Macro

2007 Fall Q I.1. Taxation 8105

- Rep. Consumer w/ $\sum_{t=0}^{\infty} \beta^t u(c_t)$
- Rep. firm : AK
- BC : $\sum_t p_t (c_t + x_t) \le \sum_t [r_t (1 - \tau_{kt}) k_t] + T_0$
  ↖ lump-sum in t=0

- 2 Fiscal Policy

  FP1 : $\tau^1_{k0} > 0$, $\tau^1_{kt} = \tau^1_k < 0$, $\forall t > 0$, $T^1_0 = 0$
  ↑ subsidize

  FP2 : $\tau^2_{k0} > 0$, $\tau^2_{kt} = 0$ $\forall t > 0$, $T^2_0 = \tau^2_{k0} r_0 k_0$
  ↖ revenue back.

(a) $\boxed{\text{Claim}}$ Under FP2, TDCE is P.O.

- u : str. inc in c ⇒ BC holds w/ equality
- u : str concave ⇒ FOC is sufficient since the choice set is convex
  Solution is unique.
- u : Inada ⇒ Interior solution
- u : diff'

Consumer's problem :

$$\begin{cases} \max_{\{c_t, k_{t+1}\}} \sum_t \beta^t u(c_t) \\ \text{s.t.} \quad \sum_t p_t (c_t + x_t) = \sum_t [r_t (1 - \tau_{kt}) k_t] + T_0 \\ \qquad x_t = k_{t+1} - (1-\delta) k_t \\ \qquad \text{nonnegativity}, \ k_0 \text{ given.} \end{cases}$$

FOC :

wrt c_t : $\beta^t u'(c_t) = p_t \lambda$

wrt k_{t+1} : $p_t = p_{t+1}(1-\delta) + r_{t+1}(1 - \tau_{kt+1})$ $\quad \forall t = 0, 1, \dots$

Under FP2, this becomes $u'(c_t) = \beta u'(c_{t+1}) [A(1 - \tau_{kt+1}) + 1 - \delta]$ (by MP cond.)

$\Leftrightarrow u'(c_t) = \beta u'(c_{t+1}) [A + 1 - \delta]$

This is equivalent to the FOC's for the similar economy w/o tax.
By FWT, the allocation is P.O.

(Proof 2) Proof of FWT

Assume u is str. inc in c. ← Only one assumption

Let $\{\hat{c}_t, \hat{x}_t, \hat{k}_{t+1}\}_{t=0}^{\infty}$ be the TDCE under FP 2.

Since u is str. inc, HHBC holds w/ equality:

$$\sum_{t=0}^{\infty} \hat{p}_t (\hat{c}_t + \hat{x}_t) = \sum_{t=0}^{\infty} \hat{r}_t (1-\tau_{kt}) \hat{k}_t + T_0,$$

where $(\hat{p}_t, \hat{r}_t)$ is an eqm price.

Suppose $\{\hat{c}, \hat{x}, \hat{k}\}$ is not P.O. Then $\exists$ a feasible allocation $\{\tilde{c}, \tilde{x}, \tilde{k}\}$ s.t.

$$\sum \beta^t u(\tilde{c}_t) > \sum \beta^t u(\hat{c}_t).$$

Since this is feasible, $\forall t$, $\tilde{c}_t + \tilde{x}_t = A\tilde{k}_t$

Assume $\hat{p}_t > 0$ $\forall t$. Then $\hat{p}_t(\tilde{c}_t + \tilde{x}_t) = \hat{p}_t A \tilde{k}_t = \hat{r}_t \tilde{k}_t$ (by MP cond.)

Thus $\sum_{t=0}^{\infty} \hat{p}_t (\tilde{c}_t + \tilde{x}_t) = \sum_{t=0}^{\infty} \hat{r}_t \tilde{k}_t = \sum_{t=0}^{\infty} \hat{r}_t (1-\tau_{kt}) \tilde{k}_t + T_0$ ↯ $\{\hat{c}, \hat{x}, \hat{k}\}$ is TDCE.

⇒ budget feasible

(b) Compare the growth rate of consumption under 2 different FP

From FOC, $i = 1, 2$

Euler: $u'(c_t^i) = \beta u'(c_{t+1}^i)[1 - \delta + A(1-\tau_{kt+1}^i)]$

$$\Leftrightarrow \quad c_t^{i\,-\sigma} = \beta c_{t+1}^{i\,-\sigma} [1 - \delta + A(1-\tau_{kt+1}^i)] \quad \forall t = 0, 1, \ldots$$

$$\Rightarrow \quad \frac{c_{t+1}^i}{c_t^i} \equiv \gamma_{t+1}^i = \{\beta [1 - \delta + A(1-\tau_{kt+1}^i)]\}^{1/\sigma}$$

Thus, $\gamma_{t+1}^1 > \gamma_{t+1}^2$, $\forall t$ ($\because \tau_{kt+1}^1 < 0$ $\forall t$)

OR.

$$\frac{c_{t+1}^1 / c_{t+1}^2}{c_t^1 / c_t^2} = \left(\frac{1 - \delta + A(1-\tau_{kt+1}^1)}{1 - \delta + A} \right)^{1/\sigma} > 1$$

$$\frac{c_t^1}{c_t^2} = \underbrace{k^t}_{k} \frac{c_0^1}{c_0^2}$$) → page 2

(c) Prove $c_0^2 > c_0^1$

Sps $c_0^2 \le c_0^1$

Since $\gamma_{t+1}^1 > \gamma_{t+1}^2$ $\forall t = 0, 1, \ldots$, $c_t^2 < c_t^1$ $\forall t = 1, 2, \ldots$

Thus $\sum_{t=0}^{\infty} \beta^t u(c_t^2) < \sum_{t=0}^{\infty} \beta^t u(c_t^1)$

Also $\{c_t^1\}$ is feasible since it is TDCE allocation.

This contradicts that $\{c_t^2\}$ is P.O. ($g_t = 0$ $\forall t$)

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How to derive the ratio c_t^1 / c_t^2 ?

$\boxed{1}$ DP Form :

state y : cash - in - hand

control c, k'

$$\begin{cases} v(y) = \max\limits_{c, k'} u(c) + \beta v(y') \\ \text{s.t.} \quad c + k' = y \\ \quad y' = A(1-\tau)k' + (1-\delta)k' \\ \quad \text{nonnegativity.} \\ \quad y_0 \equiv A(1-\tau_0)k_0 + (1-\delta)k_0 + T_0 \end{cases}$$

Why do we need y ?

- to make the problem stationary.

If k is the state variable, the problem in period 0 is different from $t \geq 1$ because of the taxation.

In this case, we have to take τ as a state too.

Can we do this ?

- Condition 1 : tax is const. for all $t \geq 1$.
- Condition 2 : period - by - period BC is satisfied.

If so, we can eliminate the price by replacing it by $A = r_t / p_t$ (MP condition).

<u>Claim</u> Policy fcn's are HD 1. I.e., $c = g_c(y) = y \cdot g_c(1)$, $k' = g_{k'}(y) = y \cdot g_{k'}(1)$.

pf)

Let $\tilde{c} = \{c_t\}_{t=0}^{\infty}$, $W(\tilde{c}) = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$. Then $W(\lambda \tilde{c}) = \lambda^{1-\sigma} W(\tilde{c})$.

Let $\Pi(y_0)$ be the budget - feasible set given y_0. Then $\Pi(\lambda y_0) = \lambda \Pi(y_0)$

Thus, if $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ solves the problem given y_0,

then $\{\lambda c_t^*, \lambda k_{t+1}^*\}_{t=0}^{\infty}$ solves the problem given λy_0.

Thus,

$$g_c(\lambda y) = \lambda g_c(y)$$
$$g_{k'}(\lambda y) = \lambda g_{k'}(y) \quad \square$$

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Then $c = g_c(y) = y \cdot g_c(1) \equiv y \cdot \gamma_c$

$k' = g_{k'}(y) = y \cdot g_{k'}(1) \equiv y \cdot \gamma_k$

[Euler]: $\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta\,[A(1-\tau)+1-\delta]$

$$\text{LHS} = \left(\frac{y_{t+1}\cdot\gamma_c}{y_t\ \ \gamma_c}\right)^{\sigma} = \left(\frac{[A(1-\tau)+1-\delta]\,k_{t+1}}{y_t}\right)^{\sigma}$$

$$= \left(\frac{[A(1-\tau)+1-\delta]\,y_t\cdot\gamma_k}{y_t}\right)^{\sigma}$$

$$\therefore\ \gamma_k = \beta^{\frac{1}{\sigma}}\,[A(1-\tau)+1-\delta]^{\frac{1-\sigma}{\sigma}}$$

$$k_1 = \gamma_k \cdot y_0$$

$$c_0 = y_0 - k_1$$

$$= (1-\gamma_k)\,y_0 = (1-\gamma_k)\{[A(1-\tau_0)+1-\delta]k_0 + T_0\}$$

Since $\gamma_k^{1}{}^{\text{economy}} = \beta^{\frac{1}{\sigma}}\,[A(1-\tau_k^1)+1-\delta]^{\frac{1-\sigma}{\sigma}}$

$> \beta^{\frac{1}{\sigma}}\,[A+1-\delta]^{\frac{1-\sigma}{\sigma}}$ (Assume $\sigma > 1$. Since $\frac{1-\sigma}{\sigma} < 0$ and $\tau_k^1 < 0$)

