Answer Key - PS2

Econ 8107
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QUESTION 1: (a) The maximization problem for the odd individual, denoted by superscript 1, is:

$$\max_{c^1_{2t-1}, c^1_{2t}, n^1_{2t-1}} \sum_{t=1}^{\infty} \beta^{2t-1}[\ln(c^1_{2t-1}) + 4(1 - n^1_{2t-1})] + \sum_{t=1}^{\infty} \beta^{2t}[\ln(c^1_{2t}) + 4]$$

s.t.

$$\sum_{t=1}^{\infty}(p_{2t-1}c^1_{2t-1} + p_{2t}c^1_{2t} + p_{2t-1}(1 - n^1_{2t-1})) \leq \sum_{t=1}^{\infty} p_{2t-1}$$

FOC's

$$\begin{array}{lll} (i) & c^1_{2t-1}: & \frac{\beta^{2t-1}}{c^1_{2t-1}} - \lambda^1 p_{2t-1} = 0 \\ (ii) & c^1_{2t}: & \frac{\beta^{2t}}{c^1_{2t}} - \lambda^1 p_{2t} = 0 \\ (iii) & n^1_{2t-1}: & -4\beta^{2t-1} + \lambda^1 p_{2t-1} = 0 \end{array}$$

Similarly, The maximization problem for the even individual, denoted by superscript 2, is:

$$\max_{c^2_{2t-1}, c^2_{2t}, n^2_{2t}} \sum_{t=1}^{\infty} \beta^{2t-1}[\ln(c^2_{2t-1}) + 4] + \sum_{t=1}^{\infty} \beta^{2t}[\ln(c^2_{2t}) + 4(1 - n^2_{2t})]$$

s.t.

$$\sum_{t=1}^{\infty}(p_{2t-1}c^2_{2t-1} + p_{2t}c^2_{2t} + p_{2t}(1 - n^2_{2t})) \leq \sum_{t=1}^{\infty} p_{2t}$$

FOC's

$$\begin{array}{lll} (iv) & c^2_{2t-1}: & \frac{\beta^{2t-1}}{c^2_{2t-1}} - \lambda^2 p_{2t-1} = 0 \\ (v) & c^2_{2t}: & \frac{\beta^{2t}}{c^2_{2t}} - \lambda^2 p_{2t} = 0 \\ (vi) & n^2_{2t}: & -4\beta^{2t} + \lambda^2 p_{2t} = 0 \end{array}$$

The Market Clearing conditions are:

$$c^1_{2t-1} + c^2_{2t-1} = n^1_{2t-1}$$
$$c^1_{2t} + c^2_{2t} = n^2_{2t}$$

From the above FOC's we have the following consumption levels for both agents:

Odd agent-

$$c_{2t-1}^1 = \frac{1}{4}$$
$$c_{2t}^1 = \beta\frac{p_{2t-1}}{4p_{2t}}$$

Even agent-

$$c_{2t-1}^2 = \frac{p_{2t}}{4\beta p_{2t-1}}$$
$$c_{2t}^2 = \frac{1}{4}$$

Using market clearing and odd agent's budget constraint, we have

$$c_{2t-1}^1 + \frac{p_{2t}}{p_{2t-1}}c_{2t}^1 = n_{2t-1}^1 = c_{2t-1}^1 + c_{2t-1}^2$$

substituting in the values for c_{2t-1}^1,c_{2t}^1, and c_{2t-1}^2,

$$\frac{1}{4} + \frac{\beta}{4} = n_{2t-1}^1 = \frac{1}{4}(1 + \frac{p_{2t}}{\beta p_{2t-1}})$$

Solving for $\frac{p_{2t-1}}{p_{2t}}$, we have

$$\frac{p_{2t-1}}{p_{2t}} = \frac{1}{\beta^2}$$

Odd agent-

$$c_{2t-1}^1 = \frac{1}{4}$$
$$c_{2t}^1 = \frac{1}{4\beta}$$

Even agent-

$$c_{2t-1}^2 = \frac{\beta}{4}$$
$$c_{2t}^2 = \frac{1}{4}$$

And, aggregate consumption becomes:

$$C_{2t-1} = Jc_{2t-1}^1 + Jc_{2t-1}^2 = J\frac{1}{4} + J\frac{\beta}{4} = J\frac{1+\beta}{4}$$

$$C_{2t} = Jc_{2t}^1 + Jc_{2t}^2 = J\frac{1}{4} + J\frac{1}{4\beta} = J\frac{1+\beta}{4\beta}$$

