Hidden Actions, Risk-Taking, and Uncertainty Shocks*

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Abstract

This paper explores a new mechanism through which increased uncertainty lowers aggregate output. In this mechanism, increased uncertainty causes endogenous active destruction of production units because of agency problems. This is in contrast with inactive waiting predicted in the real options theory. To demonstrate the mechanism, I develop a continuous-time dynamic matching model between principals and agents with long-term optimal contracts. In the model, a principal and an agent form a team to run risky projects and adjust their level of risk-taking. Because the agent can divert project payoffs, this agency relationship becomes hard to maintain when higher uncertainty increases the variance of the risky projects and generates more fluctuations in project outcomes. Therefore, teams close to termination break up instantaneously when uncertainty increases, causing an immediate reduction in aggregate output since it takes time to set up new teams. In addition, the average level of risk-taking declines among remaining teams because teams with a history of low output take less risk in response to the exogenous increase in project riskiness, in order to reduce the probability of costly separation. This further reduces aggregate output because low-risk projects have low average returns.

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1 Introduction

Uncertainty about the future appears to discourage real economic activities.[1] Previous literature explores mechanisms through which increased uncertainty lowers output by applying the real options theory.[2] In the real options view, firms become inactive and stop hiring new workers or investing in new projects. The theory also implies that firms will stop firing existing employees or closing existing factories when uncertainty increases. Therefore, this view needs to rely on a high exogenous job separation rate and capital depreciation rate to explain a reduction in factor inputs when uncertainty increases. The question is whether there is any channel through which increased uncertainty triggers endogenous job separation or factory closure.[3] In this paper, I explore such a new mechanism, which I call the "fragile agency" view of uncertainty shocks, focusing on agency problems derived from asymmetric information.

To demonstrate the fragile agency view, I develop a dynamic matching model between principals and agents with optimal long-term contracts. My starting point is the model by DeMarzo and Sannikov (2006), which is a continuous-time principal-agent model with endogenous termination of contract, where a principal and an agent run risky projects while the agent can undertake hidden actions to divert project payoffs. I extend their model in three ways. First, I embed the agency problem in an economy with a simple matching structure between unmatched principals and agents. It enables us to analyze macroeconomic consequences of individual long-term contracts. Second, I incorporate uncertainty shocks, that is, uniform changes in the

[1]Bloom (2009) shows a stark correlation between aggregate output and a stock-market volatility index, which is a proxy to uncertainty in the economy. Also, Bloom, Floetotto, and Jaimovich (2009) document that a number of proxies to uncertainty are strongly counter-cyclical.

[2]The seminal papers that study the real options theory are McDonald and Siegel (1986), Dixit and Pindyck (1994), and Abel and Eberly (1996). Recently, Stokey (2008) studies economics of inactions due to fixed costs, and also explains useful continuous-time methods. Bloom (2009) and Bloom, Floetotto, and Jaimovich (2009) apply the real options theory to study uncertainty shocks.

[3]According to the Job Openings and Labor Turnover Survey (JOLTS), the hiring rate in total nonfarm employment declined from 3.9% to 3.2% from December 2006 to December 2008 in the United States. The firing rate increased from 1.3% to 1.9% during the same period. The change in the firing rate was of the same size as in the hiring rate. The unemployment rate increased from 4.4% to 7.4% during the same period.

variance of the risky projects. Finally, I add a project portfolio choice that changes both the mean return and the variance of the risky projects. Hence, we can analyze the optimal response to the exogenous increase in project riskiness caused by higher uncertainty.

There are two channels through which increased uncertainty lowers aggregate output in this model. First, the agency relationships become hard to maintain when higher uncertainty increases the variance of the risky projects and generates more fluctuations in project outcomes. Therefore, contracts close to termination break up instantaneously when uncertainty increases. It triggers an immediate reduction in aggregate output since it takes time to set up new contracts.

Second, the average level of risk-taking declines in response to the exogenous increase in project riskiness. This is because contracts with a history of low output take less risk to reduce the probability of costly termination of the contract. This further reduces aggregate output because low-risk projects have low average returns.[4]

The primary contribution of this paper is to provide a new insight to uncertainty shocks, by showing that increased uncertainty can generate endogenous active destruction of production units such as job separation and factory closure.[5] This is a sharp contrast to the real options view, in which firms wait to fire workers and/or to close factories since the option value of waiting is high when uncertainty increases. In the fragile agency view, increased uncertainty directly lowers the principal's value of the contract because increased fluctuations in project outcomes worsen agency problems. One of the responses to such a reduction in the value of the contract is to immediately terminate the current contract and start over with another agent. Another response is to take less risk to reduce fluctuations in project outcomes so that they can reduce the probability of costly termination. Which

[4]One of the interesting applications of the theory in this paper is a financial crisis. The implied volatility index typically jumps up in a financial crisis. In response to such a rise in uncertainty, financial institutions tend to take less risk, as is known as the flight to quality. See, for example, Bernanke, Gertler, and Gilchrist (1996) and Caballero and Krishnamurthy (2008). The implication that aggregate output declines at the intensive margin due to less risk-taking is fairly consistent with the fact that there is a decline in measured productivity in a typical financial crisis, although measured productivity went up in the 2007-2008 financial crisis in the United States.

[5]A slight different way to model marketplace uncertainty is a rise in cross-sectional dispersion of firm-level credit risk, which worsens adverse selection problems in financial markets. See Gilchrist, Sim, and Zakrajsek (2009) and Shourideh and Zetlin-Jones (2009).

action to take depends on the history of project outcomes.

Recently, Arellano, Bai, and Kehoe (2010) also explore the mechanism that increased uncertainty triggers less risk-taking and thus lowers output due to a risk-return trade-off.[6] In their model, when uncertainty increases, firms take less risk to reduce the probability of costly default. This mechanism works because of the assumption of a reduced-form incomplete market in their model, whereas it works as the results of the optimal contract in the presence of agency problems in my model.

This paper also contributes to the literature of continuous-time agency models with endogenous termination.[7] I construct a tractable way to embed the optimal contracting problem in a general-equilibrium environment. The tractability stems from the fact that I can avoid having an additional state variable in the optimal contracting problem, by assuming a constant arrival rate of a new match for unmatched principals, and assuming a much larger number of agents than principals that results in a constant outside-option value for agents. The way I incorporate aggregate states of the economy follows Piskorski and Tchistyi (2010). My computational strategy follows the Appendix of DeMarzo, Fishman, He, and Wang (2009). The addition of a project portfolio choice results in a similar formula as in Merton (1973), because of a similar mean-variance trade-off.

The rest of the paper is organized as follows. Section 2 presents the model economy. Section 3 derives the optimal contract. Section 4 discusses the results. Section 5 concludes. All the proofs of lemmas and propositions and other technical issues are in Appendix.

[6] Note also that many papers in the literature of economic development explore the idea that less risk-taking leads to lower TFP growth in developing countries due to unstable financial systems. For example, Obstfeld (1994), Acemoglu and Zilibotti (1997), and recently, Aghion, Angeletos, Banerjee, and Manova (2010).

[7] The literature includes Biais, Mariotti, Plantin, and Rochet (2007), Biais, Mariotti, and Rochet (2010), DeMarzo and Sannikov (2006), DeMarzo, Fishman, He, and Wang (2009), He (2008), Hoffmann and Pfeil (2010), Piskorski and Tchistyi (2010), and Sannikov and Phillipon (2007). Note that there are significant differences between the models with and without endogenous termination, in terms of the nature of models. Li (2010) studies a continuous-time agency problem similar to the one in this paper without endogenous termination. His model does not have a matching environment between unmatched principals and agents because of no termination.

2 Model

Following the setup in DeMarzo and Sannikov (2006), I build a continuous-time dynamic principal-agent model, where a principal and an agent form a team to run risky projects. The agent can divert project payoffs for his/her own benefit because project payoffs are not publicly observable.

There are three additions to the model in DeMarzo and Sannikov (2006). First, I set up a dynamic matching structure between principals and agents who are out of contracts. This structure endogenously determines the value at termination of contract. Second, I incorporate an aggregate uncertainty shock as a change in aggregate state that follows a two-state Markov switching process, following Piskorski and Tchistyi (2010) and DeMarzo, Fishman, He, and Wang (2009). Finally, I introduce a control variable that changes the mean return and the volatility of the risky projects, in order to allow for endogenous responses to exogenous changes in project riskiness. Due to a risk-return trade-off, low-risk projects have low average returns.

In this economy, aggregate output fluctuates because of endogenous terminations and risk-taking choices. Endogenous terminations of contracts reduce the number of active production units, and hence, reduce aggregate output at the extensive margin because there is a delay to set up a new contract. Risk-taking choices in ongoing contracts alter the mean flow of output, and thus, change aggregate output at the intensive margin; high average risk-taking increases output, and low risk-taking decreases output.

I set up the primitives of the model economy in this section, and derive the optimal long-term contract in section 3.

2.1 Environment

Time is continuous and the time horizon is infinite. There are a continuum of principals with measure M_P and agents with measure $M_A \gg M_P$. Both are infinitely lived and risk neutral. The discount rate is r for principals and is $\rho > r$ for agents. Agents are more impatient than principals. A principal and an agent can form a team to run risky projects that generate a random flow of output.

