then $A+B=\{z|z=a+b$ for some $a\in A$, $b\in B\}$ is also closed.*

Proof. Consider $\{z_n\}_{n\in\mathbb{N}} \subseteq A+B$, $z_n \to z$. If $\{z_n\}_{n\in\mathbb{N}} \subseteq A+B$ then $z_n = a_n + b_n$ for some $a_n \in A$, $b_n \in B$. Note that $\{a_n\}_{n\in\mathbb{N}} \subseteq A$, where A is compact. Then there exists a subsequence $\{a_{k_n}\}_{n\in\mathbb{N}} \subseteq A$ with $a_{k_n} \to a \in A$. Then, since $z_{k_n} = a_{k_n} + b_{k_n}$, $b_{k_n} = z_{k_n} + a_{k_n}$. It follows that $b_{k_n} \to z-a$. But $\{b_{k_n}\}_{n\in\mathbb{N}} \subseteq B$, where B is closed, then $z-a\in B$.

Then $z = a + (z-a)$ where $a \in A$ and $(z-a)\in B$. Then $z \in A+B$. ■

Remark 6 *The compactness of one of the sets is needed. For example, consider $A=\left\{(x,y)\,|y \geq \frac{1}{x}, x>0\right\}$ and $B=\left\{(x,y)\,|y\geq -\frac{1}{x}, x>0\right\}$. Then $A+B=\{(x,y)\,|y>0\}$ is open.*

Example 7 *Let (X,d) be a metric space, $f,g : X \to \mathbb{R}$. Then*

$$\sup_{x\in X}\,[f(x)+g(x)] \leq \sup_{x\in X} f(x) + \sup_{x\in X} g(x).$$

Proof. Consider $L = \sup_{x\in X}\,[f(x)+g(x)]$. Then, $\forall\varepsilon>0$, there exists x_ε^* such that

$$f(x_\varepsilon^*) + g(x_\varepsilon^*) > L - \varepsilon.$$

But $f(x_\varepsilon^*) \leq \sup_{x\in X} f(x)$ and $g(x_\varepsilon^*) \leq \sup_{x\in X} g(x)$. Then

$$\sup_{x\in X} f(x) + \sup_{x\in X} g(x) > L - \varepsilon.$$

Since we can do this $\forall\varepsilon>0$, then

$$\sup_{x\in X} f(x) + \sup_{x\in X} g(x) \geq L = \sup_{x\in X}\,[f(x)+g(x)]\,.$$

■

Example 8 *Let (S,Σ) be a measurable space. Consider a sequence of functions $f_n : S \to \mathbb{R}$, measurable, and $f : S\to\mathbb{R}$ defined as $f(S) = \lim_{n\to\infty} f_n(S)$. Then f is measurable.*

Proof. We need to show that $A = \{s|f(s)\leq a\} \in \Sigma$.

Consider

$$A_{nk} = \left\{s|f_n(s) \leq a + \frac{1}{k}\right\} \in \Sigma \ \forall n,k\in\mathbb{N}$$

$$B_{Nk} = \bigcap\nolimits_{n=N}^{\infty} A_{nk} \in \Sigma$$

$$B_k = \bigcup\nolimits_{N=1}^{\infty} B_{Nk} \in \Sigma$$

$$B = \bigcap\nolimits_{k=1}^{\infty} B_k \in \Sigma.$$

We now claim that $A=B$. Note that

$$A = \bigcap\nolimits_{k=1}^{\infty}\bigcup\nolimits_{N=1}^{\infty}\bigcap\nolimits_{n=N}^{\infty}\left\{s|f_n(s)\leq a+\frac{1}{k}\right\}.$$

"$\subseteq$"

Take $s\in A$. Then $f(s)\leq a$, and $\forall k\in\mathbb{N}$, there exists some $N\in\mathbb{N}$ such that $\forall n\geq N$, $f_n(s)\leq a+\frac{1}{k}$. Then $\forall k\in\mathbb{N}$, there exists some $N\in\mathbb{N}$ such that $\forall n\geq N$,

$$s\in A_{nk} = \left\{s\in S|f_n(s)\leq a+\frac{1}{k}\right\}.$$

Now, $\forall k\in\mathbb{N}$, there exists some $N\in\mathbb{N}$

$$s\in\bigcap\nolimits_{n=N}^{\infty}\left\{s\in S|f_n(s)\leq a+\frac{1}{k}\right\}.$$

And $\forall k \in \mathbb{N}$

$$s \in \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \left\{ s \in S | f_n(s) \leq a + \frac{1}{k} \right\},$$

$$s \in \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \left\{ s \in S | f_n(s) \leq a + \frac{1}{k} \right\}.$$

"$\supseteq$"
Suppose that $s \in B$. Then

$$s \in \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \left\{ s \in S | f_n(s) \leq a + \frac{1}{k} \right\}.$$

Then $\forall k \in \mathbb{N}$ there exists some $N \in \mathbb{N}$ such that $\forall n \geq N$,

$$f_n(s) \leq a + \frac{1}{k}.$$

Then $\forall k \in \mathbb{N}$, $f(s) \leq a + \frac{1}{k}$. Then $f(s) \leq a$, and it follows that $s \in A$. ■