(b) From part (a), we have

$$\frac{p_{2t-1}}{p_{2t}} = \frac{1}{\beta^2}$$

therefore, the interest rate going from odd periods to even periods (i.e. $t = 1 \rightarrow t = 2$) is

$$r_{2t-1} = \frac{p_{2t-1}}{p_{2t}} - 1 = \frac{1}{\beta^2} - 1$$

In order to obtain the interest rate for even periods going to odd($t = 2 \rightarrow t = 3$), consider

$$\frac{p_{2t+1}}{p_{2t}} = \beta\frac{c_{2t}^2}{c_{2t+1}^2} = \beta\frac{\frac{1}{4}}{\frac{p_{2t}}{4\beta p_{2t-1}}} = \beta^2\frac{p_{2t-1}}{p_{2t}}$$

and since $\frac{p_{2t-1}}{p_{2t}} = \frac{1}{\beta^2}$, we have

$$\frac{p_{2t+1}}{p_{2t}} = 1 = \frac{p_{2t}}{p_{2t+1}}$$

Thus, the interest rate is

$$r_{2t} = \frac{p_{2t}}{p_{2t+1}} - 1 = 1 - 1 = 0$$

(c) Since τ enters only into the budget constraint of the agents, the FOC's are unaffected. Using market clearing and odd agent's budget constraint, we have

$$c_{2t-1}^1 + c_{2t-1}^2 = n_{2t-1}^1 = c_{2t-1}^1 + \tau + \frac{p_{2t}}{p_{2t-1}}c_{2t}^1$$

substituting in c_{2t-1}^1, c_{2t-1}^2, and c_{2t}^1

$$\frac{1}{4}(1 + \frac{p_{2t}}{\beta p_{2t-1}}) = n_{2t-1}^1 = \frac{1}{4} + \tau + \frac{\beta}{4}$$

Solving for $\frac{p_{2t-1}}{p_{2t}}$, we have

$$\frac{p_{2t-1}}{p_{2t}} = \frac{1}{\beta(\beta + 4\tau)}$$

Similar to part (b), we have

$$\frac{p_{2t+1}}{p_{2t}} = \beta^2 \frac{p_{2t-1}}{p_{2t}} = \frac{\beta}{\beta + 4\tau}$$

or

$$\frac{p_{2t}}{p_{2t+1}} = \frac{\beta + 4\tau}{\beta}$$

In order to determine what τ will give constant interest rates, set the interest rates for opposing periods equal to each other and solve for τ. That is,

$$\frac{p_{2t-1}}{p_{2t}} = \frac{p_{2t}}{p_{2t+1}}$$

$$\frac{1}{\beta(\beta + 4\tau)} = \frac{\beta + 4\tau}{\beta}$$

Then,

$$(\beta + 4\tau) = 1$$

and,

$$\tau = \frac{1 - \beta}{4}$$

QUESTION 2: (a) The FOC's for agent j's problem are the following:

$$\begin{aligned}
\frac{1}{c_1^j - a} &= \lambda^j \\
\frac{\beta \phi_s}{c_{2s}^j - a} &= \lambda^j p_{2s} \\
c_1^j + \sum_s p_{2s} c_{2s}^j &= 1 + \sum_s p_{2s} y_{2s}^j
\end{aligned}$$

Market clearing condions:

$$\sum_j c_1^j = J$$

$$\sum_j c_{2s}^j = \sum_j y_{2s}^j, \forall s$$

From the first order conditions, we find

$$(c_{2s}^j - a) p_{2s} = \beta \phi_s (c_1^j - a)$$

Summing over j,

$$p_{2s} = \beta \phi_s \frac{J - Ja}{\sum_j y_{2s}^j - Ja}$$

Inserting the first two FOC's into the budget constraint,

$$\frac{1}{\lambda^j} = \frac{1 - a + \sum_s p_{2s}(y_{2s}^j - a)}{1+\beta}$$

Define $\theta_j \equiv \frac{1}{\lambda^j}\frac{1}{J-Ja}$.