$= \gamma_k^2$,

$$c_0^1 = (1-\gamma_k^1)\{[A(1-\tau_{k0}^1)+1-\delta]k_0\}$$

$$\wedge$$

$$c_0^2 = (1-\gamma_k^2)\{[A(1-\tau_k^2)+1-\delta]k_0 + \tau_{k0}^2 r_0 k_0\}$$

$$= (1-\gamma_k^2)\ \ [A+(1-\delta)]k_0$$ ($p_0 = 1$ normalising)

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$\boxed{2}$ Solving BC:

In eqm. BC holds w/ equality.

$$\sum_{t=0}^{\infty} p_t (c_t + x_t) = \sum_{t=0}^{\infty} (r_t (1-\tau_t) k_t + T_0$$

$$\Leftrightarrow \sum_{t=0}^{\infty} (p_t c_t + p_t k_{t+1}) = \sum_{t=0}^{\infty} [r_t (1-\tau_t) + (1-\delta) p_t] k_t + T_0 \quad (*)$$

FOC wrt k_{t+1}: $p_t = p_{t+1} (1-\delta) + r_{t+1} (1-\tau_{t+1})$

Thus

$$(*) \Leftrightarrow \sum_{t=0}^{\infty} p_t c_t = [r_0 (1-\tau_0) + (1-\delta)] k_0 + T_0 \quad (*)' \quad \left(\begin{array}{c}\text{Normalize} \\ p_0 = 1\end{array}\right)$$

FOC wrt k_{t+1} $\Rightarrow$

$$p_t / p_{t+1} = 1-\delta + \frac{r_{t+1}}{p_{t+1}} (1-\tau_{t+1})$$

$$= 1-\delta + A(1-\tau_{t+1}) \quad (\text{MP cond.})$$

$$\Rightarrow p_t = [A(1-\tau) + 1-\delta]^{-1} p_{t-1} \quad \left(\begin{array}{c}\text{since } \tau_{t+1} = \tau \\ \forall t > 0\end{array}\right)$$

$$= [A(1-\tau) + 1-\delta]^{-t} \underbrace{p_0}_{=1}$$

Euler $\Rightarrow$

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} [A(1-\tau) + 1-\delta]$$

$$\Rightarrow c_t = \{\beta [A(1-\tau) + 1-\delta]\}^{\frac{1}{\sigma}} c_{t-1}$$

$$= \{\beta [A(1-\tau) + 1-\delta]\}^{\frac{t}{\sigma}} c_0$$

Thus

$$(*)' \Leftrightarrow \sum_{t=0}^{\infty} \beta^{\frac{t}{\sigma}} [A(1-\tau) + 1-\delta]^{t\left(\frac{1-\sigma}{\sigma}\right)} c_0 = [A(1-\tau_0) + (1-\delta)] k_0 + T_0$$

Assume $\beta^{\frac{1}{\sigma}} [A(1-\tau) + 1-\delta]^{\frac{1-\sigma}{\sigma}} < 1$. $\leftarrow$ $\exists$ eqm alloc' then automatically satisfied.

Then

$$c_0 \frac{1}{1 - \beta^{\frac{1}{\sigma}} [A(1-\tau) + 1-\delta]^{\frac{1-\sigma}{\sigma}}} = [A(1-\tau_0) + (1-\delta)] k_0 + T_0$$

$$\therefore c_0^i = \left\{1 - \beta^{\frac{1}{\sigma}} [A(1-\tau^i) + 1-\delta]^{\frac{1-\sigma}{\sigma}}\right\} y_0^i$$

$\llcorner$ same as $\boxed{1}$

- Heterogeneous economy, idiosyncratic risk.
- Huggett economy w/ no-borrowing constraint.

(See 2007 Final Q2)

(a) A stationary eqm is a value fcn v, policy fcns a', c, eqm interest rate r, and the stationary dist. λ^* s.t.

(i) Optimization : given r, a', c solve

$$\begin{cases} v(a,y) = \max_{a',c} u(c) + \beta\left\{\frac{1}{2}v(a',1-\varepsilon) + \frac{1}{2}v(a',1+\varepsilon)\right\} \\ \text{s.t.} \quad c + a' = (1+r)a + y \\ \qquad a' \geq 0. \end{cases}$$

and v is the associated value fcn

(ii) Mkt clear :

$$\int_{\mathbb{R}_+ \times \{1-\varepsilon, 1+\varepsilon\}} a'(a,y)\, d\lambda^*(a,y) = 0$$

(iii) Stationarity of λ^* : $\forall \mathcal{A} \subseteq \mathbb{R}_+$, $\Upsilon \subseteq \{1-\varepsilon, 1+\varepsilon\}$

$$\lambda^*(\mathcal{A} \times \Upsilon) = \int_{\mathbb{R}_+ \times \{1-\varepsilon, 1+\varepsilon\}} Q((a,y), \mathcal{A} \times \Upsilon)\, d\lambda^*(a,y)$$

where $Q((a,y), \mathcal{A} \times \Upsilon) = \sum_{y' \in \Upsilon} I(a'(a,y) \in \mathcal{A})\, \underbrace{\pi(y',y)}_{=1/2}$

From Mkt clearing cond. and no-borrowing const $(a' \geq 0)$,

$$a'(a,y) = 0 \quad \text{a.s}$$

Thus by BC and $b_0 = 0$,

$$c(a,y) = y \qquad \forall a \in \mathbb{R}_+,\ y \in \{1-\varepsilon, 1+\varepsilon\}$$

Therefore the asset dist. :

$$\lambda^*(a) = \int_\Upsilon \lambda^*(a,y)\,dy = \begin{cases} 0 & \text{if } a > 0 \\ 1 & a = 0 \end{cases}$$

Euler : $$u'(c) \geqq \beta(1+r)\, E\, u'(c')$$

$$\Leftrightarrow \quad c^{-\gamma} \geq \beta(1+r)\, E\, c'^{-\gamma}$$

$$\Leftrightarrow \quad y^{-\gamma} \geq \beta(1+r)\left[\tfrac{1}{2}(1-\varepsilon)^{-\gamma} + \tfrac{1}{2}(1+\varepsilon)^{-\gamma}\right] \quad \text{(by policy fcn.)}$$

Since Euler holds for all $y \in \{1-\varepsilon, 1+\varepsilon\}$,

$$r^* \le \frac{2}{\beta}\left[\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{\gamma} + 1\right]^{-1} - 1$$

Any $r \in [0, r^*]$ satisfies EE (suff.). Thus that is consistent w/ eqm.

(b) Mkt clearing cond:

asset : $t-1$: $\int_{A\times Y} a'(a,y)\, d\lambda^* = 0$.

t : $\int_{A\times Y} a_t'(a,y)\, d\lambda^* = g > 0$ should finance $(a'>0) \Rightarrow r^* > 0$

$t+1$: $\int_{A\times Y} a'(a,y)\, d\lambda_t = 0$ repaid :

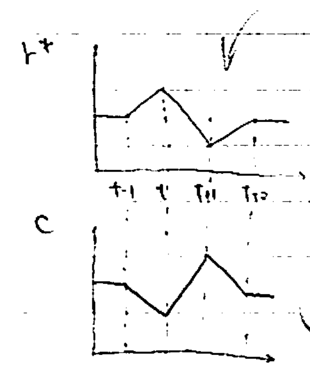
$t+2$: $\int_{A\times Y} a'(a,y)\, d\lambda^* = 0$

good : $t-1$: $\int c(a,y)\, d\lambda^* = \int y\, d\lambda^*$

t : $\int c_t(a,y)\, d\lambda^* + g = \int y\, d\lambda^*$

$t+1$: $\int c_{t+1}(a,y)\, d\lambda_t = \int y\, d\lambda^* + (1+r_t^*)\, g$

$t+2$: $\int c_{t+2}(a,y)\, d\lambda^* = \int y\, d\lambda^*$



(c) At time t, 2 types of agents $(0, 1+\varepsilon)$, $(0, 1-\varepsilon)$

So. r^* is large enough to get

$$\begin{cases} a'(0, 1+\varepsilon) > 0 \\ a'(0, 1-\varepsilon) \ge 0 \end{cases}$$

The agents w/ $(0, 1+\varepsilon)$ can smooth their cons. across t and $t+1$.

$\Rightarrow$ better off

$\Rightarrow$ weakly P.O.

If g is large enough s.t.

$$g > \underbrace{(1+\varepsilon)\cdot \lambda^*(0, 1+\varepsilon)}_{\text{wealth of rich people}}.$$

Then gov't needs to finance g by rising r^* s.t.

$$\begin{cases} a'(0, 1+\varepsilon) > 0 \\ a'(0, 1-\varepsilon) > 0 \end{cases}$$

All agents can smooth their cons. $\rightarrow$ better off

$\rightarrow$ Strictly P.O.

- Finite Period DP

(a) Set up DP :

$$\begin{cases} \lambda_t : & \text{consumption share of wealth at } t \\ 1-\lambda_t : & \text{asset} \quad '' \\ \alpha_t : & \text{risk asset share of whole asset at } t \\ 1-\alpha_t : & \text{risk free asset} \quad '' \end{cases}$$

HH problem :

$$\begin{cases} V_t(w_t) = \max\limits_{\substack{0\le\lambda_t\le 1 \\ \alpha_t}} u(\lambda_t w_t) + \beta \int V_{t+1}\left[(1-\lambda_t)\overset{w_t}{\wedge}(\alpha_t R + (1-\alpha_t)R_f)\right] F(dR) & \text{for } t=0,\cdots,T-1 \\ V_T(w_T) = \max\limits_{0\le\lambda_T\le 1} u(\lambda_T w_T) & \text{for } t=T. \end{cases}$$

(b) $\boxed{\text{Claim}}$ α_t is const. over time

pf

V_T is HD $(1-\sigma)$, since $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$.
Thus V_{T-1} is also HD $(1-\sigma)$, and so on.
Thus by induction, V_t is HD $(1-\sigma)$ $\forall t=0,\dots,T$.