There is a matching structure between principals and agents. The team production requires agent's special management skill and a new idea created by principals. Unmatched principals take time to create a new idea that is required for a new match, without producing anything. This waiting time for a new idea is random and follows an exponential distribution with a constant arrival rate α. Unmatched principals who have created a new idea can immediately pick an agent searching for a new match, if any, and start the team production. Since ideas cannot be recycled, the waiting time makes termination of team costly.

On the other hand, unmatched agents can produce at home and generate flow h of output. They also can search for a new match, and generate flow $u < h$ of output. The inequality reflects the cost of searching for a new match. Both principals and agents are anonymous, and thus, the probability of getting a new match for searching agents, denoted by π_t, is the fraction of the measure of searching agents over the measure of unmatched principals who have created an idea:

$$\pi_t = \min\left\{\frac{\alpha(M_P - K_t)}{L_t}, 1\right\}, \tag{1}$$

where K_t is the measure of active matches, and L_t is the measure of searching agents. K_t evolves according to

$$\mathrm{d}K_t = \alpha(M_P - K_t)\mathrm{d}t - K_t \times (\text{termination rate})\ \mathrm{d}t + (\text{jump term}),$$

where (termination rate) and (jump term) are determined as the results of the optimal contracting problem.

This matching structure is different from a standard search-and-matching setting, and results from a simplification for tractability. The assumption of a much larger measure of agents than principals ensures that there is always a larger measure of searching agents than the measure of unmatched principals who have created a new idea, and therefore, ensures that π_t is always less than one. It leads to a constant agent's value outside a match. The constant arrival rate of a new idea assures that the principal's value at termination of team only depends on aggregate shocks, and does not depend on the measures of unmatched principals and agents.

2.2 Team Production

If a principal and an agent form a team, they produce a random flow of output that follows the process below:

$$\mathrm{d}Y_t = (\mu(a_t) - b_t)\mathrm{d}t + a_t\sigma_t\mathrm{d}W_t,$$

where Y_t is cumulative output from the team production at time t, $a_t \in [0, 1]$ is the level of risk-taking, $\mu(a_t)$ is the mean return associated with a_t, b_t is a hidden action by the agent, W_t is a standard Wiener process that represents idiosyncratic shocks and is independent across matches, and σ_t is the instantaneous volatility of the output.[8]

I specify $\mu(a_t)$ by

$$\mu(a_t) \quad = \quad m_L + (m_H - m_L)a_t, \qquad m_L < m_H,$$

with an interpretation that a_t is a project portfolio choice, which exhibits a risk-return trade-off. They can adjust their level of risk-taking, as they change their project portfolio between safe, low-return projects and risky, high-return projects. I exclude short-selling by assuming that there is an upper bound for a_t.

Agent's hidden action b_t generates output X_t, which can be privately consumed:

$$\mathrm{d}X_t = \lambda b_t \mathrm{d}t, \qquad b_t \geq 0,$$

where $\lambda \in (0, 1)$ is a parameter that reflects the cost of cash diversion. Non-negativity of b_t implies that the agent cannot secretly generate Y_t at the cost of X_t.[9]

The volatility σ_t of idiosyncratic shocks changes according to the state of the economy. There are two states of the economy: normal and crisis states, denoted by $s_t \in \{\mathcal{N}, \mathcal{C}\}$. It follows a two-state Markov-switching process. Let N_t be a counting process over regime switches, whose arrival rate is $\eta_t \in \{\eta_\mathcal{N}, \eta_\mathcal{C}\}$. Note that η_t depends on the state of the economy so that the probability of getting into the crisis state can be different from the probability of getting out of the crisis state. The volatility σ_t takes two values, $\{\sigma_\mathcal{N}, \sigma_\mathcal{C}\}$ with $\sigma_\mathcal{N} < \sigma_\mathcal{C}$, according to the state of the economy. This creates uncertainty shocks in this economy.

[8] In this specification, dY_t becomes negative sometimes, resulting in negative production. This can be interpreted as the case where the production technology requires consumption goods as an input in advance, and ends up with producing less than the input with some probability. There exists a discrete-time model with this type of production technology that converges to the continuous-time production technology in this paper. See Biais, Mariotti, and Rochet (2010) for a clear exposition of a limiting argument from a discrete time setting to a continuous time setting.

[9] Also, b_t is assumed to be a *cadlag*, i.e., right-continuous-left-limit, function of time for integrability. The non-negativity may be removed by applying the same argument in DeMarzo and Sannikov (2006), who allow for agent's hidden saving.

2.3 Preferences

Both the agent and the principal are risk neutral and their instantaneous utility function is $u(c) = c$. The discount rate is ρ for the agent and r for the principal, where $\rho > r$.

The agent's total expected payoff in a team at time t is

$$V_t = \mathrm{E_t}\left[\int_t^{\tau} e^{-\rho(u-t)}(\mathrm{d}C_u + \lambda b_u \mathrm{d}u) + e^{-\rho(\tau-t)}U_\tau\right], \tag{2}$$

where τ is a stopping time that is the termination date of the team, C_t is the cumulative consumption of the agent at time t, U_t is the agent's value at termination of team at time t. I assume $\mathrm{d}C_t \geq 0$, that is, agent's publicly observable consumption flow must be non-negative for all t. It reflects the limited liability of the agent.

The principal's total expected payoff at time t is given by

$$J_t = \mathrm{E_t}\left[\int_t^{\tau} e^{-r(u-t)}(\mathrm{d}Y_u - \mathrm{d}C_u) + e^{-r(\tau-t)}S_\tau\right]. \tag{3}$$

where S_t is the principal's value at termination of team. The matching structure in this economy endogenously determines the values at termination of team in a forward looking way.

2.4 Information, Commitment, and Bargaining Power

The information structure in a team is as follows. The agent observes everything, whereas the principal observes Y_t, σ_t, and a_t, but not W_t, b_t, or X_t. The story behind this assumption is that the principal cannot observe agent's action a_t and b_t, but can figure out a_t from realizations of Y_t for a very short time interval.[10] The key information asymmetry is that the principal cannot identify whether b_t or W_t creates fluctuations in Y_t.

I assume that principals can commit, but agents cannot. Agents are free to leave a team to take an outside option. One-sided commitment implies $V_t \geq U_t$ for all t.

Lastly, I assume that principals can make a take-it-or-leave-it offer to a newly matched agent, and thus, they start a new match by promising agents with the value where the principal's value is maximized.

[10]This is because of the property of the Wiener process. This is not the case if we use a Poisson process, instead. Note that b_t cannot be figured out even though we can observe an infinitely many observations of Y_t in a very short time.

2.5 Endogenous Termination Values

Before discussing the optimal contracting problem, I explain how termination values are endogenously determined in this economy because it simplifies the exposition hereinafter. First of all, I restrict parameters such that $rS_t + \rho U_t < \mu(1) = m_H$ holds, to make sure that the team production is profitable to run without agency problems. Next, I explain that the agent's value at termination is constant over time, i.e., $U_t = U$, and that the principal's value at termination only takes two values depending on aggregate state, i.e., $S_t \in \{S_{\mathcal{N}}, S_{\mathcal{C}}\}$.

By the assumption of a much larger measure of agents than principals, the probability of getting a new match for searching agents can be so small that unmatched agents are indifferent between home production and searching for a new match. Thus, we have

$$h = \rho U_t = u + \pi_t (V_{0,t} - U_t), \tag{4}$$

where $V_{0,t}$ is the initial promised utility in a new match at time t, which is determined as the results of the optimal contracting problem in the next section. Therefore, we have $U_t = U := h/\rho$.

This enhances the tractability of the optimal contracting problem. The value of contracts does not depend on the measures of unmatched agents and principals if U_t is constant and the arrival rate α of a new match for principals is also constant. It turns out that we can describe the optimal contract by two value functions for the principal, $\{J_{\mathcal{N}}(V), J_{\mathcal{C}}(V)\}$, where V is the current promised utility for the agent.

As a result, the principal's value at termination is endogenously determined in two values, $\{S_{\mathcal{N}}, S_{\mathcal{C}}\}$, such that

$$rS_s = \alpha(J_s(V_{0,t}) - S_s) + \eta_s(S_{-s} - S_s), \qquad s \in \{\mathcal{N}, \mathcal{C}\}, \tag{5}$$

where subscript $-s$ is for the other aggregate state, and the assumption of full bargaining power for the principal leads to

$$V_{0,t} = \max_V J_s(V), \tag{6}$$

if aggregate state at time t is $s \in \{\mathcal{N}, \mathcal{C}\}$.

3 Optimal Contract

This section derives the optimal contract to form a team between a principal and an agent in the economy described in Section 2. First, I discuss the first best contract and the optimal contract without hidden actions, but with limited commitment. Next, I derive the optimal contract in the baseline model. The derivation of the optimal contract basically follows DeMarzo and Sannikov (2006, DS hereinafter). Finally, I describe how the optimal contract works.