Now it is verified that $\theta_j > 0$ and $\sum_j \theta_j = 1$.

By assumption, $a < 1$ and $a < \min_s \sum_j y_{2s}^j/J$. This implies $\theta_j = 0$ iff $c_1^j - a = 0$. But given the log utility function, $\theta_j = 0$ would contradict optimality. Hence we have $\theta_j > 0$ for every j.
And

$$\sum_j \frac{1}{\lambda^j} = \frac{J - Ja + \sum_s p_{2s}(\sum_j y_{2s}^j - Ja)}{1+\beta} = J - Ja$$

Which implies $\sum_j \theta_j = 1$.

In this way we have the following:

$c_1^j = \frac{1}{\lambda^j} + a = \theta_j(J - Ja) + a$
$c_{2s}^j = \frac{\beta\phi_s}{\lambda^j P_s} + a = \theta_j(\sum_j y_{2s}^j - Ja) + a$

(b) An equilibrium is a tuple $\{(c_1^j, (c_{2s}^j)_s), (A_i^j)_{j=1}^J)_{i=1}^J, (P_i)_{i=1}^J\}$ such that:

(I) Given $(P_j)_{j=1}^J, (c_1^i, (c_{2s}^i)_s, (A_j^i)_{j=1}^J)$ solves agent i's problem

$$\max \ln(c_1^i - a) + \beta \sum_s \phi_s \ln(c_{2s}^i - a)$$

$$s.t. \begin{cases} c_1^i + \sum_j P_j A_j^i \leq 1 \\ c_{2s}^i \leq y_{2s}^i + \sum_j A_j^i y_{2s}^j \ \ \forall s \\ \text{nonnegativity for consumption} \end{cases}$$

(II) Market clearing

$$\begin{aligned} \sum_i c_1^i &= J \\ \sum_i c_{2s}^i &= \sum_i y_{2s}^j, \ \forall s \\ \sum_i A_j^i &= 0, \forall j \end{aligned}$$

In this setting, we only have J different risky assets which means we can not construct S state-contigent claims from those J assets.

(c) The FOC's for the agent j's problem are the following:

$$\frac{1}{c_1^j} = \mu_1^j$$

$$\frac{\beta\phi_s}{c_{2s}^j} = \mu_{2s}^j$$

$$P_i\mu_1^j = \sum_s \mu_{2s}^j y_{2s}^i$$

If $a = 0$, then we have $c_1^j = \theta^j J$ and $c_{2s}^j = \theta^j \sum_j y_{2s}^j$.

Define:

$$\begin{cases} A_i^j = \theta^j \text{if } i \neq j, \;\; A_i^j = \theta^j - 1 \text{ if } i = j \\ P_i = \beta J \sum_s \phi_s \frac{y_{2s}^i}{\sum_j y_{2s}^j} \end{cases}$$

Claim: If $a = 0$, prices and asset holdings as defined above, together with the allocations define in part (a), constitute an equilibrium.

First we check the market clearing condition for assets (here we use $\sum_{j=1}^J \theta^j = 1$):

$$\sum_j A_i^j = \theta^j - 1 + \sum_{j\neq i}\theta^j = \theta^j - 1 + (1-\theta^j) = 0 \;\forall i$$

Next we check the sufficient FOC's:
Notice that the FOC's hold with multipliers $\mu_1^j = \lambda^j$, and $\mu_{2s}^j = \lambda^j p_{2s}$. Also by construction $P_i = \beta J \frac{\theta^j}{\theta^j}\sum_s \frac{y_{2s}^i}{\sum_j y_{2s}^j}\phi_s = \beta c_1^j \sum_s \frac{y_{2s}^i}{c_{2s}^i}\phi_s$.