Then we focus on $W=1$ case :

$$V_t(1) = \max_{\substack{0\le\lambda_t\le 1 \\ \alpha_t}} u(\lambda_t) + \beta \int V_{t+1}\left[(1-\lambda_t)(\alpha_t R + (1-\alpha_t)R_f)\right] F(dR)$$

$$= \max u(\lambda_t) + \beta \int (1-\lambda_t)^{1-\sigma} (\alpha_t R + (1-\alpha_t)R_f)^{1-\sigma} V_{t+1}(1) F(dR)$$

$$= \max u(\lambda_t) + \beta (1-\lambda_t)^{1-\sigma} V_{t+1}(1) \int (\alpha_t R + (1-\alpha_t)R_f)^{1-\sigma} F(dR)$$

Therefore,

$$\alpha_t = \alpha^* = \arg\max_{\alpha} \int (\alpha R + (1-\alpha)R_f)^{1-\sigma} F(dR)$$

does not depend on t

$\therefore$ α_t is constant over time.

Q.E.D.

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2007 Fall Q II.1. Durable Goods

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- Single consumer
- output = perishable cons. goods + inv. in a durable goods ; $w = c_t + d_{xt}$
- d_t : stock of durables enters u.
- $$\begin{cases} \max_{c_t, d_t, d_{xt}} & \sum_t \beta^t \{u_1(c_t) + u_2(d_t)\} \\ \text{s.t.} & c_t + d_{xt} \le w \\ & d_{t+1} \le (1-\delta) d_t + d_{xt} \\ & \text{nonnegativity}, \ d_0 \text{ given} \end{cases}$$

u_1, u_2 str inc. cont

(a) Cond. for Canonical Form

Canonical Form
$$\begin{cases} \max_{\{d_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} F(d_t, d_{t+1}) \\ \text{s.t.} & d_{t+1} \in \Gamma(d_t) \\ & d_0 \text{ given} \end{cases}$$

$$F(d_t, d_{t+1}) := u_1[w - (d_{t+1} - (1-\delta) d_t)] + u_2(d_t)$$
$$\Gamma(d_t) := \{ d_{t+1} \in \mathbb{R}_+ \mid w - (d_{t+1} - (1-\delta) d_t) \ge 0 \}$$

Condition : (for the solution (max) to exist : F bdd $\Rightarrow u_1, u_2$ bdd)

to eliminate c_t (constraint is binding) $\Rightarrow u_1, u_2$ str. inc

(b) Bellman eqn

$$V(d) = \max_{d'} u_1\{w - [d' - (1-\delta) d]\} + u_2(d) + \beta V(d')$$

$$\text{s.t.} \quad d' \in \Gamma(d) \equiv \{ d' \in \mathbb{R}_+ \mid w - (d' - (1-\delta) d) \ge 0 \}$$

(c) V str. inc & str. concave / Cond. on u_1 & u_2 AND Proof.

○ V : str. increasing (by Thm 4.7)

(A4.4) F bdd & cont $\Rightarrow$ u_1 and u_2 are bounded and continuous.

(A4.5) $F(\cdot, d')$ is str. inc $\Rightarrow$ u_1 and u_2 are nondecreasing and at least one of them is strictly increasing. (#1)

pf.

(See pf of Thm 4.7)

nondecreasing ← str increasing

Suff' to show that $T[C'(X)] \subseteq C''(X)$.

Let $f \in C'(X)$, and $d, \tilde{d} \in \mathbb{R}_+$ be $d > \tilde{d}$.

Since T is monotone : $w + (1-\delta)d > w + (1-\delta)\tilde{d}$, $f \in C'(X)$ and (#1) holds,

$$(Tf)(d) > (Tf)(\tilde{d})$$

where $(Tf)(d) = \max_{d' \in \Gamma(d)} \left[u_1(w - d' + (1-\delta)d) + u_2(d) + \beta f(d')\right]$.

Thus, $T[C'(X)] \subseteq C''(X)$ and hence $v \in C''(X)$. □

○ V is str. concave (by Thm 4.8)

(A4.4) —〃—

(A4.7) F str concave $\Rightarrow$ u_1 and u_2 are concave and at least one of them is str. concave. (##)

pf:

(See pf of Thm 4.8) ← weakly concave ← str. concave.

Suff' to show that $T[C'(X)] \subseteq C''(X)$.

Let $f \in C'(X)$ and $d_\theta \equiv \theta d + (1-\theta)\tilde{d}$, $d \neq d'$, $\theta \in (0,1)$.

Let $d' \in \Gamma(d)$ and $\tilde{d}' \in \Gamma(\tilde{d})$ attain $(Tf)(d)$ and $(Tf)(\tilde{d})$ respectively.

$d'_\theta = \theta d' + (1-\theta)\tilde{d}'$. Then

$$\begin{aligned}(Tf)(d_\theta) &\geq u_1(w - d'_\theta + (1-\delta)d_\theta) + u_2(d_\theta) + \beta f(d'_\theta) \\ &> \theta\left[u_1(w - d' + (1-\delta)d) + u_2(d) + \beta f(d')\right] \quad (\text{since } d'_\theta \in \Gamma(d_\theta)) \\ &\quad + (1-\theta)\left[u_1(w - \tilde{d} + (1-\delta)\tilde{d}) + u_2(\tilde{d}) + \beta f(\tilde{d}')\right] \quad \text{by (\#\#)} \\ &= \theta (Tf)(d) + (1-\theta)(Tf)(\tilde{d})\end{aligned}$$

Thus, $T[C'(x)] \subseteq C''(X)$ and hence $v \in C''(X)$ □

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(d)

policy for d'

Envelope cond. : $V'(d) = u_1'[w - g^*(d) + (1-\delta)d](1-\delta) + u_2'(d)$

FOC : $u_1'[w - g^*(d) + (1-\delta)d] = \beta V'(g^*(d))$

$\Rightarrow$ Euler : $u_1'[w - g^*(d) + (1-\delta)d] = \beta\{u_1'[w - g^*[g^*(d)] + (1-\delta)g^*(d)](1-\delta) + u_2'[g^*(d)]\}$

★ We need an extra assumption for the following parts :

u_1 and u_2 are strictly concave.

(e) $\boxed{\text{Claim}}$ $\exists!$ ss $d^* > 0$.

★ Pf Existence

Since u_1 and u_2 are strictly concave,

$\exists\ d^*$ s.t.

$$\underbrace{u_1'(w - \delta d^*)[1 - \beta(1-\delta)]}_{\text{increasing in } d^*} = \underbrace{\beta u_2'(d^*)}_{\text{decreasing in } d^*}.$$

Then let

$c^* \equiv w - \delta d^*$,

$x^* = \delta d^*$.

Then we have

$d' = (1-\delta)d^* + x^* = d^*$

Since EE is satisfied by construction

$\Leftrightarrow u_1'(w - \delta d^*) = \beta[u_1'(w - \delta d^*)(1-\delta) + u_2'(d^*)]$

and feasibility is satisfied by construction

$c^* + x^* = w$,

$\{c^*, x^*, d^*\}$ is the solution of the problem given $d_0 = d^*$.

Thus,

d^* is a steady state value of the stock.

□

(pf) Uniqueness

Sps not ; $\exists$ d^* and d^{**} s.t. those are SS stock and $d^* \neq d^{**}$

Then by EE,

$$u_1'(w - \delta d^*)[1-\beta(1-\delta)] = \beta u_2'(d^*) , \text{ and}$$
$$u_1'(w - \delta d^{**})[1-\beta(1-\delta)] = \beta u_2'(d^{**})$$

Since u_1' is strictly increasing in d

and u_2' " decreasing " ,

this is a contradiction.

Thus SS value of the stock is unique. $\square$

(pf) $d^* > 0$.

Sps not. Then $d^* = 0$.

Since d^* is SS, if $d_0 = d^* = 0$, then $d_t = d^* = 0$, $\forall t$.

So $d_1 = d^* = 0$.

However, since u_2 satisfies Inada condition,

d_1 cannot be zero. This is a contradiction. $\square$

(f) $\boxed{\text{Claim}}$ Policy fcn's are increasing ($c^*(d)$, $g^*(d) = d'$)

(pf) Since u_1 and u_2 are str. concave, so is V by (c).

Suppose g^* is not increasing.

Then if $d > \tilde{d}$, then $g^*(d) < g^*(\tilde{d})$.

FOC:

$$u_1'(c^*(d)) = \beta V'(g^*(d))$$

Then

$$\beta V'(g^*(d)) > \beta V'(g^*(\tilde{d})) \qquad (V \text{ is str concave})$$
$$\Leftrightarrow \quad u_1'(c^*(d)) > u_1'(c^*(\tilde{d}))$$
$$\Leftrightarrow \quad c^*(d) < c^*(\tilde{d}) \qquad (u_1 \text{ is str concave})$$

By the feasibility, d and $\tilde{d}$ must satisfy

$$c^*(d) + g^*(d) = w + (1-\delta)d,$$
$$c^*(\tilde{d}) + g^*(\tilde{d}) = w + (1-\delta)\tilde{d}$$

This is a contradiction. Thus g^* is increasing, i.e. $g^*(d) \geq g^*(\tilde{d})$

By the same argument, we get $c^*(d) \geq c^*(\tilde{d})$.