The definition of the contract space is as follows. The contract (a_t, b_t, C_t, τ) determines planned risk-taking schedule a_t, planned hidden action b_t, agent's cumulative compensation C_t, and termination date τ based on the history of realized project outcomes $Y = \{Y_t\}_{t\geq 0}$ and the history of aggregate state $N = \{N_t\}_{t\geq 0}$. Technical assumptions on the contract space are summarized in section A.1 in Appendix.

3.1 First Best

In the first best case, there is no hidden action and the agent can commit to the contract. In this case, it is optimal to take maximum risk ($a_t = 1$) to enjoy the highest mean output, and not to terminate the contract forever ($\tau = \infty$). The agent with promised value V receives initial lump-sum payment V, and no flow payment thereafter. The value function of the principal is

$$J_s(V) = \frac{\mu(1)}{r} - V, \tag{7}$$

for each $s \in \{\mathcal{N}, \mathcal{C}\}$. Any new contract starts with $V = U$ and all contracts last forever.

Note that the value of the contract does not depend on aggregate state at all. This is because uncertainty shocks do not change the mean output of the team.

3.2 No Hidden Action and Limited Commitment

Here, I consider the case where there is still no hidden action, but the agent cannot commit to the contract. In this case, it is still optimal to take maximum risk ($a_t = 1$) to enjoy the highest mean output, and not to terminate the contract forever ($\tau = \infty$). Any new contract starts with $V = U$ and all contracts last forever.

The difference from the first best case is that the principal needs to give the agent the right incentive to stay with the contract. Therefore, the promised value must be greater than or equal to U. Hence, the agent with promised value V receives initial lump-sum payment $V - U$, and constant flow payment ρU thereafter. The value function of the principal is

$$J_s(V) = \frac{\mu(1) - \rho U}{r} - (V - U) \quad < \quad \frac{\mu(1)}{r} - V.$$

for each $s \in \{\mathcal{N}, \mathcal{C}\}$. The value is lower than in the first best case because of $\rho > r$.

The limited commitment requires the principal to delay the payment to the agent, relative to the first best case. Such delayed payments are costly for the principal because the discount rate of the agent is higher than the discount rate of the principal. In other words, the gap in the discount rates makes it costly to delay the payment to the agent because promise-keeping requires to increase the delayed payment at a higher rate than the discount rate of the principal. Therefore, the gap creates the tension between incentive provision and the timing of payments, which is a key to understanding the baseline model in the next subsection.

The basic idea is that incentive provision requires the principal to delay the payment to the agent; however, it is costly to do so because of the difference in the discount rates. Therefore, in the baseline case there is an optimal threshold, below which the agent is never paid, and at which the agent receives a transfer so that the promised value stay at the threshold. Such an extreme payment schedule stems from linear preferences, and the same is true in DS.

As a preview of the results, Figure 1 shows the Pareto frontiers without aggregate shocks for each case: first best, limited commitment only, and the baseline case. The principal's value is lower in the baseline case than in the limited commitment case because of the cost of delayed payment and costly termination. In the baseline case, the agent's promised utility V fluctuates along with the Pareto frontier due to incentive compatibility constraints, and reflects at an upper boundary B, which is determined as the results of the optimal contracting problem. The contract will terminate when the agent's promised value reaches a lower boundary U, which is the agent's outside-option value.

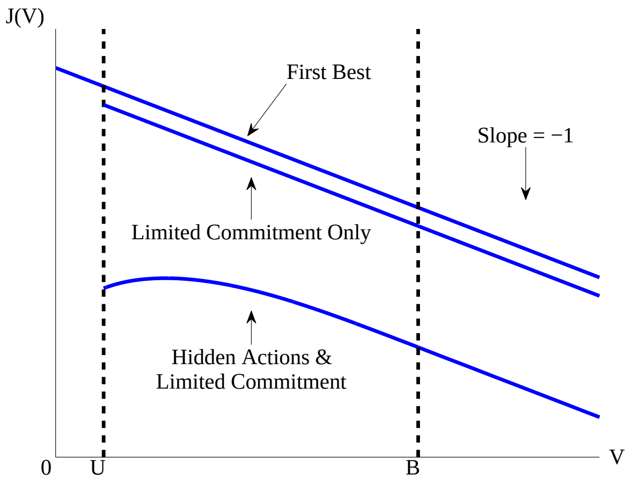

Figure 1: The Pareto frontiers.

No regime switch. B is an endogenous threshold determined as the results of the optimal contracting problem. There is no such threshold in the first best and limited commitment only cases. Above B, all three frontiers have a slope of -1.

3.3 Hidden Actions and Limited Commitment

This is the baseline case where we consider both hidden actions and limited commitment. The optimal contract maximizes the principal's expected payoff subject to the constraints needed to incentivize the agent not to divert cash flow and not to leave the contract. First of all, observe that $b_t = 0$ is always optimal because $\lambda < 1$.

The agent's promised utility is V_t can be written in a differential form as follows:

$$dV_t = \underbrace{\rho V_t \mathrm{d}t - \mathrm{d}C_t}_{\text{promise-keeping}} + \underbrace{g_t[\mathrm{d}Y_t - \mu(a_t)\mathrm{d}t]}_{\text{incentive provision}} + \underbrace{\Delta_t[\mathrm{d}N_t - \eta_t \mathrm{d}t],}_{\text{adjustment for regime switches}} \tag{8}$$

where g_t is the sensitivity to the change in current output, and Δ_t is the sensitivity to the change in aggregate state. The agent's promised utility V_t evolves according to this accounting equation, starting from an initial value V_0.

The incentive compatibility in this problem turns out to be fairly simple. The agent who steals b dollars can privately consume λb dollars, and decreases the current output by b dollars. According to (8), it decreases the agent's future promised value by $g_t b$. Therefore, incentive compatibility requires $g_t \geq \lambda > 0$. In fact, Lemma 2 in Appendix shows that $g_t \geq \lambda$ is a necessary and sufficient condition for incentive compatibility, as it is ture for the case with DS.

The participation constraints require $V_t \geq U$ for all t. Note that $V_t = U$ is not incentive compatible, and thus, termination date τ satisfies $\tau = \min\{t | V_t = U\}$. This is because incentive compatibility requires V_t to fluctuate together with output process Y_t, and hence, does not allow V_t to stay at U. Intuitively, the principal cannot further punish the agent if V_t is already set at the outside-option value U, and hence, cannot give the right incentive to the agent.

In summary, the optimal contracting problem is formulated as follows.[11] Given some $V_0 \geq U$, the principal solves

$$\max_{(a_t, C_t, g_t, \Delta_t, \tau)} \mathrm{E}_0 \left[\int_0^\tau e^{-rt}(\mathrm{d}Y_t - \mathrm{d}C_t) + e^{-r\tau} S_\tau \right]$$

$s.t.$

$$(PK) \quad \mathrm{d}V_t = \rho V_t \mathrm{d}t - \mathrm{d}C_t + g_t[\mathrm{d}Y_t - \mu(a_t)\mathrm{d}t] + \Delta_t[\mathrm{d}N_t - \eta_t \mathrm{d}t],$$

$$(IC) \quad g_t \geq \lambda,$$

$$(RC) \quad dY_t = \mu(a_t)dt + a_t \sigma_t dW_t,$$

for all t, and

$$(PC) \quad \tau = \min\{t | V_t = U\}, \quad V_t \geq U.$$

I solve this problem by following the same procedure in DS. First, focus on Markov contracts characterized by two value functions. Next, to formulate the Hamilton-Jacobi-Bellman equation, assume in advance that the value functions are concave and that there are thresholds for the promised utility, below which no payment to the agent is optimal. Finally, after obtaining the solution of the HJB equation, verify that the assumptions made in advance

[11] I formally derive this problem from the original problem in Section A.2 in Appendix.

are, in fact, satisfied by the solution, and that the optimal Markov contract is the solution to the optimal contracting problem above.

The optimal contract, derived from the procedure above, is characterized as follows.

Proposition 1 The contract that maximizes the principal's expected payoff given the promised value $V_0 \geq U$ to the agent takes the following form.

(i) The principal's expected value can be obtained from a set of concave functions, $\{J_s(V)\}_{s\in\{N,C\}}$, each of which satisfies

$$rJ_s(V) = \max_{\substack{a \in [0,1], \\ g \geq \lambda, \\ \Delta \geq -V+U}} \mu(a) + (\rho V - \eta_s \Delta)J_s'(V) + \frac{1}{2}g^2\sigma_s^2(a)J_s''(V) + \eta_s(J_{-s}(V+\Delta) - J_s(V)). \quad (9)$$

for $V \in (U, B_s)$, and satisfies

$$J_s(V) = J_s(B_s) - (V - B_s). \quad (10)$$

for $V \geq B_s$, as well as boundary conditions $J_s(U) = S_s$ with (5), where upper threshold B_s satisfies $J_s'(B_s) = -1$ and $J_s''(B_s) = 0$.

(ii) No hidden action is planned: $b_t = 0$.

(iii) Initial promised value $V_{0,t}$ is given by (6).

(iv) The agent's promised value V_t evolves according to (8) with $g_t = \lambda$.