In order to check that the budget constraint holds, we do the following algebra:

$$c_1^j + \sum_i A_i^j P_i = c_1^j + \sum_{i=1}^J \sum_s p_{2s} y_{2s}^i A_i^j = c_1^j + (\theta^j - 1)\sum_s p_{2s} y_{2s}^j + \theta^j \sum_s \sum_{i\neq j} p_{2s} y_{2s}^i =$$

$$= c_1^j - \sum_s p_{2s} y_{2s}^j + \theta^j \sum_{i=1}^J \sum_s p_{2s} y_{2s}^i = c_1^j - \sum_s p_{2s} y_{2s}^j + \sum_s p_{2s} \sum_{j=1}^J \theta^j y_{2s}^j =$$

$$c_1^j - \sum_s p_{2s} y_{2s}^j + \sum_s p_{2s} c_{2s}^j = 1$$

Where the result in the last line comes from the budget constraint in the complete markets economy.

Finally we need to check the state-s constraint in the agent j's problem. The following equation will be useful:

$$\sum_{i=1}^{J} A_i^j y_{2s}^i = \theta^j \sum_{i \neq j} y_{2s}^i + (\theta^j - 1) y_{2s}^j$$

Now consider some state s:

$$c_{2s}^j = \theta^j \sum_{i=1}^{J} y_{2s}^i = \theta^j y_{2s}^j + \sum_{i \neq j} y_{2s}^i \theta^j = \sum_{i=1}^{J} A_i^j y_{2s}^i + (1 - \theta^j) y_{2s}^j + \theta^j y_{2s}^j =$$

$$= \sum_{i=1}^{J} A_i^j y_{2s}^i + y_{2s}^j$$

Hence the state-s constraint is satisfied.

(d) The proof breaks down when we are checking the state-s constraint: If $a \neq 0$, then $c_{2s}^j = \theta^j \sum_i y_{2s}^i + (1 - \theta^j J)a$. So the problem in the proof would be to construct a portfolio that pays off a constant amount (risk-free) in all states.

(e) An equilibrium is a tuple $\{(c_1^j, (c_{2s}^j)_s), (A_i^j)_{j=1}^J)_{i=1}^J, b^j, q, (P_i)_{i=1}^J\}$ such that:

(I) Given $(P_j)_{j=1}^J$ and q,$(c_1^i, (c_{2s}^i)_s, (A_j^i)_{j=1}^J, b^j)$ solves agent j's problem

$$\max \ln(c_1^j - a) + \beta \sum_s \phi_s \ln(c_{2s}^j - a)$$

$$s.t. \begin{cases} c_1^j + \sum_i P_i A_i^j \leq 1 \\ c_{2s}^j \leq y_{2s}^j + \sum_i A_i^j y_{2s}^i + b^j \ \ \forall s \\ \text{nonnegativity for consumption} \end{cases}$$

(II) Market clearing

$$\begin{aligned} \sum_i c_1^i &= J \\ \sum_i c_{2s}^i &= \sum_i y_{2s}^j, \ \forall s \\ \sum_i A_j^i &= 0, \forall j \end{aligned}$$

$$\sum_i b^i = 0$$

(f) The FOC's for the agent j's problem are the following:

$$\frac{1}{c_1^j - a} = \mu_1^j$$

$$\frac{\beta\phi_s}{c_{2s}^j - a} = \mu_{2s}^j$$

$$P_i \mu_1^j = \sum_s \mu_{2s}^j y_{2s}^i$$

$$q\mu_1^j = \sum_s \mu_{2s}^j$$

Define:

$$\begin{cases} A_i^j = \theta^j \text{ if } \; i \neq j, \quad A_i^j = \theta^j - 1 \text{ if } i = j \\ P_i = \sum_s p_{2s} y_{2s}^i \\ q = \sum_s p_{2s} \\ b^j = a(1 - \theta^j J) \end{cases}$$

Claim: Prices, bond holdings and asset holdings as defined above, together with the allocations defined in part (a), constitute an equilibrium.

First we check the sufficient FOC's:

Notice that the first two FOC's hold with multipliers $\mu_1^j = \lambda^j$, and $\mu_{2s}^j = \lambda^j p_{2s}$. Also the pricing equations hold:

$$P_i = \frac{\sum_s \beta\phi_s (J - Ja) y_{2s}^i}{\sum_j y_{2s}^j - Ja} = \sum_s p_{2s} y_{2s}^i$$

$$q = \frac{\sum_s \beta\phi_s (J - Ja)}{\sum_j y_{2s}^j - Ja} = \sum_s p_{2s}$$

The market clearing condition for goods is trivially satisfied, and the condition regarding assets can be showed exactly as in part (c). Since $\sum_j \theta^j = 1$, also the market clearing for bonds is satisfied.

