1. Problem Set 2

1. There are $2J$ agents in an infinite horizon economy. J agents belong to the odd group. These agents are productive in odd periods and unproductive in even periods. The other J agents belong to the even group. They are productive in even periods and unproductive in odd periods. Productive agents are able to convert x units of their time into x units of consumption for any non-negative value of x; unproductive agents are unable to convert their time into any output.

The agents have preferences:

$$\sum_{t=1}^{\infty} \beta^{t-1}\{\ln(c_t) + 4(1 - n_t)\}, 1 > \beta > 1/3$$

where n_t is the amount of time spent working in a period, and c_t is the amount of consumption in a period.

In period 0, agents trade a complete set of date-contingent claims to consumption and time.

a. What is the equilibrium sequence of aggregate consumption? (You can find an analytical formula for it.)

b. What is the equilibrium sequence of risk-free rates in this economy?

c. Now, suppose there is a government who imposes a lump-sum tax τ on the odd agents in the odd periods; the government transfers the proceeds of this tax to the even agents. For what value of τ do interest rates become constant?

2. This question concerns incomplete markets. Consider an economy with two periods and J agents. In period 1, each agent is endowed with one unit of consumption. There are S states of the world in period 2; the probability of state s occurring is ϕ_s. In period 2,agent j is endowed with $y_{2S}^j > 0$ units of consumption in state s. Agents have identical preferences of the form:

$$\ln(c_1^j - a) + \beta \sum_s \phi_s \ln(c_{2s}^j - a), a \geq 0$$

I assume that $J < S$, and that $a < 1$ and $a < \min_s \sum_{j=1}^J y_{2s}^j / J$.

Suppose first that in period 1, agents can trade current consumption and a complete set of state-contingent claims to consumption in period 2. Let $(c_1^{j*}, (c_{2s}^{j*})_{s=1}^S)$ be an equilibrium allocation of consumption in this economy.

a. Prove that there exists a vector $(\theta_1, ..., \theta_J)$ such that:

$$\begin{aligned} \theta_j &> 0 \text{ for all } j \text{ and } \sum_{j=1}^J \theta_j = 1 \\ c_1^{j*} &= \theta^j (J - Ja) + a \\ c_{2s}^{j*} &= \theta^j (\sum_{j=1}^J y_{2s}^j - Ja) + a \end{aligned}$$

Suppose next that agents cannot trade all state-contingent claims. Rather, they can only trade current period consumption and J assets that pay off in period 2. Asset j is the same as the endowment of agent j because it has payoff y_{2s}^j in state s. The assets are available in zero net supply. (This means that the total amount of each asset is zero.) Each agent is initially endowed with zero units of each asset.

b. Write down a definition of equilibrium in this setting. Why do we say that markets are "incomplete" in this setting?

c. Prove that if the preference parameter $a = 0$, then $(c_1^{j*}, (c_{2s}^{j*})_{s=1}^S)$ is an equilibrium allocation of consumption in this economy.

d. Explain why your proof in part (c) doesn't work if $a > 0$.

Now suppose that in period 1, agents trade current consumption, the J "endowment" assets described in part (a), and a risk-free asset that pays off one unit of consumption in all states. Again, all assets are available in zero net supply, and each agent is initially endowed with 0 units of each asset.

e. Write down a definition of equilibrium in this economy.

f. Prove that $(c_1^{j*}, (c_{2s}^{j*})_{s=1}^S)$ is an equilibrium in this economy, for any positive value of a.

3. Consider an economy with 3 periods, numbered $0, 1, 2$, and $2J$ agents. (Period 0 is a trading period.) There are S states of the world in period two; the probability of state s occurring is π_s. The agents have identical utility functions:

$$u(c_1^j) + \beta \sum_s \pi_s u(c_{2s}^j), \text{ where } u', -u'' > 0$$

where c_1^j is consumption in period 1, and c_{2s}^j is consumption in period 2, state s. They begin life with y_1 units of consumption in period 1. They can split this endowment into consumption, c_1^j, and investment, i_1^j. For half of the agents, investing i_1^j units of investment generates $R^f i_1^j$ units of period two consumption, where R^f is riskfree (that is, independent of s).

For the other J agents, investing i_1^j units of investment produces $R_s i_1^j$ units of consumption in period 2. Their technology is risky, because R_s depends on s.

In period 0, the agents trade period 1 consumption and S state-contingent claims to period two consumption.

a. Suppose $\Sigma_s \pi_s R_s = R^f$. What is the equilibrium allocation of consumption and investment across agents?

b. Suppose $R_s > R^f$ for all s. What is the equilibrium allocation of consumption and investment across agents? Is the price of a bundle of claims that has payoff 1 in all states of the world bigger than, smaller than, or equal to $1/R^f$?

4. Consider a TJ period economy with J agents. Agent j has a utility function given by:

$$\sum_{t=1}^{TJ} \beta^{t-1} u^j(c_t^j, G_t),\ \ u_c^j, -u_{cc}^j > 0$$

In period t, each agent is endowed with one unit of consumption. The government sets $G_t = 1$ in every period. It finances this consumption by taxing agent j one unit of consumption in period t, where $t = (sJ + j)$ for some $s = 0, 1, 2, ..., (T - 1)$. In other words, the taxation burden rotates around the various agents.

In this economy, agents trade claims to consumption in an Arrow-Debreu economy in period 0. What is the equilibrium sequence of interest rates in this economy?

5. Consider a riskless economy with 2 agents and 2 periods. Both agents have identical preferences over period 1 and period 2 consumption of the form:

$$(c_1)^2 + (c_2)^2$$

The total amount of consumption in each period is 1.

a. Completely characterize the set of Pareto optimal allocations in this economy.

b. For each $\omega \in [0, 1]$, find the set of allocations that solve:

$$\max_{(c_1^1, c_2^1, c_1^2, c_2^2)} \omega[(c_1^1)^2 + (c_2^1)^2] + (1-\omega)[(c_1^2)^2 + (c_2^2)^2]$$

$$s.t.\ c_1^1 + c_1^2 \leq 1$$

$$c_2^1 + c_2^2 \leq 1$$

$$c_1^1, c_2^1, c_1^2, c_2^2 \geq 0$$

whee c_t^j stands for consumption of agent j in period t.

c. Discuss the implications of your answers in parts (a) and (b).