Thus c^* is also increasing. $\square$

Macro 2007 Fall Q II . 1 Durable Goods

(g) $\boxed{\text{Claim}}$ The system is globally stable.

pf Only for $d_0 < d^*$ case.

Since $d^* > 0$ i.e., 0 cannot be ss, we have
$$g^*(0) > 0.$$
Let $g^*(0) = d_1$
Since $0 < d^*$, we have
$$d_1 = g^*(0) < g^*(d^*) = d^*.$$
Thus $0 < d_1 < d^*$.
Since $d_1 < d^*$ and ss is unique, $g^*(d_1) \neq d_1$.
Also, since $0 < d_1$ and g^* increasing,
$$d_1 = g^*(0) \leq g^*(d_1).$$
Thus $g^*(d_1) > d_1$. Let $d_2 = g^*(d_1)$.

By the same argument, we have $\{d_t\}_{t=0}^{\infty}$ s.t.
$$\forall t \quad 0 < d_t < d^*$$
$$\text{and} \quad d_t < d_{t+1}.$$

Thus, $\{d_t\}$ is converging to d^*.

Fix this seq $\{d_t\}$.

$\forall d$ s.t. $0 < d < d^*$, $\exists\, i, i+1$ s.t. $d_i < d < d_{i+1}$
Thus $g^*(d_i) < g^*(d) < g^*(d_{i+1})$
But since $g^*(d_i) = d_{i+1}$, we must have
$$g^*(d) > d_{i+1} > d.$$
Also since $d < d^*$, we have
$$g^*(d) < g^*(d^*) = d^*.$$
Therefore, we get
if $0 \leq d < d^*$, then $d < g^*(d) < d^*$.
Thus, $d, g^*(d), g^* \circ g^*(d), \cdots$ is converging to d^*.
$\Rightarrow$ system is globally stable.
(locally) only for $d < d^*$.

- open economy.

(a)

$$\begin{aligned}
\Delta c_t &= c_t - c_{t-1} \\
&= \frac{r}{1+r}\left[A_t + y_t + \frac{1}{\cancel{r}}\bar{y} - \left(A_{t-1} + y_{t-1} + \frac{1}{\cancel{r}}\bar{y}\right)\right] \\
&= \frac{r}{1+r}\left[(1+r)(A_{t-1} + y_{t-1} - c_{t-1}) + y_t - A_{t-1} - y_{t-1}\right] \\
&= \frac{r}{1+r}\left[r(A_{t-1} + y_{t-1}) - \underbrace{r\left(A_{t-1} + y_{t-1} + \frac{1}{r}\bar{y}\right)}_{(1+r)c_{t-1}} + (1+r)\pi + y_t\right] \\
&= \frac{r}{1+r}(y_t - \bar{y}) + r\pi .
\end{aligned}$$

(b) Euler :

$$u'(c_t) = \underbrace{\beta(1+r)}_{1} E_t u'(c_{t+1})$$

$$\Leftrightarrow \quad e^{-\gamma c_t} = E_t e^{-\gamma c_{t+1}}$$

By the expression in (a)

$$\begin{aligned}
e^{-\gamma c_t} &= E_t e^{-\gamma\left[c_t + \frac{r}{1+r}(\overset{\bar{y}+\varepsilon_{t+1}}{\overset{\|}{y_{t+1}}} - \bar{y}) + r\pi\right]} \\
&= e^{-\gamma c_t} \cdot e^{-\gamma r \pi} \cdot \underbrace{E_t e^{-\frac{\gamma r}{1+r}\varepsilon_{t+1}}}_{= e^{\frac{\gamma^2 r^2}{(1+r)^2}\frac{1}{2}\sigma^2}}
\end{aligned}$$

$$\because \; -\frac{\gamma r}{1+r}\varepsilon \sim N\left(0, \left(\frac{\gamma r}{1+r}\right)^2 \sigma^2\right)$$

$$\Leftrightarrow \quad e^{-\gamma r \pi + \frac{\gamma^2 r^2}{(1+r)^2}\frac{1}{2}\sigma^2} = 1$$

$$\Leftrightarrow \quad -\gamma r \pi + \frac{\gamma^2 r^2}{(1+r)^2}\frac{1}{2}\sigma^2 = 0$$

$$\Leftrightarrow \quad \pi = \frac{\gamma r \sigma^2}{2(1+r)^2} > 0$$

With this π, the result in (a) satisfies EE.

(c) $E(y_t) = \bar{y}$ stationary

$Var(y_t) = \sigma^2$ stationary $\leftarrow$ iid across agents

$$E(c_t - c_{t-1}) = \frac{r}{1+r}\left(\underbrace{E y_t}_{\bar{y}''} - \bar{y}\right) + r\pi$$

$$= r\pi$$

$$\therefore E(c_t) = r\pi t.$$

Macro

2007 Spring QI.1 Different Discount Factor

- Ak model w/ different discount factor (CE w/ two types of agents)
- Derivation of c_0.

See 2007 Fall QI.1 p.2.3.

$$
\begin{aligned}
c_0^1 &= \left[1 - \beta_1^{\frac{1}{\sigma}} (A+1-\delta)^{\frac{1-\sigma}{\sigma}}\right] (A+1-\delta) k_0 \\
&< \left[1 - \beta_2^{\frac{1}{\sigma}} (A+1-\delta)^{\frac{1-\sigma}{\sigma}}\right] (A+1-\delta) k_0 \\
&= c_0^2
\end{aligned}
$$

pacient $\rightarrow$ consume less in period 0

Macro 2007 spring Q II . 1 Infinitely lived consumers and dynamic programming

8105

- Representative Consumer
- Leisure in his utility

(a) Formulate DP

$$v(k) = \max_{c,\,l,\,k'} \; \gamma \log c + (1-\gamma) \log (1-l) + \beta\, v(k')$$

s.t. $c + k' \leq \theta k^{\alpha} l^{1-\alpha}$

nonnegativity.

(b)

FOC

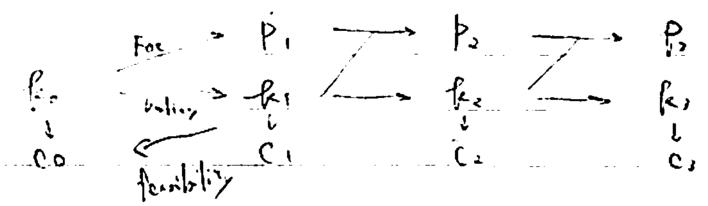
(c) FOC wrt k

- Price

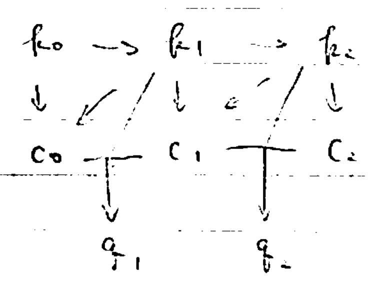
(d) FOC wrt

↳ Price

(c)



(d)



Macro 2007 Spring Q II . 2 Ramsey with Government in the Utility Function

8105

- Ramsey Problem w/ Gov't in the utility. $u(c_t, l_t, g_t)$
- 1 consumer + 1 firm / single sector growth model
- Finance by τ_{nt} and τ_{kt}
- $\tau_{kt} \to 0$? Chamley - Judd characterization

(a) A TDCE is a quantity $\{c_t, l_t, n_t, x_t, k_{t+1}\}_{t=0}^{\infty} \equiv \Omega_{HH}$ for HH

~ $\{n_t^f, k_t^f\}_{t=0}^{\infty} \equiv \Omega_f$ for firm

a price system $\{p_t, w_t, r_t\}_{t=0}^{\infty} \equiv \mathbb{P}$,

a fiscal policy $\{g_t, \tau_{nt}, \tau_{kt}\}_{t=0}^{\infty} \equiv \mathbb{G}$, and

profit $\{\pi_t\}_{t=0}^{\infty}$

Normal F → Put π_t in the def of TDCE

such that (i) given $\mathbb{P}$ and $\mathbb{G}$, Ω_{HH} solves

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, l_t, g_t)$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} p_t(c_t + x_t) \leq \sum_{t=0}^{\infty} [r_t(1-\tau_{kt})k_t + w_t(1-\tau_{nt})n_t + \pi_t]$$

$$k_{t+1} = (1-\delta)k_t + x_t \quad \forall t$$

$$n_t + l_t = 1 \quad \forall t$$

nonnegativity, k_0 given.

(ii) given $\mathbb{P}$, Ω_f solves

$$\max \sum_{t=0}^{\infty} \{p_t F(k_t^f, n_t^f) - r_t k_t^f - w_t n_t^f\}$$

s.t. nonnegativity

(iii) Mkt clear :

$$k_t = k_t^f, \quad n_t = n_t^f \quad \forall t$$

$$c_t + x_t + g_t = F(k_t, n_t)$$

(iv) GBC :

$$\sum_{t=0}^{\infty} p_t g_t = \sum_{t=0}^{\infty} (r_t \tau_{kt} k_t + w_t \tau_{nt} n_t).$$

(v) Profit

$$\pi_t = p_t F(k_t^f, n_t^f) - r_t k_t^f - w_t n_t^f \quad \forall t$$

IC : $\sum_{t=0}^{\infty} \beta^t (U_{ct} c_t - U_{\ell t} n_t) = \frac{U_{c0}}{1+\tau_{c0}} [(1-\delta)(1+\tau_{x0}) + F_{k0}(1-\tau_{k0})] k_0$

(b) <u>Asymptotic Behavior of Ramsey Tax system</u>

Assume U satisfies the Inada condition $\to$ solution is interior
F is CRS $\to$ $\pi_t = 0, \forall t$.