(v) No payment is given ($\mathrm{d}C_t = 0$) if $V_t \in [U, B_s)$. If $V_t = B_s$, then payment $\mathrm{d}C_t$ causes V_t to stay at B_s.

(vi) Termination date is given by $\tau = \min\{t | V_t = U\}$.

(vii) Planned adjustment Δ_t in response to regime switches satisfies

$$\Delta_t = \Delta_s(V_t) := \begin{cases} -V_t + U & \text{if} \quad U < V_t \leq \underline{V_s} \\ [J_{-s}']^{-1}(J_s'(V_t)) - V_t & \text{if} \quad \underline{V_s} < V_t \leq B_s \end{cases} \quad (11)$$

where $\underline{V_s}$ is, if any, a threshold that solves

$$J_{-s}'(U) = J_s'(\underline{V_s}).$$

(viii) Planned risk-taking level a_t satisfies

$$a_t = a_s(V_t) := \begin{cases} 0 & \text{if} \quad U < V_t \leq F_s \\ \frac{m_H - m_L}{-J_s''(V_t)\lambda^2\sigma_s^2} & \text{if} \quad V_t \leq A_s \\ 1 & \text{if} \quad A_s < V_t \leq B_s \end{cases}$$

where F_s and A_s are, if any, thresholds that solve

$$\begin{aligned} rJ_s''(F_s) &= m_L + (\rho F_s - \eta_s \Delta_s(F_s))J_s'(F_s) + \eta_s(J_{-s}(F_s + \Delta_s(F_s)) - J_s(F_s)), \\ J_s''(A_s) &= \frac{m_H - m_L}{-\lambda^2\sigma^2}. \end{aligned}$$

First of all, properties (ii) - (vi) are basically the same as in DS. The equality $g_t = \lambda$ in property (iv) stems from the concavity of the value functions. Property (v) reflects what I describe in Section 3.2. The agent's promised utility V_t has to fluctuate according to the fluctuations in current output Y_t because of incentive compatibility. Thus, V_t sometimes reaches the outside-option value U after a period of low output realizations. Hence, the principal would not like to pay to the agent because doing so reduces the promised value and increases the probability of costly termination. However, such a delayed payment is costly because the discount rate is higher for the agent than the principal. The optimal strategy with linear preferences is to pay nothing when the promised utility is low and to start paying when the promised utility is high enough.

Next, observe that (9) in property (i) has two deviations from DS: risk-taking choices a and a jump term associated with Δ. The optimal level of a and Δ are determined by the first order conditions in the case of interior solutions, which are described in property (vii) and (viii). The promised value process also has a jump term according to regime switches, as in (8).

To demonstrate what the optimal contract looks like, I first explain the case without regime switches. Figure 2 shows an example of the Pareto frontier with the optimal level of risk-taking in the case without regime switches. As is shown in property (viii), the optimal contract requires less risk-taking as the promised utility goes down. Note that this is the case where the contract always requires a positive level of risk-taking. I focus on such cases because there will be no endogenous termination otherwise.

Next, Figure 3 shows the Pareto frontier with regime switches. The first order condition in (9) with respect to Δ gives

$$J_{-s}'(V + \Delta) = J_s'(V), \tag{12}$$

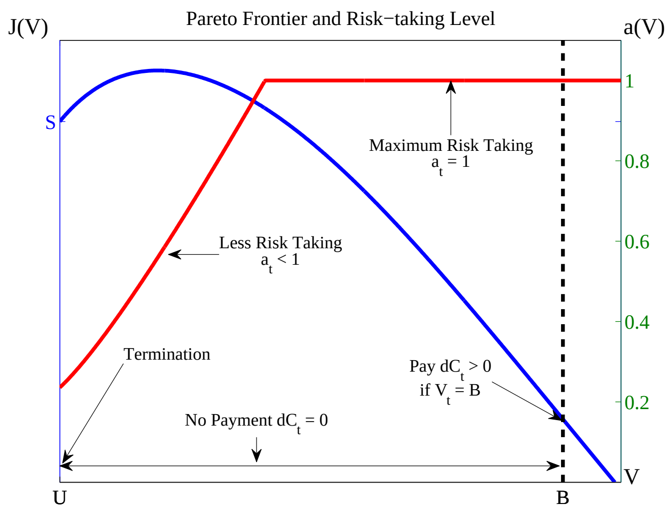

Figure 2: The optimal contract with risk-taking choices.

No regime switch. The y-axis on the left shows the principal's value of the contract. The y-axis on the right shows the optimal level of risk-taking.

for each $s \in \{\mathcal{N}, \mathcal{C}\}$, if Δ is an interior solution. This means that the optimal adjustment in response to regime switches is to move to the point at which the slope is the same. The intuition is that the principal's marginal cost of promising the value after the regime switch must be equal to the marginal cost of promising the previous value before the regime switch.

The important case is the one where there is no point that has the same slope. Since the value functions are concave and have a slope of negative one at the upper boundary, this happens only when the point to jump to is lower than the outside-option value U. In this case, the contract terminates instantaneously when the regime switches. Therefore, increased uncertainty leads to active destruction of contracts with low promised utilities, specifically below $\underline{V_s}$ defined in property (vii).

This active destruction is a sharp contrast to what the real options theory predicts, that is, inactive waiting when uncertainty increases. The intuition is that higher uncertainty directly lowers the value of contract by increasing fluctuations in project outcomes, and thus, low value contracts with low promised utilities become no longer profitable. Note that there is no value of waiting in this economy since the output process is independently and

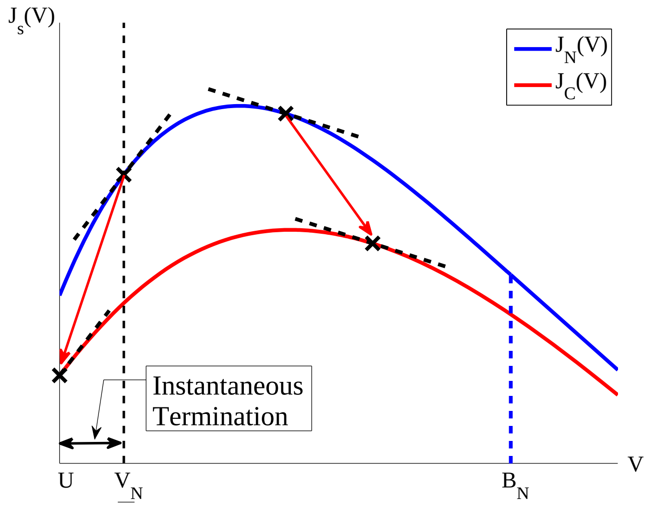

Figure 3: The optimal adjustment of promised utilities
in response to regime switches.

$\underline{V_{\mathcal{N}}}$ is the value at which the slope of $J_{\mathcal{N}}(V)$ is the same as the slope of $J_{\mathcal{C}}(V)$ at termination point U. Contracts with promised utilities below $\underline{V_{\mathcal{N}}}$ will be immediately terminated when uncertainty increases.

identically distributed for each time t.

4 Results

This section discusses the implications of the optimal contract. The first subsection shows comparative statics for the model without risk-taking choices or aggregate uncertainty shocks. Then, I conduct a numerical experiment with a certain set of parameters. The purpose of the experiment is for a coherent illustration of the model economy. Next, I use the Kolmogorov's forward equation to analyze the dynamics of the promised utility distribution. The last subsection shows the results of a transition analysis using the method introduced in the previous subsection.

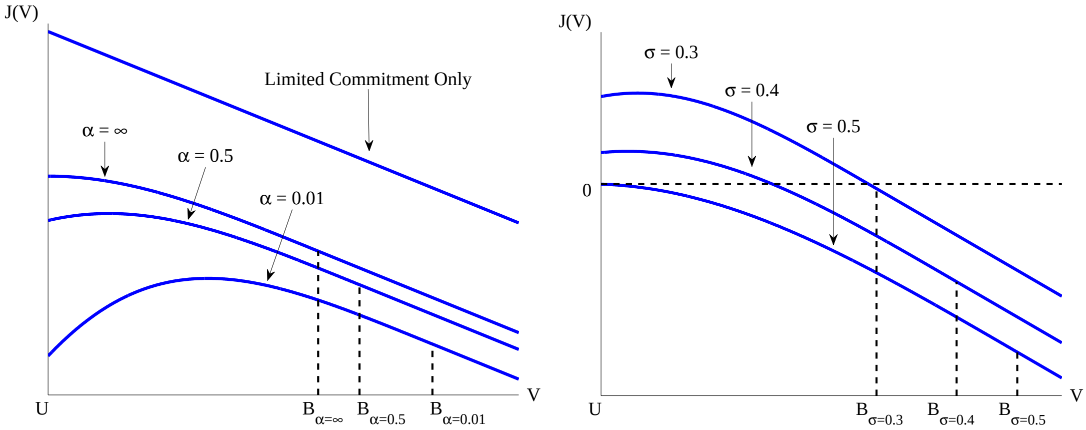

Figure 4: Comparative statics with respect to α and σ.

