Risk Aversion and Expected-Utility Theory

Matthew Rabin (2000)
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"The report of my death was an exaggeration."
– The parrot.

- Expected utility theory is vastly used in economics to explain choice under uncertainty.

- The paper shows that within the expected-utility model, anything but virtual risk neutrality over modest stakes implies unrealistic risk aversion over large stakes.

- Rabin (2000) and Rabin and Thaler (2001) conclude that expected-utility theory gives absurd results under the calibrations they perform. In short, they recommend that *"it is time for economists to recognize that expected utility is an ex-hypothesis."*

The calibrations

Let our agent be an expected-utility maximizer over wealth w, with von Neumann – Morgenstern preferences $U(w)$. The agent likes money and is risk-averse: for all w, $U(w)$ is strictly increasing and weakly concave. Nothing else is assumed about $U(w)$.

Suppose that for some range of initial wealth levels and for some $g > \ell > 0$, he rejects bets of losing $\$\ell$ or gaining $\$g$, each with 50% chance.

(We'll go into detail about the calibrations and the theorem in a minute. Really.)

Example [Rabin and Thaler (2001)]

Suppose Johnny is a risk-averse expected utility maximizer, and that he will always turn down the 50-50 gamble of losing \$10 or gaining \$11. What else can we say about Johnny?

Specifically, what is the biggest Y such that we know Johnny will turn down a 50-50 lose \$100 / win \$$Y$ bet?

(a) \$110.

(b) \$221.

(c) \$2,000.

(d) \$20,242.

(e) \$1.1 million.

(f) \$2.5 billion.

(g) Johnny will reject the bet no matter what Y is.

(h) We need more information about Johnny's utility function.

The following table shows how Johnny would behave for different initial assumptions about our bets:

Table I [Rabin(2000)]

If averse to 50-50 lose \$100 / gain \$$g$ bets for all wealth levels, will turn down 50-50 lose \$$L$ / gain \$$G$ bets; G's entered in table.

L	$g = \$101$	$g = \$105$	$g = \$110$	$g = \$125$
\$400	400	420	550	1,250
\$600	600	730	990	∞
\$800	800	1,050	2,090	∞
\$1,000	1,010	1,570	∞	∞
\$2,000	2,320	∞	∞	∞
\$4,000	5,750	∞	∞	∞
\$6,000	11,810	∞	∞	∞
\$8,000	34,930	∞	∞	∞
\$10,000	∞	∞	∞	∞
\$20,000	∞	∞	∞	∞

(Note that ∞ means precisely that: ∞.)

Now suppose our agent gives us the same info as Table I when her lifetime wealth is below \$300,000. The agent has a very *peculiar* behavior for an initial wealth level of \$290,000:

Table II – selected columns [Rabin(2000)]

Table I replicated, for initial wealth level \$290,000, when ℓ/g behavior is only known to hold for $w \leq 300,000$.

L	$g = \$101$	$g = \$110$	$g = \$125$
\$400	400	550	1,250
\$600	600	990	36 billion
\$800	800	2,090	90 billion
\$1,000	1,010	718,190	160 billion
\$2,000	2,320	12,210,880	850 billion
\$4,000	5,750	60,528,930	9.4 trillion
\$6,000	11,510	180 million	89 trillion
\$8,000	19,920	510 million	830 trillion
\$10,000	27,780	1.3 billion	7.7 quadrillion
\$20,000	85,750	160 billion	540 quadrillion

(Note that I've chopped the $g = 105$ column ... but you get the idea.)

Now suppose we assign a constant absolute risk aversion (CARA) utility function. Yet, our agent is still behaving *very bizarrely*:

Table III – selected rows [Rabin(2000)]

If a person has CARA utility function and is averse to 50/50 lose $\$\ell$ / gain $\$g$ bets for all wealth levels, then (*i*) he has coefficient of absolute risk aversion no smaller than ρ and (*ii*) invests $\$X$ in the stock market when stock yields are normally distributed with mean real value return 6.4% and standard deviation 20%, and bonds yield a riskless return of 0.5%.

ℓ/g	ρ	X
\$100 / \$101	0.0000990	\$14,899
\$100 / \$110	0.0009084	\$1,639
\$100 / \$150	0.0032886	\$449
\$1,000 / \$1,050	0.0000476	\$30,987
\$1,000 / \$1,500	0.0003288	\$4,497
\$1,000 / \$2,000	0.0004812	\$3,067
\$10,000 / \$11,000	0.0000090	\$163,889
\$10,000 / \$12,000	0.0000166	\$88,855
\$10,000 / \$15,000	0.0000328	\$44,970
\$10,000 / \$20,000	0.0000481	\$30,665

The intuition (and the elusive theorem)

Suppose you have an initial wealth of w, and you reject a 50-50 lose \$10 / gain \$11 gamble because of diminishing marginal utility of wealth. Then, it must be that

$$U(w+11) - U(w) \leq U(w) - U(w-10)$$

Hence, on average you value each of the dollars between w and $w+11$ by at most 10/11 as much as you, on average, value each of the dollars between $w-10$ and w.

By concavity, this implies that you value the dollar $w+11$ at most 10/11 as much as you value the dollar $w-10$. Iterating, if you have the same risk aversion to the lose \$10 / gain \$11 bet at wealth level $w+21$, then you value dollar $w+21+11 = w+32$ by at most 10/11 as you value dollar $w+21-10 = w+11$, which means you value dollar $w+32$ by at most $10/11 \times 10/11 \approx 5/6$ as much as dollar $w-10$.