Next we check that the budget constraint holds for any agent j:

$$c_1^j + \sum_i A_i^j P_i + qb^j = c_1^j - \sum_s p_{2s} y_{2s}^j + \theta^j \sum_{i=1}^J \sum_s p_{2s} y_{2s}^i + \sum_s p_{2s}(a(1 - \theta^j J)) =$$

$$= c_1^j + \sum_s p_{2s}[\theta^j \sum_{i=1}^J y_{2s}^i + a(1 - \theta^j J)] - \sum_s p_{2s} y_{2s}^j =$$

$$= c_1^j + \sum_s p_{2s} c_{2s}^j - \sum_s p_{2s} y_{2s}^j = 1$$

Again the last line comes from the budget constraint in the complete markets economy.

The last set of conditions to be checked is the state-s constraint in the agent j's problem:

$$y_{2s}^j + \sum_i A_i^j y_{2s}^i + b^j = b^j + \theta^j \sum_{i=1}^J y_{2s}^i = a(1-\theta^j J) + \theta^j \sum_{i=1}^J y_{2s}^i = c_{2s}^j$$

QUESTION 3: Let J_1 be the set of agents with the risk-free investment technology and J_2 be the set of those with the risky investment technology. Then an equilibrium is a tuple $((c^j, i^j)_{j=1}^{2J}, (p_{2s})_{s=1}^S)$ such that:

(I) For each $j \in J_1$, (c^j, i^j) solves

$$\max_{(c^j, i^j)} u(c_1^j) + \beta \sum_s \pi_s u(c_{2s}^j)$$

subject to

$$c_1^j + \sum_s p_{2s} c_{2s}^j + i_1^j \le y_1 + \sum_s p_{2s} R^f i_1^j$$

and non-negativity.

(II) For each $j \in J_2$, (c^j, i^j) solves

$$\max_{(c^j, i^j)} u(c_1^j) + \beta \sum_s \pi_s u(c_{2s}^j)$$

subject to

$$c_1^j + \sum_s p_{2s} c_{2s}^j + i_1^j \le y_1 + \sum_s p_{2s} R_s i_1^j$$

and non-negativity.

(III) There holds

$$\sum_j c_1^j + \sum_j i_1^j = 2Jy_1$$

and

$$\sum_j c_{2s}^j = \sum_{j \in J_1} R^f i_1^j + \sum_{j \in J_2} R_s i_1^j$$

for all s.

(a) Let $c_1^* \in (0, y_1)$ solve

$$\frac{1}{R^f} = \beta \frac{u'(R^f(y_1 - c_1^*))}{u'(c_1^*)}.$$

Assuming u' continuous and $\lim_{c \to 0} u'(c) = \infty$, a solution exists. Also, by strict concavity of u, the function $g(c_1^*) \equiv \beta \frac{u'(R^f(y_1 - c_1^*))}{u'(c_1^*)}$ is strict increasing. This implies the existence of a unique solution to the equation above.

Claim : $((c^j, i^j)_{j=1}^{2J}, (p_{2s})_{s=1}^S)$ defined by

$$
\begin{aligned}
c_1^j &= c_1^* \quad (\forall j) \\
c_{2s}^j &= R^f(y_1 - c_1^*) \quad (\forall j, s) \\
i_1^j &= 2(y_1 - c_1^*) \quad (\forall j \in J_1) \\
i_1^j &= 0 \quad (\forall j \in J_2) \\
p_{2s} &= \beta \frac{u'(R^f(y_1 - c_1^*))}{u'(c_1^*)} \pi_s \quad (\forall s)
\end{aligned}
$$

is an equilibrium.