Construct a Ramsey Problem.

$$\begin{cases} \text{HH FOC w.r.t } c_t : & \beta^t U_{ct} - \lambda p_t = 0 & \text{— (1)} \\ \quad \sim \quad \ell_t : & \beta^t U_{\ell t} - \lambda w_t (1-\tau_{nt}) = 0 & \text{— (2)} \\ \quad \sim \quad k_{t+1} : & p_t + r_{t+1}(1-\tau_{kt+1}) - p_{t+1}(1-\delta) = 0 & \text{— (3)} \\ \text{Firm's FOC} : & r_t / p_t = F_{kt} & \text{— (4)} \\ \quad \sim & w_t / p_t = F_{nt} & \end{cases}$$

Then HHBC becomes — Assume U is str. inc.

$$\sum p_t [c_t + k_{t+1} - (1-\delta) k_t] = \sum [r_t (1-\tau_{kt}) k_t + w_t (1-\tau_{nt}) n_t]$$

$$\Leftrightarrow \sum [p_t c_t - w_t (1-\tau_{nt}) n_t] = \sum \{[r_t (1-\tau_{kt}) + p_t (1-\delta)] k_t - p_t k_{t+1}\}$$

Substitute $(p_0 = 1)$ $\quad p_t = \frac{\beta^t U_{ct}}{U_{c0}}$ by (1), $\quad w_t(1-\tau_{nt}) = \frac{\beta^t U_{\ell t}}{U_{c0}}$ by (2).

Then, $\quad \text{LHS} = \frac{1}{U_{c0}} \sum_{t=0}^{\infty} \beta^t [U_{ct} c_t - U_{\ell t} n_t]$.

Also $\quad \text{RHS} = [r_0 (1-\tau_{k0}) + p_0 (1-\delta)] k_0$

$$+ \sum_{t=0}^{\infty} [r_{t+1}(1-\tau_{kt+1}) + p_{t+1}(1-\delta) - p_t] k_{t+1} \quad = 0 \quad \text{by (3)}$$

(since $\lim p_t k_{t+1} = 0$ by TVC)

$$= [F_{k0}(1-\tau_{k0}) + 1 - \delta] k_0 \quad \text{by (4)}$$

Σ
$t = 1$

GBC redundant!

Thus, [Ramsey Problem] : λ : multiplier on (IC).

$$\max_{\substack{c_t, n_t, k_t \\ g_t, \tau_{nt}, \tau_{kt}}} \sum_{t=1}^{\infty} \beta^t W(c_t, 1-n_t, g_t; \lambda) + W_0(c_0, 1-n_0, g_0, k_0; \lambda)$$

s.t. $\quad n_t \in [0,1], \quad c_t + g_t + k_{t+1} - (1-\delta) k_t \leq F(k_t, n_t)$

where $\quad W(c_t, 1-n_t, g_t; \lambda) = U(c_t, 1-n_t, g_t) - \lambda (U_{ct} c_t - U_{nt} n_t)$

$$W_0(c_0, 1-n_0, g_0, k_0; \lambda) = U(c_0, 1-n_0, g_0) + \lambda \{ k_0 [F_k(k_0, n_0)(1-\tau_{k0}) + 1 - \delta] - [U_{c0} c_0 - U_{n0} n_0] \}$$

Euler :

$$W_{ct} = \beta W_{ct+1} [1 - \delta + F_{kt+1}]$$

In ss, $\quad 1 = \beta [1 - \delta + F_{k\infty}]$

Euler from consumer's problem

$$U_{ct} = \beta U_{ct+1} [1 - \delta + (1-\tau_{kt+1}) F_{kt+1}]$$

In ss, $\quad 1 = \beta [1 - \delta + (1-\tau_{k\infty}) F_{k\infty}] \quad$ Thus $\tau_{kt} \to 0$.

Macro

2006 Fall Q I.1 Taxes on Firms

- def of TDCE w/ taxes on firms
- equivalent setup w/ taxes on HH

A TDCE is a quantity $\{c_t, x_t, l_t, k_{t+1}\}_{t=0}^{\infty} \equiv \Omega_{HH}$ for HH,

$\qquad$ — " — $\{l_t^i, k_t^i\}_{t=0}^{\infty} \equiv \Omega_f^i$, $i=1,\dots,J$. for firms

prices $\{p_t, r_t, w_t\}_{t=0}^{\infty}$, and

fiscal policy $(\{g_t\}_{t=0}^{\infty}, \tau)$

such that

(i) given prices, Ω_{HH} solves

$$\max \quad u(\{c_t\}, \{l_t\})$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} p_t(c_t + x_t) \leq \sum_{t=0}^{\infty} (r_t k_t + w_t l_t)$$

$$k_{t+1} = (1-\delta)k_t + x_t \qquad \forall t$$

$$0 \leq l_t \leq 1, \text{ nonnegativity.}$$

(ii) given prices, Ω_f^i $i=1,\dots,J$ solves

$$\max \quad p_t(1-\tau)F_t^i(k_t^i, l_t^i) - r_t k_t^i - w_t l_t^i$$

$$\text{s.t.} \quad \text{nonnegativity.}$$

(iii) Mkt clear: $\forall t$,

$$k_t = \sum_{i=1}^{J} k_t^i, \quad l_t = \sum_{i=1}^{J} l_t^i$$

$$c_t + x_t + g_t = \sum_{i=1}^{J} F_t^i(k_t^i, l_t^i)$$

(iv) GBC:

$$\sum_{t=0}^{\infty} p_t g_t = \sum_{t=0}^{\infty} \sum_{i=1}^{J} p_t(1-\tau)F_t^i(k_t^i, l_t^i)$$

Generalized by

- nonseparable utility fcn
- time-dependent production fcn.
- consolidated GBC

Not in general

- time endowment
- no transfer

Assume the usual ass'ns to make FOC suffient and sol'n unique.
CRS tech $\to$ representative firm

Eqm charaterization :

GBC redundant

$$\begin{cases} U_c(t) = \beta U_c(t+1)\left[1-\delta + F_k(t+1)(1-\tau)\right] & \text{Intertemporal EE} \\ \dfrac{U_n(t)}{U_c(t)} = F_n(t)(1-\tau) & \text{Intratemporal EE} \\ c_t + g_t + k_{t+1} - (1-\delta)k_t = F(k_t, n_t) & \text{Feasibility} \end{cases}$$

Since $\{g_t\}$ is exogenous, eqm allocation is characterized by these 3 eqn
(3 unknown, 3 eqn)

FOC w/ HH tax system (general case)

$$\begin{cases} \beta^t U_c(t) - \lambda p_t(1+\tau_{ct}) = 0 \\ \beta^t U_n(t) - \lambda w_t(1-\tau_{nt}) = 0. \\ p_t(1+\tau_{xt}) = p_{t+1}(1+\tau_{xt+1})(1-\delta) + r_{t+1}(1-\tau_{kt+1}) \\ r_t/p_t = F_k(t), \quad w_t/p_t = F_n(t) \end{cases}$$

Eqm characterization :

$$\begin{cases} U_c(t) = \beta U_c(t+1)\dfrac{1+\tau_{ct}}{1+\tau_{ct+1}}\left[\dfrac{1+\tau_{xt+1}}{1+\tau_{xt}}(1-\delta) + \dfrac{1-\tau_{kt+1}}{1+\tau_{xt}}F_k(t+1)\right] \\ \dfrac{U_c(t)}{U_c(t)} = F_n(t)\dfrac{1-\tau_{nt}}{1+\tau_{ct}} \\ c_t + g_t + k_{t+1} - (1-\delta)k_t = F(k_t, n_t) \end{cases}$$

Then if we set $\tau_{it} = \tau_i \quad \forall i \in \{c, n, x, k\}$

and $\dfrac{1-\tau_k}{1+\tau_x} = 1-\tau = \dfrac{1-\tau_n}{1+\tau_c}$,

then the 3 eqn are the same in both econ. $\to$ eqm allocations are the same.

Macro 2006 Fall Q II.1 DP in an economy w/ leisure

8105

- PP w/ leisure in the utility
- guess and verify
- Sequential eqm
- AD eqm
- See spring 2007 Q II.1

(a) Bellman eqn

$$v(k) = \max_{c,\ell,k',x} \log c + \gamma \log x + \beta v(k')$$

s.t. $c + k' \leq \theta k^{\alpha} \ell^{1-\alpha}$

$x + \ell \leq 1$

nonnegativity. $k_0 < \bar{k}_0$

Guess & Verify

(b) Str. inc. of log $\Rightarrow$ all constraints hold w/ equality. / Inada $\rightarrow$ ignore the nonnegativity

Guess :

$$v(k) = a_0 + a_1 \log k$$

$$\ell(k) = \bar{\ell}.$$

Then $x(k) = 1 - \bar{\ell}$.