4.1 Comparative Statics for A Simplified Economy

I first consider a simplified economy without risk-taking choices or aggregate uncertainty shocks. The purpose is to demonstrate how the matching structure interacts with this type of contracts, in order to enhance the understanding of the matching structure in the baseline model. Specifically, I show comparative statics with respect to the arrival rate of new ideas α and the variance of risky projects σ.

The left panel of Figure 4 shows the Pareto frontiers with different values of the arrival rate of new ideas α. The principal's value of the contract goes down as α goes down. This is because the expected cost of termination increases since expected waiting time increases as α decreases. Note that it does not converges to the limited commitment case even if α goes to ∞. This is because of the cost of delayed payments needed to keep the agent from undertaking hidden actions.

The right panel of Figure 4 shows the Pareto frontiers with different values of the variance of risky projects σ. The principal's value of the contract goes down as σ goes up. The reason is the following. Take an optimal contract at some level of σ. Making σ higher increases the probability that V_t reaches two costly points, U and B, and hence, increases the expected cost for the principal given the same contract. The intuition is that higher σ increases both the expected costs of terminating the contract and rewarding the agent. In response, the optimal contract with higher σ sets B at a larger value to offset such an increase in the probability of incurring costs. However, larger

Table 1: The set of parameters in the numerical experiment

Parameter	**Value**
Principal's discount rate	$r = 0.02$
Agent's discount rate	$\rho = 0.04$
The mean return of risky projects	$m_H = 0.15$
The mean return of safe projects	$m_L = 0.02$
The variance in the Normal state	$\sigma_{\mathcal{N}} = 0.09$
The variance in the Crisis state	$\sigma_{\mathcal{C}} = 0.27 = 3\sigma_{\mathcal{N}}$
Effective cash diversion rate	$\lambda = 0.9$
The outside option	$U = 0.1$
New idea arrival rate	$\alpha = 0.5 = 1/(2 \text{ years})$.
Crisis arrival rate	$\eta_{\mathcal{N}} = 0.05 = 1/(20 \text{ years})$
Recovery arrival rate	$\eta_{\mathcal{C}} = 0.2 = 1/(5 \text{ years})$

B results in more delay in the payment to the agent, which is socially costly due to a gap in the discounting factors. Therefore, the surplus of the match in the optimal contract decreases as σ increases.

When σ is too large, the principal's value becomes always below zero. In this case, any contract is no longer profitable enough to start. That is, there is an upper bound for σ, above which no contract will be made. Note that $J(U)$ cannot be less than zero, and if it is zero, then the value is only attained by terminating the contract instantaneously when they make a new contract.

4.2 Numerical Experiment

In the following subsections, I show the results with a fixed set of parameters to demonstrate what the economy looks like and how individual contracts affect the macroeconomy. Table 1 summarizes the set of parameters that I use. The purpose of this numerical experiment is not a quantitative analysis, but an illustration of a coherent overview of the model economy.

Figure 5 shows the solution $(J_N(V), J_C(V))$ and $(\Delta_N(V), \Delta_C(V))$ under the set of parameters summarized in Table 1. As for $\Delta_s(V)$, I show $V+\Delta_s(V)$, rather than $\Delta_s(V)$ itself, to show the point to jump to after an uncertainty shock.

The agent's value will be spread out when aggregate state moves from

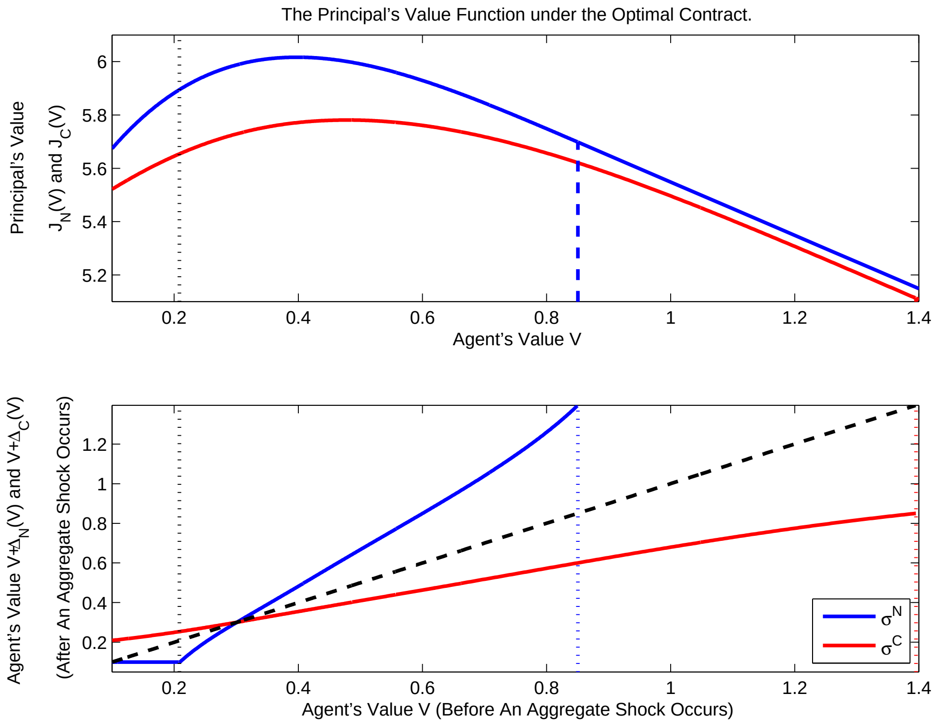

Figure 5: The Pareto frontiers and the change in the agent's value when uncertainty increases.

Use $r = 0.02, \rho = 0.04, m_H = 0.15, m_L = 0.02, \sigma_N = 0.09, \sigma_C = 0.27, \lambda = 0.9,\ U = 0.1, \alpha = 0.5, \eta_N = 0.05, \eta_C = 0.5$.

the normal state to the crisis state. Contracts with V_t lower than $\underline{V_N}$ will be immediately terminated when uncertainty increases. If V_t is low but higher than $\underline{V_N}$, then they keep the contract but decrease the agent's values. If V_t is high enough, then they increase V_t. The principal's value decreases when uncertainty increases.

Note that the mean return of the risky projects itself does not change across aggregate states. The reason why the contracts with $V_t \leq \underline{V_N}$ become no longer profitable is that higher uncertainty increases the possibility of costly termination in the presence of agency problems. Note also that, without agency problems, increased uncertainty does not affect aggregate output at all.

Figure 6 shows the optimal level of risk-taking in the model for each

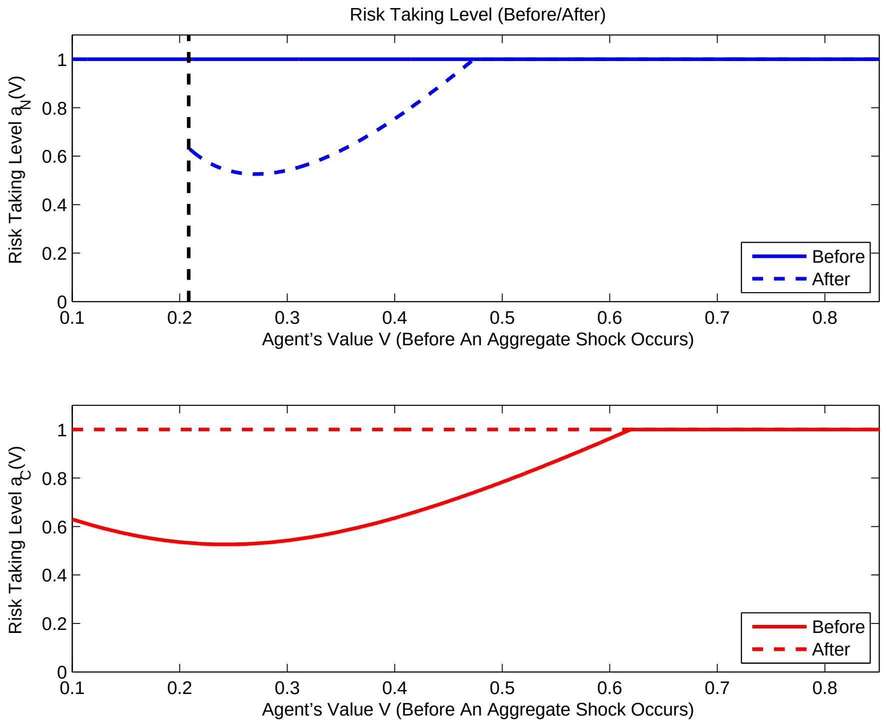

Figure 6: The optimal risk-taking level before/after uncertainty increases.

Use $r = 0.02, \rho = 0.04, m_H = 0.15, m_L = 0.02, \sigma_N = 0.09, \sigma_C = 0.27, \lambda = 0.9, U = 0.1, \alpha = 0.5, \eta_N = 0.05, \eta_C = 0.5$. Contracts will be immediately terminated if the agent's values are below the vertical dotted line.

aggregate state. With the set of parameters here, the optimal contract does not require to take less risk in the normal state. In the crisis state, on the other hand, the optimal contract requires the agent to take less risk if the agent's promised utility is lower than A_C defined in property (viii) of Proposition 1. Figure 6 also shows how the optimal level changes in response to an uncertainty shock. The optimal risk-taking level goes down in a region with low agent's values after an uncertainty shock.