Indeed, the market clearing conditions are clearly satisfied. Also, the households' optimization conditions for $j \in J_1$ are satisfied because the Kuhn-Tucker conditions for the Lagrangian

$$
L^j = u(c_1^j) + \beta \sum_s \pi_s u(c_{2s}^j) + \lambda^j \left\{ y_1 + \sum_s p_{2s} R^f i_1^j - c_1^j - \sum_s p_{2s} c_{2s}^j - i_1^j \right\}
$$

are satisfied with multiplier $\lambda^j = u'(c_1^*)$, and the problem has a concave objective function and a convex constraint set.

Now consider the Lagrangian for $j \in J_2$:

$$
L^j = u(c_1^j) + \beta \sum_s \pi_s u(c_{2s}^j) + \lambda^j \left\{ y_1 + \sum_s p_{2s} R^f i_1^j - c_1^j - \sum_s p_{2s} c_{2s}^j - i_1^j \right\} + \mu^j i_1^j
$$

Using $\sum_s R_s \pi_s = R^f$, we see that the Kuhn-Tucker conditions hold with multiplier $\mu^j = 0$ for $j \in J_2$.

(b) Let $c_1^* \in (0, y_1)$ solve

$$
1 = \beta \sum_s \frac{u'(R_s(y_1 - c_1^*))}{u'(c_1^*)} R_s \pi_s.
$$

As in part (a), it is assumed that a solution exists.

Claim: $((c^j, i^j)_{j=1}^{2J}, (p_{2s})_{s=1}^S)$ defined by

$$
\begin{aligned}
c_1^j &= c_1^* \quad (\forall j) \\
c_{2s}^j &= R_s(y_1 - c_1^*) \quad (\forall j, s) \\
i_1^j &= 0 \quad (\forall j \in J_1) \\
i_1^j &= 2(y_1 - c_1^*) \quad (\forall j \in J_2) \\
p_{2s} &= \beta \frac{u'(R_s(y_1 - c_1^*))}{u'(c_1^*)} \pi_s \quad (\forall s)
\end{aligned}
$$

is an equilibrium.

Indeed, the market clearing conditions are clearly satisfied. Also, the households' optimization conditions for $j \in J_2$ are satisfied because the Kuhn-Tucker conditions for the Lagrangian

$$L^j = u(c_1^j) + \beta \sum_s \pi_s u(c_{2s}^j) + \lambda^j \left\{ y_1 + \sum_s p_{2s} R_s i_1^j - c_1^j - \sum_s p_{2s} c_{2s}^j - i_1^j \right\}$$

are satisfied with multiplier $\lambda^j = u'(c_1^*)$, and the problem has a concave objective function and a convex constraint set. The optimization conditions for $j \in J_1$ are also satisfied because the Kuhn-Tucker conditions for the Lagrangian

$$L^j = u(c_1^j) + \beta \sum_s \pi_s u(c_{2s}^j) + \lambda^j \left\{ y_1 + \sum_s p_{2s} R^f i_1^j - c_1^j - \sum_s p_{2s} c_{2s}^j - i_1^j \right\} + \mu^j i_1^j$$

are satisfied with multipliers $\lambda^j = u'(c_1^*)$ and $\mu^j = u'(c_1^*)(1 - \sum_s p_{2s} R^f)$ (μ^j is the multiplier on the non-negativity constraint of i_1^j), and the problem has a concave objective function and a convex constraint set.

Here, by the definition of p_{2s} and the condition $R_s > R^f$ all s, we have

$$1 = \sum_s p_{2s} R_s > \sum_s p_{2s} R^f$$

so

$$\sum_s p_{2s} < \frac{1}{R^f}.$$

Thus the price of a bundle of claims that has payoff 1 in all states of the world is smaller than $1/R^f$.

QUESTION 4: Assuming interior solution, it suffices to consider agent j's FOC's. Every agent j solves the following problem for a given $\{p_t\}_{t=1}^{TJ}$:

$$\max_{\{c_t^j\}_{t=1}^{TJ}} \sum_{t=1}^{TJ} \beta^{t-1} u^j \left(c_t^j, G_t\right)$$

$$s.t. \begin{cases} \sum_{t=1}^{TJ} p_t c_t^j \le \sum_{t=1}^{TJ} p_t - \sum_{s=0}^{T-1} p_{sJ+j} \\ c_t^j \ge 0, \quad G_t = 1 \forall t \end{cases}$$