Thus you will value the $w+210th$ dollar by at most 40 percent as much as dollar $w-10$, and the $w+900th$ dollar by at most 2 percent as much as dollar $w-10$. This rate of deterioration for the value of money is absurdly high, and hence leads to absurd risk aversion.

It is with this logic that we are able to prove the following theorem (and corollary):

Theorem Suppose that for all w, $U(w)$ is strictly increasing and weakly concave. Suppose that there exists $\bar{w} > \underline{w}$, $g > \ell > 0$, such that for all $w \in [\underline{w}, \bar{w}]$, $0.5U(w-\ell)+0.5U(w+g) < U(w)$. Then for all $w \in [\underline{w}, \bar{w}]$, for all $x > 0$,

(a) If $g \leq 2\ell$, then

$$U(w) - U(w-x) \geq 2 \sum_{i=1}^{k^*(x)} \left(\frac{g}{\ell}\right)^{i-1} r(w)$$

if $w - \underline{w} + 2\ell \geq x \geq 2\ell$, and

$$U(w) - U(w-x) \geq 2\left[\sum_{i=1}^{k^*(w-\underline{w}+2\ell)} \left(\frac{g}{\ell}\right)^{i-1} r(w)\right] + [x - (w - \underline{w} + \ell] \left(\frac{g}{\ell}\right)^{k^*(w-\underline{w}+2\ell)} r(w)$$

if $x \geq w - \underline{w} + 2\ell$.

(b)

$$U(w+x) - U(w) \leq \sum_{i=0}^{k^{**}(x)} \left(\frac{\ell}{g}\right)^{i} r(w)$$

if $x \leq \bar{w} - w$, and

$$U(w+x) - U(w) \leq \sum_{i=0}^{k^{**}(\bar{w})} \left(\frac{\ell}{g}\right)^{i} r(w) + [x - \bar{w}] \left(\frac{\ell}{g}\right)^{k^{**}(\bar{w})} r(w)$$

if $x \geq \bar{w} - w$,

where, letting int(y) denote the smallest integer less than or equal to y, $k^*(x) \equiv int(x/2\ell)$, $k^{**}(x) \equiv int((x/g)+1)$, and $r(w) \equiv U(w) - U(w-\ell)$.

So we have that expected utility theory gives absurd results. What can we conclude? Rabin (2000) says:

> What is empirically the most firmly established feature of risk preferences, *loss aversion*, is a departure from expected utility theory that provides a direct explanation for modest-scale risk aversion (...) Variants of this or other models of risk attitudes can provide useful alternatives to expected utility theory that can reconcile plausible risk attitudes over large stakes with nontrivial risk aversion over modest stakes.

But wait, because if Rabin is mad, then Rabin and Thaler (2001) get even ... at a parrot:

> We feel much like the customer in the pet shop, beating away at a dead parrot. For nearly 50 years, economists have been fending of researchers who have identified clear departures from expected utility theory. (...) The expected utility model clearly has "beautiful plumage." But when the model is plainly wrong and frequently misleading, at some point economists must conclude that the plumage doesn't enter into it. Even the obstinate shopkeeper finally admitted the parrot was dead and conceded: "I had better replace it, then."

Should we burn the house down in outrage? Some authors say no. LeRoy (2003) gives three important points:

- Rabin (2000) and Rabin and Thaler (2001) show that if an agent rejects a 50-50 gamble of win-11 / lose-10 gamble, he should reject a 50-50 gamble of losing 100 while winning ∞.

- Do people actually reject these bets? Let $U(x)$ be $-e^{-\alpha x}$. Then, the expected payoff of 365 independent repetitions is 182.5; the standard deviation is 200.6. But this is way better than what we normally get in the stock market!

- Every day individuals mantain positions in risky portfolios that are worse than Rabin-Thaler's win-11 / lose-10 gamble. Then, in terms of what they actually do, individuals accept Rabin-Thaler's gamble. Contradiction!

Also, you knew that Rubinstein (2006) *had to say something* about this result:

> Should we be as concerned with an absurd conclusion reached from sound assumptions? (...) Does an absurd conclusion require us to abandon an economic model? (...) Do we economists take our own findings seriously? (...) Unlike parrots, human beings have the ability to invent new ways to reason that will clash with any theory (...) I doubt there is any set of assumptions that does not produce absurd conclusions when applied to circumstances far removed from the context in which they were conceived.

Yet another way to evaluate this result is given by Palacios-Huerta and Serrano (2006): They note that Rabin's claim is $(p \wedge q) \Rightarrow r$, where p is "risk-averse expected utility maximizer", q is "rejects modest gamble X", and r is "rejects large gamble Y". After some new calibrations, they conclude:

> The assumption that a person turns down gambles where she loses $100 or gains $110 for *any* initial wealth level implies that the coefficient of relative risk aversion must go to infinity when wealth goes to infinity, while the assumption that a 50-50 lose $100 / gain $105 bet is turned down for any lifetime wealth level less than $350,000 implies a value of the same coefficient no less than 166.6 at $350,000.

So does q really hold? They don't think so. And they sum it up very well:

In a more recent paper, Rabin and Thaler (2001) continue to drive home the theme of the demise of expected utility and compare expected utility to a dead parrot from a Monty Python show. To the extent that all their arguments are based on the calibrations in Rabin (2000), the expected utility parrot may well be saying that "the report of my death was an exaggeration."