I summarize the consequences of an uncertainty shock in Proposition 2.

Proposition 2 The optimal contract responds to an uncertainty shock as follows.

(i) It reduces the principal's value:

$$J_{\mathcal{N}}(V_t) > J_{\mathcal{C}}(V_t + \Delta_{\mathcal{N}}(V_t)). \tag{13}$$

(ii) It changes the agent's value V_t depending on the current value; in particular, it triggers instantaneous termination for contracts with $V_t \leq \underline{V_{\mathcal{N}}}$.

(iii) It reduces the average level of risk-taking a_t.

(iv) It leads to an immediate reduction in aggregate output both at the intensive and extensive margins.

4.3 Distribution of the Agent's Values

One of the advantages in continuous-time models is that we can characterize the distribution of state variables via partial differential equations.[12] That is, we can obtain a deterministic transition of the distribution of the agent's value, as a function of time and V_t given a sample path of aggregate states. Using this method, I conduct a transition analysis and show results in the next subsection.

Let $f_s(t,V)$ denote the density of the agent's value V at time t in state s. Based on the dynamics of V_t formulated as (8) and $g_t = \lambda$, $f_s(t,V)$ satisfies the following Kolmogorov forward equation.

$$\frac{\partial}{\partial t} f_s(t,V) = -\frac{\partial}{\partial V}\left[(\rho V_t - \eta_s \Delta_s(V)) f_s(t,V)\right] + \frac{1}{2}\frac{\partial^2}{\partial V^2}\left[\lambda^2 \sigma_s^2 a_s^2(V) f_s(t,V)\right], \tag{14}$$

with boundary conditions as follow:[13]

$$f_s(t,U) = 0, \tag{15}$$

$$-(\rho B_s - \eta_s \Delta_s(B_s)) f_s(t,B_s) + \frac{1}{2}\frac{\partial}{\partial V}\left[\lambda^2 \sigma_s^2 a_s^2(V) f_s(t,V)\right]\Bigg|_{V=B_s} = 0. \tag{16}$$

[12] Brunnermeier and Sannikov (2010) argue that this aspect of continuous-time models is more attractive than log-linearization around the steady state if we analyze large shocks, such as a financial crisis.

[13] See Chapter 3.8 in Stokey (2008). For appropriate boundary conditions, see Chapter 5.7 in Cox and Miller (1976).

Note that $f_s(t, V)$ is continuous in V due to continuous fluctuations in V_t due to the Wiener process W_t. An instantaneous outflow of terminated contracts at time t is determined by $\frac{1}{2}\frac{\partial}{\partial V}\left[\lambda^2\sigma_s^2 a_s^2(V) f_s(t, V)\right]\Big|_{V=U}$. Since there is an inflow at $V_s^* = \arg\max_V J_s(V)$, the first derivative of $f_s(t, V)$ is not continuous at V_s^*.[14]

When aggregate state changes, optimal adjustment $\Delta_s(V_t)$ will be applied. Let $F_s(t, V)$ be the distribution of the agent's value V at time t in state s. If aggregate state switches from state s to state $-s$ at time t, then we have

$$F_{-s}(t, V + \Delta_s(V)) = F_s(t, V), \tag{17}$$

or equivalently,

$$F_{-s}(t, V) = F_s(t, V + \Delta_{-s}(V)), \tag{18}$$

for $V > \underline{V_s}$.[15] Hence, we obtain

$$f_{-s}(t, V) = f_s(t, V + \Delta_{-s}(V))(1 + \Delta'_{-s}(V)), \tag{19}$$

for $V > \underline{V_s}$, which can be used to compute the density $f_s(t, V)$ right after the aggregate state changes.

4.4 Transition Analysis

This subsection shows the results of a transition analysis to demonstrate how aggregate output changes when an uncertainty shock hits the economy. The method in section 4.3 allows me to conduct a transition analysis given a fixed sample path of aggregate states.

I conduct the following transition exercise. First, set the initial distribution of all the agent's values of contracts in the economy at the limit distribution in the normal state, which is the distribution that arises if there is no change in aggregate state for a long time. Next, at period 2, aggregate state changes to the crisis state. Finally, it comes back to the normal state at period 4. One period corresponds to one year.

Figure 6 shows a time series of aggregate output in the transition. Aggregate output declines immediately when the economy is hit by an uncertainty shock. There is a jump at the point of recovery because ongoing contracts start taking more risk to earn the higher mean output.

[14] This can be seen by integrating (14) from U to V_s^* and from V_s^* to B_s. Also, see Luttmer (2007) for a similar argument.

[15] Note that for $V > \underline{V_s}$, if $V' = V + \Delta_s(V)$ then $V = V' + \Delta_{-s}(V')$ because Δ_s adjusts V so that the first derivative of the value functions are equalized.

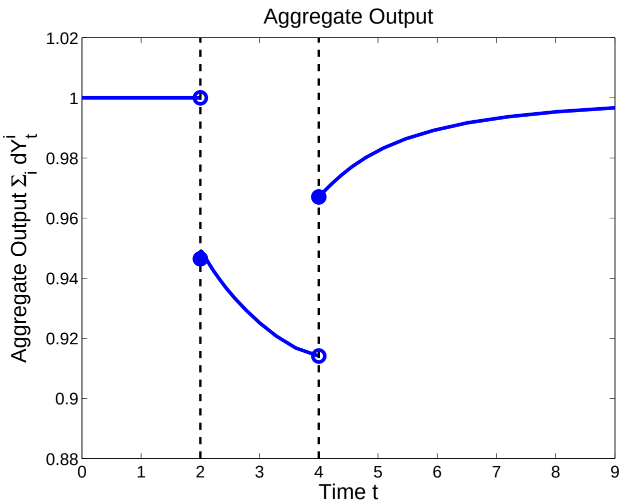

Figure 7: Time series of aggregate output in the transition exercise.

Starting from the distribution in the normal state without regime switches after a while. At period 2, uncertainty increases. At period 4, uncertainty comes down. Use $r = 0.02, \rho = 0.04, m_H = 0.15, m_L = 0.02, \sigma_N = 0.09, \sigma_C = 0.27, \lambda = 0.9,\ U = 0.1, \alpha = 0.5, \eta_N = 0.05, \eta_C = 0.5$.

5 Conclusion

Due to agency problems, increased uncertainty lowers aggregate output in two ways; (1) increased fluctuations in project outcomes lower the value of contracts and cause active termination for fragile contracts, and (2) less risk-taking in response to higher uncertainty lowers output due to a mean-variance trade-off. I demonstrate these effects by constructing a continuous-time principal-agent model with a tractable dynamic matching structure, uncertainty shocks, and a control over levels of risk taking. I conduct a transition analysis and show how an uncertainty shock affects aggregate output. Higher uncertainty lowers output both at the intensive and extensive margins.

The theory in this paper is abstract, and therefore, we can apply it to various situations. Matches between a principal and an agent could be jobs, larger production units, or firms. The principal-agent relationship could be between managers and workers or between financial institutions and entrepreneurs. The point is that the fragility of agency problems caused by asymmetric information creates another channel through which uncertainty shocks can generate fluctuations in aggregate output.

This "fragile agency" view of uncertainty shocks has a sharp contrast to the real options view of uncertainty shocks. Higher uncertainty increases the value of waiting in the real options view, whereas higher uncertainty directly lowers the value of production units in the fragile agency view. This difference results in different implications, for example, for job separation. In the real options view, increased uncertainty makes firms stop hiring and firing. In the fragile agency view, it can induce active job separation.[16]

We may want to set up a richer matching structure than the one in this paper, in order to evaluate the potential importance of the fragile agency view for the business cycles. One possibly tractable way is to use a standard search-and-matching structure. To solve such a richer model, we have to numerically compute several partial differential equations iteratively, approximating some of nonlinear functions.

An important and fundamental question is why idiosyncratic riskiness sometimes dramatically increases in the economy. In this paper, I take it as granted that there are uncertainty shocks that change idiosyncratic riskiness. One of the economic examples of uncertainty shocks is a financial crisis, as we observe a large increase in the implied volatility index of the stock market in the 2007-2008 U.S. financial crisis. We could argue that a financial crisis increases so-called counter-party risk, and thus, results in an increase in idiosyncratic riskiness. A further exploration of this point would be an interesting and important research topic.

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Appendix

A.1 Technical Assumptions on the Contract Space

Let $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ be the minimal right-continuous filtration of $Y = \{Y_t\}_{t\geq 0}$ and $N = \{N_t\}_{t\geq 0}$. I define the contract space by a set of stochastic processes (a_t, b_t, C_t, τ) such that

- $\{a_t\}_{0\leq t\leq \tau}$ takes values in $\in [0,1]$ and is adapted to $\mathcal{F}$.
- $\{b_t\}_{0\leq t\leq \tau}$ is *cadlag* (right-continuous-left-limit), non-negative, and adapted to $\mathcal{F}$.
- $\{C_t\}_{0\leq t\leq \tau}$ is *cadlag*, nondecreasing, adapted to $\mathcal{F}$, and square-integrable, i.e.,

$$\mathrm{E}\left[\left(\int_0^\infty e^{-\rho u}\mathrm{d}C_u\right)^2\right] < \infty.$$

- τ is a stopping time adapted to $\mathcal{F}$.