Then from the agent's FOC we get $p_t = \frac{1}{\lambda} \beta^{t-1} u_c^j \left(c_t^j, 1\right)$, implying that

$$\frac{p_t}{p_{t+1}} = \frac{1}{\beta} \frac{u_c^j \left(c_t^j, 1\right)}{u_c^j \left(c_{t+1}^j, 1\right)}.$$

We normalize $p_1 = 1$ and guess that $\frac{p_t}{p_{t+1}} = \frac{1}{\beta}$. Next we verify the guess. In equilibrium, from the agents budget constraint we obtain using our guess (assuming $\beta \in (0,1)$:

$$\sum_{t=1}^{TJ} \beta^{t-1} c_t^j = \sum_{t=1}^{TJ} \beta^{t-1} - \sum_{s=0}^{T-1} \beta^{sJ+j-1}$$

implying that for all t

$$\begin{aligned} \frac{1-\beta^{TJ}}{1-\beta} c_t^j &= \frac{1-\beta^{TJ}}{1-\beta} - \beta^{j-1} \frac{1-\beta^{TJ}}{1-\beta^J} \\ c_t^j &= 1 - \beta^{j-1} \frac{1-\beta}{1-\beta^J} \end{aligned}$$

which in turn implies that $u_c^j\left(c_t^j, 1\right) = u_c^j\left(c_{t+1}^j, 1\right)$. Moreover,

$$\sum_{j=1}^{J} c_t^j = \sum_{j=1}^{J} \left[1 - \beta^{j-1} \frac{1-\beta}{1-\beta^J}\right] = J - \frac{1-\beta}{1-\beta^J} \frac{1-\beta^J}{1-\beta} = J - 1$$

implying that the market clearing constraint is satisfied too. Thus the guess was correct and the equilibrium sequence of interest rates in this economy is $\left\{\frac{1}{\beta}\right\}_{t=1}^{TJ}$.

QUESTION 5: (a) First notice that the utility functions are strictly convex: $\frac{\partial U}{\partial c_t^i} > 0, \frac{\partial^2 U}{\partial (c_t^i)^2} < 0 \forall\ i = 1, 2,\ t = 1, 2$.

Drawing a Edgeworth box, it is easy to check that the set of Pareto optimal allocations is the following (it does not contain any point in the interior of the box):

$$\begin{aligned}(c_1^1, c_1^2, c_2^1, c_2^2) = \{(c, 1-c, 1, 0) | c \in [0,1]\} \bigcup \{(c, 1-c, 0, 1) | c \in [0,1]\} \bigcup \\ \bigcup \{(1, 0, c, 1-c) | c \in [0,1]\} \bigcup \{(0, 1, c, 1-c) | c \in [0,1]\}\end{aligned}$$

(b) Since the return function is convex, the FOC's are not sufficient. Additionally, we know that the solutions are found at the corners regardless the value of ω.

The solution will depend on the set in which ω belongs in the following partition: $\{[0, \frac{1}{2}), (\frac{1}{2}, 1], \{\frac{1}{2}\}\}$

1. If $\omega \in (\frac{1}{2}, 1]$: $(c_1^1, c_1^2, c_2^1, c_2^2) = (1, 0, 1, 0)$
2. If $\omega \in [0, \frac{1}{2})$: $(c_1^1, c_1^2, c_2^1, c_2^2) = (0, 1, 0, 1)$
3. If $\omega \in \{\frac{1}{2}\}$: $(c_1^1, c_1^2, c_2^1, c_2^2) = \{(1, 0, 1, 0), (0, 1, 0, 1), (1, 0, 0, 1), (0, 1, 1, 0)\}$

(c) If U is concave and the feasibility set is convex, then for any Pareto optimal allocation $\exists\ \omega \in [0,1]$ such that the Pareto optimal allocation solves the social planner problem. This question shows a counter example of this theorem in the case that U is not concave.