Bellman eqn :

$$v(k) = \max_{k'} \log(\theta k^{\alpha} \bar{\ell}^{1-\alpha} - k') + \gamma \log(1-\bar{\ell}) + \beta(a_0 + a_1 \log k') \quad (*)$$

FOC wrt k' (as usual, sufficient) :

$$\frac{-1}{\theta k^{\alpha} \bar{\ell}^{1-\alpha} - k'} + \beta a_1 \frac{1}{k'} = 0$$

$$\Leftrightarrow \quad k' = \frac{\beta a_1}{1+\beta a_1} \theta k^{\alpha} \bar{\ell}^{1-\alpha}$$

★ Envelope cond. :

$$v'(k) = \frac{1}{\theta k^{\alpha} \bar{\ell}^{1-\alpha} - k'(k)} \alpha \theta k^{\alpha-1} \bar{\ell}^{1-\alpha}$$

$$\Leftrightarrow \quad \frac{a_1}{k} = \frac{1+\beta a_1}{\theta k^{\alpha} \bar{\ell}^{1-\alpha}} \alpha \theta k^{\alpha-1} \bar{\ell}^{1-\alpha} \qquad \text{since } v(k) = a_0 + a_1 \log k$$

$$\Leftrightarrow \quad a_1 = (1+\beta a_1)\alpha$$

$$\therefore \quad a_1 = \frac{\alpha}{1-\alpha\beta}$$

Thus Bellman becomes :

$$a_0 + \frac{\alpha}{1-\alpha\beta}\log k = \log(1-\alpha\beta)\theta k^{\alpha}\bar{l}^{1-\alpha} + \gamma\log(1-\bar{l}) + \beta\left(a_0 + \frac{\alpha}{1-\alpha\beta}\log \alpha\beta\theta k^{\alpha}\bar{l}^{1-\alpha}\right)$$

$$\therefore\ a_0 = \frac{1}{1-\beta}\left\{\log(1-\alpha\beta)\theta\bar{l}^{1-\alpha} + \gamma\log(1-\bar{l}) + \frac{\alpha\beta}{1-\alpha\beta}\log\alpha\beta\theta\bar{l}^{1-\alpha}\right\}$$

$$= \frac{1}{1-\beta}\left\{\gamma\log(1-\bar{l}) + \frac{1}{1-\alpha\beta}\log\theta\bar{l}^{1-\alpha} + \log(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\log\alpha\beta\right\}$$

Using these results, we get a maximization problem

$$v(k) = \max_{\bar{l}}\ \log(1-\alpha\beta)\theta k^{\alpha}\bar{l}^{1-\alpha} + \gamma\log(1-\bar{l}) + \frac{\alpha\beta}{1-\alpha\beta}\log\alpha\beta\theta k^{\alpha}\bar{l}^{1-\alpha} + \frac{\beta}{1-\beta}\left\{\gamma\log(1-\bar{l}) + \frac{1}{1-\alpha\beta}\log\theta\bar{l}^{1-\alpha} + \log(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\log\alpha\beta\right\}$$

$$= \text{const.} + \max_{\bar{l}}\ (1-\alpha)\left(1 + \frac{\beta}{1-\beta}\frac{1}{1-\alpha\beta} + \frac{\alpha\beta}{1-\alpha\beta}\right)\log\bar{l} + \gamma\left(1+\frac{\beta}{1-\beta}\right)\log(1-\bar{l})$$

FOC wrt $\bar{l}$:

$$\frac{1-\alpha}{(1-\alpha\beta)(1-\beta)}\frac{1}{\bar{l}} - \frac{\gamma}{1-\beta}\frac{1}{1-\bar{l}} = 0$$

$$\iff \bar{l} = \frac{1-\alpha}{1-\alpha+\gamma(1-\alpha\beta)} \qquad \left(\in(0,1) \text{ since } \begin{array}{l}\alpha\in(0,1)\\ \beta\in(0,1)\end{array}\right)$$

Therefore,

$$\begin{cases} v(k) = \underset{\text{above}}{a_0} + \dfrac{\alpha}{1-\alpha\beta}\log k \\ k'(k) = \alpha\beta\theta k^{\alpha}\bar{l}^{1-\alpha} \\ c(k) = (1-\alpha\beta)\theta k^{\alpha}\bar{l}^{1-\alpha} \qquad (\text{feasibility}) \\ l(k) = \bar{l} \\ x(k) = 1-\bar{l} \end{cases}$$

Macro 2006 Fall Q II. 1 DP in an economy w/ leisure

8105

(c) Mkt Structure [- Mkts open at each period.
- There are 3 mkts ; 2 spot mkt c_t, l_t , 1 forward mkt k_{t+1}.

A <u>sequential markets eqm</u> is a quantity $\{c_t, k_{t+1}, x_t, l_t\}_{t=0}^{\infty} \equiv \Omega_{HH}$ for HH,
" $\{k_t^f, l_t^f\}_{t=0}^{\infty} \equiv \Omega_f$ for firm,
prices $\{r_t, w_t\}_{t=0}^{\infty}$

such that

(i) given prices, Ω_{HH} solves

$$\max \sum_{t=0}^{\infty} \beta^t (\log c_t + \gamma \log x_t)$$

s.t.

$$c_t + k_{t+1} \leq r_t k_t + w_t l_t$$

$$l_t + x_t \leq 1$$

nonnegativity, $k_0 \leq \bar{k}_0$

(ii) given prices, Ω_f solves

$$\max \quad \theta k_t^{f\alpha} l_t^{f\,1-\alpha} - r_t k_t^f - w_t l_t^f \qquad \forall t$$

s.t. nonnegativity

(iii) mkt clear :

$$\left.\begin{aligned} k_t = k_t^f, \ l_t = l_t^f \\ c_t + k_{t+1} = \theta k_t^\alpha l_t^{1-\alpha} \end{aligned}\right) \forall t$$

Firm's FOC :

$$r_t = \alpha \theta k_t^{\alpha-1} l_t^{1-\alpha}$$

$$w_t = (1-\alpha) \theta k_t^\alpha l_t^{-\alpha}$$

Using the solution in part (b), we get the unique eqm (By SWT we can apply the result in (b) .

Given k_0, we can calculate $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ by $c(k), k'(k)$.
also $\{l_t, x_t\}_{t=0}^{\infty} = \{\bar{l}, 1-\bar{l}\}$.
$\{k_t^f, l_t^f\} = \{k_t, l_t\}$ (by mkt clearing)
finally $\{r_t, w_t\}_{t=0}^{\infty} = \{\alpha\theta k_t^{\alpha-1} \bar{l}^{1-\alpha}, (1-\alpha)\theta k_t^\alpha \bar{l}^{-\alpha}\}_{t=0}^{\infty}$
where k_t is obtained above.

(d) Mkt Structure $\left[\begin{array}{l}\text{- Mkts open only at time 0.} \\ \text{- There are infinite number of goods : contingent claims.}\end{array}\right.$

An <u>AD eqm</u> is a quantity $\{c_t, k_{t+1}, l_t, x_t\}_{t=0}^{\infty} = \Omega_{HH}$ for HH
& $\{k_t^f, l_t^f\}_{t=0}^{\infty} = \Omega_f$ for firm,
prices $\{p_t, r_t, w_t\}_{t=0}^{\infty}$

such that

(i) given prices Ω_{HH} solves

$$\max \sum_{t=0}^{\infty} \beta^t (\log c_t + \gamma \log x_t)$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} p_t (c_t + k_{t+1}) \le \sum_{t=0}^{\infty} (r_t k_t + w_t l_t)$$

$$l_t + x_t \le 1 \quad \forall t,$$

nonnegativity, $k_0 \le \bar{k}_0$.

(ii) given prices, Ω_f solves

$$\max \quad p_t \theta k_t^{f\,\alpha} l_t^{f\,1-\alpha} - r_t k_t^f - w_t^f l_t$$

s.t. nonnegativity.

(iii) mkt clear : $\forall t$,

$$k_t = k_t^f, \; l_t = l_t^f, \; c_t + k_{t+1} = \theta k_t^{\alpha} l_t^{1-\alpha}$$

Allocation $\rightarrow$ same as before

prices :

Firm's FOC

$$\frac{r_t}{p_t} = \alpha \theta k_t^{\alpha-1} \bar{l}^{1-\alpha}$$

$$\frac{w_t}{p_t} = (1-\alpha) \theta k_t^{\alpha} \bar{l}^{-\alpha}$$

HH FOC wrt k_{t+1} : $-p_t + r_{t+1} = 0$

$$\therefore \frac{p_{t+1}}{p_t} = \left[\alpha \theta k_{t+1}^{\alpha-1} \bar{l}^{1-\alpha}\right]^{-1}$$

Thus

$$p_t = \left[\alpha \theta \bar{l}^{1-\alpha}\right]^{-t} (k_1 \cdots k_t)^{-(\alpha-1)}$$

$$\left(\begin{array}{c}\text{normalize} \\ p_0 = 1\end{array}\right)$$

where k_t is obtained by $k'(k)$.