Note that by the square-integrability of C_t, I actually mean the square-integrability of $e^{-\rho t}C_t$, to be precise. The same convention is used throughout this paper for convenience. Since C_t is nondecreasing, the square-integrability only restricts C_t from exploding at a faster rate than the rate of $e^{-\rho t}$ on a set of sample paths of positive measure.

A.2 Formal Derivation of the Optimal Contracting Problem

This is the optimal contracting problem without any simplification.

$$\max_{(a_t,b_t,C_t,\tau)} \mathrm{E}_0\left[\int_0^\tau e^{-rt}(\mathrm{d}Y_t - \mathrm{d}C_t) + e^{-r\tau}S_\tau\right]$$

$$\begin{aligned}
s.t.\quad & t\in[0,\tau],\\
(PK)\quad & V_t = \mathrm{E}_t\left[\int_t^\tau e^{-\rho(t-u)}(\mathrm{d}C_u + \lambda b_u \mathrm{d}u) + e^{-\rho(\tau-t)}U\right],\\
(IC)\quad & V_t = \max_{\hat{b}_s\geq 0} \mathrm{E}_t\left[\int_t^{\tau(\hat{Y},N)} e^{-\rho(t-u)}\mathrm{d}\hat{C}_u + e^{-\rho(\tau(\hat{Y},N)-t)}U\right]\\
& \qquad s.t.\quad \mathrm{d}\hat{C}_u = \mathrm{d}C_u(\hat{Y},N) + \lambda\hat{b}_u\mathrm{d}u,\\
& \qquad\qquad\ \ \mathrm{d}\hat{Y}_u = (\mu(a_u) - \hat{b}_u)\mathrm{d}u + a_u\sigma_u\mathrm{d}W_u,\\
(PC)\quad & V_t \geq U,\\
(RC)\quad & \mathrm{d}Y_t = (\mu(a_t) - b_t)\mathrm{d}t + a_t\sigma_t\mathrm{d}W_t,\quad Y_0 = 0,\\
& a_t\in[0,1],\quad b_t\geq 0,
\end{aligned}$$

where I have to explicitly write the arguments of a_t, C_t, and τ in (IC), because they have to be evaluated based on reported project outcomes, $\hat{Y} = \{\hat{Y}\}_{t\geq 0}$.

First, I transform contract (a_t, b_t, C_t, τ) into a more intuitive form. While termination date τ is an intuitive notion, it is hard to see what criteria are used to decide to terminate the contract. Lemma 1 shows that τ can be represented by the first hitting time when the agent's promised value, V_t, reaches the outside option value U, with two control variables over V_t:

sensitivity of V_t to realized current output, denoted by g_t, and adjustment in V_t in response to a change in aggregate state, denoted by Δ_t.

Lemma 1 For any contract (a_t, b_t, C_t, τ), there exists a set of square-integrable processes (g_t, Δ_t) adapted to a filtration generated by Y and N such that $(a_t, b_t, C_t, g_t, \Delta_t)$ with implied τ yields the same promised value for any realization of Y and N.

By Lemma 1, we can focus on a new-form contract $(a_t, b_t, C_t, g_t, \Delta_t, \tau)$. Next, Lemma 2 establishes that incentive compatibility condition (IC) for this form of contracts can be reduced into $g_t \geq \lambda$.

Lemma 2 Contract $(a_t, b_t, C_t, g_t, \Delta_t, \tau)$ is incentive compatible, i.e., satisfies (IC) if and only if $g_t \geq \lambda$ holds.

Using Lemmas 1 and 2, we can obtain the simplified problem in the main text.

A.3 Proofs

Proof of Lemma 1 Take any contract (a_t, b_t, C_t, τ). The agent's promised value satisfies

$$\int_0^t e^{-\rho u} \mathrm{d}C_u + e^{-\rho t} V_t = \mathrm{E}_t \left[\int_0^\tau e^{-\rho u} (\mathrm{d}C_u + \lambda b_u \mathrm{d}u) + e^{-\rho \tau} U \right].$$

From the law of iterated expectations, the right hand side is a martingale with respect to a filtration generated by Y and N. In addition, this martingale is square-integrable since C_t is square-integrable by assumption. Therefore, by the martingale representation theorem, there exists a set of square-integrable adapted processes $\{g_t, \Delta_t\}$ such that

$$\int_0^t e^{-\rho u} \mathrm{d}C_u + e^{-\rho t} V_t \;=\; V_0 + \int_0^t e^{-\rho u} g_u \left[\mathrm{d}Y_u - (\mu(a_u) - b_u)\mathrm{d}u\right] + \int_0^t e^{-\rho u} \Delta_u \left[\mathrm{d}N_u - \eta_u \mathrm{d}u\right]. \tag{20}$$

By rewriting it as in a differential form, we obtain

$$dV_t = \rho V_t \mathrm{d}t - dC_t + g_t \left[\mathrm{d}Y_t - (\mu(a_t) - b_t)\mathrm{d}t\right] + \Delta_t [\mathrm{d}N_t - \eta_t \mathrm{d}t]. \tag{21}$$

Hence, the dynamics of the agent's value V_t is fully described by $(a_t, b_t, C_t, g_t, \Delta_t)$ and underlying shocks, (Y_t, N_t).

Conversely, take any $(a_t, b_t, C_t, g_t, \Delta_t)$. Construct τ by $\min\{t|V_t = U\}$ with (21) as the dynamics of V_t. Then, (a_t, b_t, C_t, τ) yields the same promised value as $(a_t, b_t, C_t, g_t, \Delta_t)$ yields. ■

Proof of Lemma 2 For ease of exposition, I assume that planned hidden action b_t is always zero. Take an arbitrary deviation $\hat{b} = \{\hat{b}_t\}_{t\geq 0}$. Let $\hat{Y}_t$ be the reported cumulative cash flows.

$$\mathrm{d}\hat{Y}_t = (\mu(a_t) - \hat{b}_t)\mathrm{d}t + a_t\sigma_t \mathrm{d}W_t. \tag{22}$$

Define $\hat{\tau} := \tau(\hat{Y}, N)$ and $\hat{C}_t := C_t(\hat{Y}, N)$. Let $\hat{V}_0$ be the agent's value associated with deviation $\hat{b}$. We have

$$\begin{aligned}
\hat{V}_0 &= \mathrm{E}_0\left[\int_0^{\hat{\tau}} e^{-\rho u}(\mathrm{d}\hat{C}_u + \lambda\hat{b}_u du) + e^{-\rho\hat{\tau}}U\right] \\
&= \mathrm{E}_0\left[V_0 + \int_0^{\hat{\tau}} e^{-\rho u}g_u\left[\mathrm{d}\hat{Y}_u - \mu(a_u)\mathrm{d}u\right] + \int_0^{\hat{\tau}} e^{-\rho u}\lambda\hat{b}_u\mathrm{d}u + \int_0^{\hat{\tau}} e^{-\rho u}\Delta_u\left[\mathrm{d}N_u - \eta_u\mathrm{d}u\right]\right] \\
&= \mathrm{E}_0\left[V_0 + \int_0^{\hat{\tau}} e^{-\rho u}g_u\sigma_u\mathrm{d}W_u + \int_0^{\hat{\tau}} e^{-\rho u}(\lambda - g_u)\hat{b}_u\mathrm{d}u + \int_0^{\hat{\tau}} e^{-\rho u}\Delta_u\left[\mathrm{d}N_u - \eta_u\mathrm{d}u\right]\right] \\
&= V_0 + \mathrm{E}_0\left[\int_0^{\hat{\tau}} e^{-\rho u}(\lambda - g_u)\hat{b}_u\mathrm{d}u\right] \\
&\leq V_0 \qquad \text{if and only if} \qquad g_t \geq \lambda \qquad a.s.
\end{aligned} \tag{23}$$

The second equality follows from (20) with $t = \tau$ and $V_\tau = U$. The forth equality follows from a version of the optimal sampling theorems, whose assumptions are satisfied because g_t and Δ_t are square-integrable. The "if" part follows from non-negativity of $\hat{b}_t$, and the "only if" part is proved by contraposition. Suppose that $g_t < \lambda$ holds on a set Ω_1 of positive measure. Then, by making $\hat{b}_t$ positive on Ω_1 and zero otherwise, the last inequality is flipped without equality.

Hence, we have establish $\hat{V}_0 \leq V_0$ if and only if $g_t \geq \lambda$ holds almost surely. The same argument holds true for $\hat{V}_t$ for any $t > 0$. ■

Proof of Proposition 1

I prove this proposition by basically following DeMarzo and Sannikov (2006, hereinafter DS), although I need to deal with several deviations from their model.