Here it is presented a proof of such theorem, and one can figure out exactly why the proof breaks down in this question. The proof refers to a bit more general social planner problem (SPP):

$$\max_{c_t{}^i} \sum_{t=0}^{\infty} \beta^t [\alpha_1 u^1(c_t^1) + \alpha_2 u^2(c_t^2)]$$

$$s.t. \begin{cases} c_t^1 + c_t^2 \leq w_t^1 + w_t^2 \ \forall \ t \geq 0 \\ c_t^i \geq 0 \ \ \forall t \geq 0, \ \ \forall i = 1,2 \end{cases}$$

Proposition: Assume U^i concave for every agent i. Let $\{\widetilde{c}_t^{i=1,2}\}_{t\geq 0}$ be a Pareto allocation. Then $\exists\ \gamma_1 \geq 0$ and $\gamma_2 \geq 0$ s.th. $\{\widetilde{c}_t^{i=1,2}\}_{t\geq 0}$ solves the SPP with weights γ_1 and γ_2.

Proof:

Step 1: Let Z be the set of all possible feasible allocations.

Claim: Z is nonempty and convex. That Z is nonempty comes from $w_t^i \geq 0\ \forall t \geq 0\,, \forall i$. Hence $\{c_t^{i=1,2}\}_{t\geq 0}$ $s.th.$ $c_t^i = 0\ \forall t \geq 0\,, \forall i$ belongs to Z. That Z is convex follows from the fact that all constraints in the SPP are linear.

Step 2: Define $\widetilde{U} = \{\widetilde{u} \in \Re^2 : \exists z \in Z\ s.th.\, \widetilde{u}^i \leq U^i(z)\}$.

Claim: $\widetilde{U}$ is nonempty.Further if U^i is concave, then $\widetilde{U}$ is convex.

That $\widetilde{U}$ is nonempty follows from non-emptiness of Z.

Convexity of $\widetilde{U}$: Take $\widetilde{u}_1$ and $\widetilde{u}_2 \in \widetilde{U}$. Then $\exists\ z_1$ and $z_2 \in Z$ s.th.:

$$\begin{cases} \widetilde{u_1}^i \leq U^i(z_1)\ \forall i \\ \widetilde{u_2}^i \leq U^i(z_2)\ \forall i \end{cases} \tag{0.1}$$

Since Z is convex, $z^\lambda \equiv \lambda z_1 + (1-\lambda) z_2 \in Z\ \ \forall \lambda \in (0,1)$.

Also by concavity of U^i, we have $U^i(z^\lambda) \geq \lambda U^i(z_1) + (1-\lambda) U^i(z_2) \geq \lambda \widetilde{u_1}^i + (1-\lambda)\widetilde{u_2}^i$.

Hence $\exists\ z \in Z$ s.th. $\lambda \widetilde{u_1}^i + (1-\lambda)\widetilde{u_2}^i \leq U^i(z)$.

Step 3:Let $\bar{z} \in Z$ be Pareto efficient.

Claim: $U(\bar{z}) \equiv (U^1(\bar{z}), U^2(\bar{z}))$ belongs to the boundary of $\widetilde{U}$.

Suppose that the claim is false. Then $\exists\ \varepsilon > 0$ s.th. $\forall v = (v^1, v^2)$ with $\|U(\bar{z}) - v\| \leq \varepsilon$ we have $v \in \widetilde{U}$.

Define: $\hat{U}^i \equiv U^i(\bar{z}) + \frac{\varepsilon}{\sqrt{2}}$.

Then $\|U(\bar{z}) - \hat{U}\| \leq \varepsilon$ and $\hat{U} \in \widetilde{U}$. But then using the definition of the set $\widetilde{U}$ we have a contradiction of the fact that $\bar{z} \in Z$ is Pareto efficient.

Step 4: By step 3 and the separating hyperplane theorem (using the fact that

$\widetilde{U}$ is convex) we have the following:
$\exists\ \gamma \in \Re^2 \backslash \{0\}$ s.th. $\gamma U(\bar{z}) \geq \gamma \widetilde{u} \ \ \forall \widetilde{u} \in \widetilde{U}$.

Step5:It remains to show that $\gamma_i \geq 0\ i = 1, 2$.
Suppose not. Without loss of generality assume $\gamma_1 < 0$. Since $U(\bar{z})$ is unbounded from bellow by construction, we can pick the first component of $\widetilde{u}$ small enough s.th. $\gamma U(\bar{z}) < \gamma \widetilde{u}$, contradicting step 4.