Macro 2006 Fall QⅡ.2 Capital Taxation with LS Rebates

8/05

- Ramsey Problem / 1 consumer + 1 firm
- gov't purchases increase the productivity
- full depreciation → feasibility : $c_t + k_{t+1} + g_{t+1} \leq A k_t^{\alpha} g_t^{1-\alpha}$
- $u(c_t)$: CRRA , No labor decision.
- Optimum / implement this by LS tax + balanced budget / TDCE / IC for RP
- limit of optimal capital tax

(a) Planner's problem :

$$\begin{cases} \max\limits_{\{c_t, k_{t+1}, g_{t+1}\}_{t=0}^{\infty}} & \sum_t \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \\ \text{s.t.} & c_t + k_{t+1} + g_{t+1} \leq A k_t^{\alpha} g_t^{1-\alpha} \quad \forall t \\ & \text{nonnegativity}, \ k_0, g_0 \text{ given} \end{cases}$$

FOC : sufficient

FOC wrt c_t : $\beta^t c_t^{-\sigma} - \lambda_t = 0$

〃 k_{t+1} : $-\lambda_t + \lambda_{t+1} A \alpha k_{t+1}^{\alpha-1} g_{t+1}^{1-\alpha} = 0$

〃 g_{t+1} : $-\lambda_t + \lambda_{t+1} A (1-\alpha) k_{t+1}^{\alpha} g_{t+1}^{-\alpha} = 0$

Then

$$\lambda_{t+1} A \alpha k_{t+1}^{\alpha-1} g_{t+1}^{1-\alpha} = \lambda_{t+1} A (1-\alpha) k_{t+1}^{\alpha} g_{t+1}^{-\alpha}$$

$$\Rightarrow \quad \alpha g_{t+1} = (1-\alpha) k_{t+1}$$

Thus

$$\beta^t c_t^{-\sigma} = \beta^{t+1} c_{t+1}^{-\sigma} A \alpha k_{t+1}^{\alpha-1} \left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} k_{t+1}^{1-\alpha}$$

$$\Rightarrow \quad c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} A \alpha^{\alpha} (1-\alpha)^{1-\alpha}$$

Therefore the optimum is characterized by

$$\begin{cases} \alpha g_{t+1} = (1-\alpha) k_{t+1} \\ c_{t+1} = [\beta A \alpha^{\alpha} (1-\alpha)^{1-\alpha}]^{\sigma} c_t \\ c_t + k_{t+1} + g_{t+1} = A k_t^{\alpha} g_t^{1-\alpha} \end{cases} \quad \Big| \ \forall t \geq 0$$

$\text{TVC}_{\times 2} = \beta^t u'(c_t) k_{t+1}$?

How to solve → AK

Macro 2006 Spring Q I.1 Redundancy of the GBC

8105

- Complete def. of TDCE
- Need firm's profit to derive Walras' law. (zero-profit (usual setting) is a result of eqm ✓)

(a) A TDCE is quantities $\{c_t^i, x_t^i, k_{t+1}^i, l_t^i\}_{t=0}^{\infty} \equiv \Omega_{HH}^i$, $i=1,\cdots,I$ for HHs,
and $\{k_t^j, l_t^j\}_{t=0}^{\infty} \equiv \Omega_f^j$, $j=1,\cdots,J$ for firms
prices $\{p_t, r_t, w_t\}_{t=0}^{\infty} \equiv \mathbb{P}$,
fiscal policy $\{g_t, \tau_t^c, \tau_t^x, \tau_t^k, \tau_t^l, \{T_t^i\}_{i=1}^{I}\}_{t=0}^{\infty}$,
profit for each HH, π^i, $i=1,\cdots,I$,

such that

(i) given $\mathbb{P}$, for $i=1,\cdots,I$, Ω_{HH}^i solves

$$\max \quad u^i(\{c_t^i\}, \{l_t^i\})$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} p_t[(1+\tau_t^c)c_t^i + (1+\tau_t^x)x_t^i]$$

★ → BC w/ profit

$$\leq \sum_{t=0}^{\infty}[r_t(1-\tau_t^k)k_t + w_t(1-\tau_t^l)l_t + T_t^i] + \pi^i$$

$$k_{t+1}^i = (1-\delta)k_t^i + x_t^i \qquad \forall t$$

nonnegativity, given k_0^i

(ii) given $\mathbb{P}$, for $j=1,\cdots,J$, Ω_f^j solves

$$\max \sum_{t=0}^{\infty}\{p_t F^j(k_t^j, l_t^j) - r_t k_t^j - w_t l_t^j\}$$

s.t. nonnegativity.

(iii) Mkt clear:

$$\left.\begin{aligned} \sum_{i=1}^{I} k_t^i &= \sum_{j=1}^{J} k_t^j \\ \Sigma_i l_t^i &= \Sigma_j l_t^j \\ \Sigma_i c_t^i + \Sigma_i x_t^i + g_t &= \Sigma_j F^j(k_t^j, l_t^j) \end{aligned}\right\} \forall t$$

★ → (iv) Profit:

$$\sum_{i=1}^{I}\pi^i = \sum_{j=1}^{J}\sum_{t=0}^{\infty}\{p_t F^j(k_t^j, l_t^j) - r_t k_t^j - w_t l_t^j\}.$$

★ → GBC w/ transfer (v) GBC:

$$\sum_{t=0}^{\infty}\left(p_t g_t + \sum_{i=1}^{I} T_t^i\right) = \sum_{t=0}^{\infty}\sum_{i=1}^{I}\{p_t\tau_t^c c_t^i + p_t\tau_t^x x_t^i + r_t\tau_t^k k_t^i + w_t\tau_t^l l_t^i\}$$

(b) Claim : Feasibility for firms
(*) BC w/ equality
Mkt clear
$\Big\} \Rightarrow$ GBC holds automatically.

pf. Agg. BC w/ equality

$$\sum_i \sum_t \{ p_t T_t^c c_t^i + p_t T_t^x x_t^i + r_t T_t^k k_t^i + w_t T_t^l l_t^i \} = \text{RHS of GBC}$$

$$= \sum_i \sum_t \{ r_t k_t^i + w_t l_t^i - p_t c_t^i - p_t x_t^i + T_t^i \} + \sum_{i=1}^{I} \pi^i$$

$$= \sum_i \sum_t \{ \sim \} + \sum_i \sum_t \{ p_t F^i(k_t^i, l_t^i) - r_t^i k_t^i - w_t l_t^i \}$$

(∵ profit condition (def))

$$= \sum_t \Big\{ p_t \Big[\sum_i F^i(k_t^i, l_t^i) - \sum_i c_t^i - \sum_i x_t^i \Big] + \sum_i T_t^i$$
$$+ r_t \sum_i k_t^i + w_t \sum_i l_t^i - r_t \sum_i k_t^i - w_t \sum_i l_t^i \Big\}$$

$$= \sum_t \{ p_t g_t + \sum_i T_t^i \}$$

(by mkt clearing condition + feasibility)

= LHS of GBC.

Thus an allocation and price system which satisfy (*) also satisfy GBC w/ equality.

Q.E.D.

Macro

2006 Spring Q I. 2 AD eqm w/ and w/o transfer payments

8105

- pure-exchange economy w/ 2 agents
- same utility / hetero. in endowment : $w^1 = (4, 2, 4, 2, \cdots)$
  $w^2 = (2, 4, 2, 4, \cdots)$
- AD eqm / AD eqm w/ tranfer
- transfer payment to impliment P.O. allocation / equal allocation

(a) - Mkt for contingent claims is open only at time 0.
- 2 agents trade with each other.

An AD eqm is a quantity $\{c_t^1, c_t^2\}_{t=0}^{\infty}$ and a price $\{q_t\}_{t=0}^{\infty}$

such that

(i) given $\{q_t\}$, for $i = 1, 2$, $\{c_t^i\}_{t=0}^{\infty}$ solves

$$\begin{cases} \max \sum_{t=0}^{\infty} \beta^t \log c_t^i \\ \text{s.t.} \quad \sum_{t=0}^{\infty} q_t c_t^i \le \sum_{t=0}^{\infty} q_t w_t^i \\ \qquad \text{nonnegativity.} \end{cases}$$

(ii) Mkt clear :

for all t, $c_t^1 + c_t^2 = w_t^1 + w_t^2 = 6$.

(b) An allocation $\{\hat{c}_t^1, \hat{c}_t^2\}_{t=0}^{\infty}$ is P.O. if

(i) (feasible) $\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2 = 6$ for all t.

(ii) (efficient) $\nexists$ feasible allocation $\{c_t^1, c_t^2\}_{t=0}^{\infty}$ s.t.

$\forall i \in \{1,2\}$, $\sum \beta^t \log c_t^i \ge \sum \beta^t \log \hat{c}_t^i$,

$\exists j \in \{1,2\}$, $\quad$ 〃 $\quad > \quad$ 〃 .

Pareto Problem : $\max \; \alpha_1 \sum \beta^t \log c_t^1 + \alpha_2 \sum \beta^t \log c_t^2$

s.t. $\forall t$, $c_t^1 + c_t^2 = 6$, nonnegativity.

FOC (sufficient) : $\alpha_1 / c_t^1 = \alpha_2 / c_t^2$

$$\Rightarrow \quad c_t^i = \frac{6\alpha_i}{\alpha_1 + \alpha_2}, \quad i = 1, 2 \qquad \text{(by feasibility)}$$

(c) An <u>AD eqm w/ transfer payment</u> is a quantity $\{c_t^1, c_t^2\}_{t=0}^{\infty}$,
a price $\{q_t\}_{t=0}^{\infty}$, and
a transfer $\{T_1, T_2\}$

such that

(i) given $\{q_t\}$, for $i=1,2$, $\{c_t^i\}_{t=0}^{\infty}$ solves

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t^i$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} q_t c_t^i \le \sum_{t=0}^{\infty} q_t w_t^i + T_i$$

nonnegativity.