[Step 1: Deriving the optimal Markov contract]

Let $\{J_s(V)\}_{s\in\{N,C\}}$ denote the principal's value under the optimal Markov contract where the agent's promised value is V in aggregate state $s \in \{N, C\}$. To obtain a usual Hamilton-Jacobi-Bellman equation, we next need to deal with compensation process C_t, since this process may not be differentiable with respect to time t.[17] Note that Lemma 2 ensures that incentive compatibility does not restrict any payment plan C_t.

As is the case with DS, the optimal Markov contract satisfies $J_s'(V) \geq -1$. This is because the principal always has the option to provide an instantaneous payment to the agent. Also, following DS, I additionally assume that $J_s(V)$ is a concave function and that there exists B_s such that

$$J_s'(B_s) = -1, \quad \text{and} \quad J_s'(V) > -1 \text{ for all } V \in [U, B_s). \tag{24}$$

These assumptions are confirmed later.

The assumptions imply that for any $V < B_s$, any positive transfer $\mathrm{d}C$ is not optimal. Using the fact that $\mathrm{d}C_t = 0$ for $V < B_s$ at the optimum, we can write the following Hamilton-Jacobi-Bellman equation, for each aggregate state $s \in \{\mathcal{N}, \mathcal{C}\}$.

$$\begin{aligned} rJ_s(V) \quad = \quad & \max_{\substack{a\,\in\,[0,1],\\ g\,\geq\,\lambda,\\ \Delta\,\geq\,-V+U}} \mu(a) + (\rho V - \eta_s\Delta)J_s'(V) + \frac{1}{2}g^2\sigma_s^2(a)J_s''(V) \\ & + \eta_s(J_{-s}(V+\Delta) - J_s(V)). \end{aligned} \tag{25}$$

On the other hand, for any $V \geq B_s$, we have

$$J_s(V) = J_s(B_s) - (V - B_s). \tag{26}$$

[17] By the assumption of $\mathrm{d}C_t \geq 0$, C_t has non-decreasing paths, which are of finite variation (Theorem 4 on page 86 in Royden, 1963). Thus, we can ignore the second order terms in Ito's formula. Even so, we cannot form a Hamilton-Jacobi-Bellman equation because it is the equation for the drift, which is the time derivative of the expected value of the process.

In this region, the optimal value is attained by an immediate move to the contract with the promised value B_s, by paying $\mathrm{d}C = V - B_s$ when $V \geq B_s$. Thus,

By concavity of $J_s(V)$, it is optimal to set $g = \lambda > 0$ for any V. That is, the agent's promised utility is sensitive to realized output to give the right incentive not to take hidden actions.

As for Δ, there are two cases: a corner solution ($\Delta = -V + U$) and an interior solution, which satisfies the first order condition:

$$J'_{-s}(V + \Delta) = J'_s(V). \tag{27}$$

That is, the optimal adjustment Δ do not change the marginal cost for promising V before and after aggregate state changes. In sum, the principal's optimal choice $\Delta_s(V)$ given V is[18]

$$\Delta_s(V) = \begin{cases} -V + U & \text{if} \quad U < V \leq \underline{V_s} \\ [J'_{-s}]^{-1}(J'_s(V)) - V & \text{if} \quad \underline{V_s} < V \leq B_s \end{cases} \tag{28}$$

where $\underline{V_s}$ is, if any, a threshold that satisfies

$$J'_{-s}(U) = J'_s(\underline{V_s}). \tag{29}$$

As for a, the first order condition is

$$\mu'(a) + \frac{1}{2} J''_s(V) \lambda^2 \sigma_s^2 (2a) = 0. \tag{30}$$

We can solve (30) for a and obtain

$$a = \frac{m_H - m_L}{-J''_s(V)\lambda^2\sigma_s^2}, \tag{31}$$

which is positive due to $J''_s(V) < 0$, but may exceeds 1 for some V. This is because the principal wants to increase risk to attain higher mean output when the probability of termination is very low. The assumption of the upper limit on a does not allow them to unlimitedly increase the mean and variance of risky projects. Note also that $a = 0$ when $1/J''_s(V) = 0$, that is, $J''_s(V) = -\infty$. Hence, we have

$$a_s(V) = \begin{cases} 0 & \text{if} \quad U < V \leq F_s \\ \frac{m_H - m_L}{-J''_s(V)\lambda^2\sigma_s^2} & \text{if} \quad V \leq A_s \\ 1 & \text{if} \quad A_s < V \leq B_s \end{cases}$$

[18]The existence of $[J'_{-s}]^{-1}$ is derived from $J''_{-s}(V) < 0$ for $V < B_{-s}$.

where F_s and A_s are, if any, thresholds that solve[19]

$$
\begin{aligned}
rJ_s''(F_s) &= m_L + (\rho F_s - \eta_s \Delta_s(F_s))J_s'(F_s) + \eta_s(J_{-s}(F_s + \Delta_s(F_s)) - J_s(F_s)),\\
J_s''(A_s) &= \frac{m_H - m_L}{-\lambda^2\sigma^2}.
\end{aligned}
$$

Therefore, the principal starts adjusting the level of risk-taking if the promised utility is low as indicated by $V < A_s$. Otherwise, it is optimal to take the highest risk to gain the highest mean output.

Given the policy function $a_s(V)$, the functional equation for $J_s(V)$ differs in possibly three regions with boarders F_s and A_s. For $U < V \leq F_s$, we obtain

$$
rJ_s(V) = r + (\rho V - \eta_s \Delta_s)J_s'(V) + \eta_s(J_{-s}(V + \Delta_s) - J_s(V)). \tag{32}
$$

For $F_s < V \leq A_s$, substituting (31) into (25), we obtain

$$
rJ_s(V) = r + (\rho V - \eta_s \Delta_s)J_s'(V) + \frac{1}{2}\frac{(\bar{\mu} - r)^2}{(-\lambda^2\bar{\sigma}_s^2 J_s''(V))} + \eta_s(J_{-s}(V + \Delta_s) - J_s(V)). \tag{33}
$$

For $A_s < V < B_s$, we have

$$
rJ_s(V) = \bar{\mu} + (\rho V - \eta_s \Delta_s)J_s'(V) + \frac{1}{2}\lambda^2\bar{\sigma}_s^2 J_s''(V) + \eta_s(J_{-s}(V + \Delta_s) - J_s(V)), \tag{34}
$$

which is similar to the case of DS except for a change in aggregate state.

As in DS, there are two terminal conditions, i.e., $J_s'(B_s) = -1$ and $J_s''(B_s) = 0$. The latter is so-called super-contact conditions. Also, we have one initial condition $J_s(U) = S_s$ where S_s satisfies (5) by the matching structure of the model. Hence, there is a unique solution of $J_s(V)$ for $V < B_s$ that satisfies (25) with these boundary conditions.[20]

[19] Note that we have $A_s < B_s$ because a boundary condition $J_s''(B_s) = 0$ implies $a_s \to \infty$ as $V \to B_s$. Since $J_s''(F_s) = -\infty$, $F_s < A_s < B_s$ holds.

[20] I numerically solve for $J_s(V)$ and $\Delta_s(V)$, following a iterative procedure described in the appendix of DeMarzo et al. (2010). As for the step I solve for $J_s(V)$ given the guess of $J_{-s}(V)$, $\Delta_s(V)$, and $\Delta_{-s}(V)$, I first guess B_s and numerically solve (25) backward with two terminal conditions, using a MATLAB® function called "ode45.m", and check if $J_s(U) = S_s$ implied by (5). Then, I update the guess by a binary search algorithm.

[Step 2: Checking if additional assumptions are satisfied]

I need to prove the concavity of the solutions to (32), (33), and (34). For (34), we can apply the same argument of DS. For (33), concavity directly follows the fact that $a_s(V)$ is strictly positive in this region. As for (32), we can first show the concavity of the solution associated with the normal state, because of $J_{\mathcal{N}}(V) > J_{\mathcal{C}}(V)$ for V in this region. The rest is the concavity of the solution to (32) in the crisis state. To prove it, I apply two Lemmas corresponding to Lemmas 4 and 6 in DS, to evaluate the second derivative of the solutions.

(Formal proof To be added)

[Step 3: Checking if the solution is the optimal contract]

We can apply the same argument in DS to prove this part. Take an arbitrary contract Π that is incentive compatible. Define the value G_t of the contract that follows Π up to time t but follows the optimal Markov contract after time t. Show G_t is a supermartingale, and is a martingale only if $g_t = \lambda$ holds for Π. Observe that the principal's value of contract Π is equal to $\lim_{t\to\infty} \mathrm{E}_0(G_{t\wedge\tau})$. Use the property of supermartingales and a version of optional sampling theorems to conclude that the value of contract Π is less than or equal to the value of the optimal Markov contract.

(Formal proof to be added)

Proof of Proposition 2

The structure of the proof is the following. First, I apply two Lemmas that correspond to Lemmas 4 and 6 in DS, to use their comparative statics results. Second, I show that the first derivative of the value function in the crisis state with respect to the variance in the crisis state $\sigma_{\mathcal{C}}$ is larger than that of the value function in the normal state, since $\sigma_{\mathcal{C}}$ directly affects $J_{\mathcal{C}}$ but indirectly affects $J_{\mathcal{N}}$ through the expectation of getting into the crisis state. This establishes property (i).

(Formal proof to be added)