(ii) Mkt clear:

$$c_t^1 + c_t^2 = w_t^1 + w_t^2 = 6 \quad \forall t$$

$$T_1 + T_2 = 0$$

FOC (sufficient): $\beta^t \frac{1}{c_t^i} - \lambda^i q_t = 0$

$$\Rightarrow \text{Euler}: \frac{\beta \frac{1}{c_{t+1}^i}}{\frac{1}{c_t^i}} = \frac{q_{t+1}}{q_t} \quad \cdots (*)$$

$$\Rightarrow \frac{\beta c_t^1}{c_{t+1}^1} = \frac{\beta c_t^2}{c_{t+1}^2} = \frac{\beta(6-c_t^1)}{(6-c_{t+1}^1)} \quad (\because \text{Mkt clear})$$

$$\therefore c_t^i = c_{t+1}^i \quad \forall t$$

From Euler,

$$q_t = \beta^t \quad (\text{normalize } q_0 = 1)$$

Thus BC becomes

$$\frac{c_0^i}{1-\beta} = \sum_{t=0}^{\infty} \beta^t w_t^i + T_i$$

for $i=1$,

$$\frac{c_0^1}{1-\beta} = 4(1+\beta^2+\beta^4+\cdots) + 2\beta(1+\beta^2+\beta^4+\cdots) + T_1$$

$$= \frac{4+2\beta}{1-\beta^2} + T_1$$

$$\therefore c_0^1 = \frac{4+2\beta}{1+\beta} + (1-\beta) T_1$$

To get

$$c_0^1 = \frac{6\alpha_1}{\alpha_1+\alpha_2},$$

$$\begin{cases} T_1 = \dfrac{1}{1-\beta}\left\{\dfrac{6\alpha_1}{\alpha_1+\alpha_2} - \dfrac{4+2\beta}{1+\beta}\right\} \\ T_2 = -T_1 \end{cases}$$

← easy to check and derive $T_1 + T_2 = 0$

Macro 2006 Spring Q I. 2 AD eqm w/ and w/o transfer payments

8105

(d) Transfer to make the eqm allocation equal

To get $c_0^1 = 3$,

$$\begin{cases} T_1 = \dfrac{1}{1-\beta}\left\{3 - \dfrac{4+2\beta}{1+\beta}\right\} \\ \quad = \dfrac{-1}{1+\beta} \\ T_2 = 1/1+\beta \end{cases}$$

endowment

$w_0^1 > w_0^2$
$\quad 4 \qquad 2$

$\hookrightarrow T_1 < 0.$

$\underbrace{4, 2}, \underbrace{4, 2}$
cycle

$\dfrac{1}{1+\beta}$

now tomorrow

(e) Welfare weight s.t. transfer is 0.

$$0 = T_1 = \frac{1}{1-\beta}\left\{\frac{6\alpha_1}{\alpha_1+\alpha_2} - \frac{4+2\beta}{1+\beta}\right\}$$

Normalize $\alpha_1 + \alpha_2 = 1$,

$$\begin{cases} \alpha_1 = \dfrac{2+\beta}{3(1+\beta)} \\ \alpha_2 = 1 - \dfrac{2+\beta}{3(1+\beta)} = \dfrac{1+2\beta}{3(1+\beta)} \end{cases}$$

$\alpha_1 > \alpha_2$
rich poor

Macro 2006 Spring Q II . 2 Capital Taxation with LS Rebates

8105

- capital tax / LS rebates
- 1 consumer
- TDCE can be a solution of a PP.

(a) A TDCE is a quantity $\{c_t, x_t, k_{t+1}\}_{t=0}^{\infty} \equiv \Omega_{HH}$ for HH,

$\sim$ $\{k_t^f, l_t^f\} \equiv \Omega_f$ for firm.

a price system $\{p_t, r_t, w_t\} \equiv \mathbb{P}$ and

a fiscal policy $\{\tau, T_t\}_{t=0}^{\infty} \equiv \mathbb{G}$

such that

(i) given $\mathbb{P}$ and $\mathbb{G}$, Ω_{HH} solves

$$\begin{cases} \max & \frac{1}{1-\beta} \sum \beta^t u(c_t) \\ \text{s.t.} & \sum_{t=0}^{\infty} p_t (c_t + x_t) \leq \sum_{t=0}^{\infty} (r_t (1-\tau) k_t + w_t + T_t) \\ & k_{t+1} \leq x_t \\ & \text{nonnegativity, given } k_0 \end{cases}$$

(ii) given $\mathbb{P}$, Ω_f solves

$$\max \sum_{t=0}^{\infty} p_t F(k_t^f, l_t^f) - r_t k_t^f - w_t l_t^f$$

s.t. nonnegativity

(iii) GBC :

$$T_t = r_t \tau k_t \quad \forall t.$$

(iv) Mkt clear :

$$k_t^f = k_t, \; l_t^f = 1 \qquad \forall t$$
$$c_t + x_t = F(k_t, l_t).$$

Assume that F is CRS $\Rightarrow$ zero profit

(b) $\boxed{\text{Claim}}$ TDCE can be a solution of a PP.

(See 2006 Spring Q II.4.)

(pf) Planning Problem:

$$
(\#)\ \begin{cases} \max\limits_{\{c_t, k_{t+1}\}} \dfrac{1}{1-\beta} \sum\limits_{t=0}^{\infty} \tilde{\beta}^t u(c_t) \\ \text{s.t.} \\ \quad c_t + k_{t+1} \leq F(k_t, 1) \quad \forall t \\ \quad \text{nonnegativity} \quad \text{given } k_0 \end{cases}
$$

where $\tilde{\beta} \equiv \beta(1-\tau)$

Let $f(k_t) \equiv F(k_t, 1)$.

FOC (sufficient):

wrt c_t : $\dfrac{\tilde{\beta}^t}{1-\beta} u'(c_t) - \lambda_t = 0$

k_{t+1} : $-\lambda_t + \lambda_{t+1} f'(k_{t+1}) = 0$

$\rightarrow$ Euler : $u'(c_t) = \tilde{\beta} u'(c_{t+1}) f'(k_{t+1})$

On the other hand, for TDCE

Euler : $u'(c_t) = \beta(1-\tau) u'(c_{t+1}) f'(k_{t+1})$

Thus TDCE allocation solves a PP (#)

TDCE is characterized by
- Euler $\rightarrow$ above
- TVC $\rightarrow$ if alloc satisfies TVC w/ β, it also satisfies TVC w/ $\tilde{\beta}$.
- Feasibility $\rightarrow$ obviously satisfied

□

* Note: PP

$$
\begin{cases} \max \dfrac{1}{1-\beta} \sum\limits_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t.} \quad c_t + k_{t+1} \leq \tilde{F}(k_t, 1) \equiv (1-\tau) F(k_t, 1) \quad \forall t \\ \quad \text{nonnegativity} \quad \text{given } k_0 \end{cases}
$$

does NOT work unlike sol Q II.4.

<u>why?</u>

$\hookrightarrow$ No gov't ex. $\rightarrow$ does not satisfies the feasibility.

Macro 2006 Spring Q II.4 Single Sector Growth Model

8105

- Ak model, $\delta = 1$
- 1 consumer
- stochastic tax process
- TDCE solves a PP.
- mean preserving spread / growth rate

(a) $\boxed{\text{Claim}}$ TDCE allocation solves PP.

PP Define history $s^t \equiv (s_0, \cdots, s_t)$

Planning Problem:

$$(\#)\begin{cases} \max\limits_{c_t, k_{t+1}} & E_0 \sum \beta^t \dfrac{c_t(s^t)^{1-\sigma}}{1-\sigma} \\ \text{s.t.} & c_t(s^t) + k_{t+1}(s^t) = (1-\tau_t(s^t)) A k_t(s^{t-1}) \\ & \text{nonnegativity, given } k_0(s^{-1}) \equiv k_0 \end{cases}$$

$\tau_t(s^t)$ → not tax, just a parameter → stochastic technology.

FOC (sufficient):

wrt $c_t(s^t)$: $\beta^t \pi(s^t) c_t(s^t)^{-\sigma} - \lambda_t(s^t) = 0$

$k_{t+1}(s^t)$: $-\lambda_t(s^t) + \sum\limits_{s^{t+1}|s^t} \lambda_{t+1}(s^{t+1}) (1-\tau_{t+1}(s^{t+1})) A = 0$

→ Euler : $c_t(s^t)^{-\sigma} = \beta E_t (1-\tau_{t+1}(s^{t+1})) A c_{t+1}(s^{t+1})^{-\sigma}$

TDCE is characterized by

- Euler
- TVC → easy to check.
- Feasibility

TDCE allocation :

consumer's problem :

$$\begin{cases} \max_{c_t, k_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t \dfrac{c_t(s^t)^{1-\sigma}}{1-\sigma} \\ \text{s.t.} \quad \sum_t \sum_{s^t} p_t(s^t)\left[c_t(s^t) + k_{t+1}(s^t)\right] \\ \qquad \le \sum_t \sum_{s^t} \left[r_t(s^t)(1-\tau_t(s^t))k_t(s^{t-1}) + w_t(s^t)\right] \\ \quad \text{nonnegativity}, \quad k_0(s^{-1}) = k_0 \text{ given.} \end{cases}$$

Then

Euler : $c_t(s^t)^{-\sigma} = \beta \; E_t \cdot (1-\tau_{t+1}(s^{t+1}))A \; c_{t+1}(s^{t+1})^{-\sigma}$

by MP cond.

Thus, this is the same as PP's Euler.

- Check feasibility :

By Mkt clearing cond, in TDCE

$$c_t(s^t) + k_{t+1}(s^t) + g_t(s^t) = A\, k_t(s^{t-1}) \qquad \forall t, s^t$$

$$\Rightarrow c_t(s^t) + k_{t+1}(s^t) + \tau_t(s^t)\frac{r_t(s^t)}{p_t(s^t)}k_t(s^{t-1}) = A\, k_t(s^{t-1})$$

(by GBC)

$$\Rightarrow c_t(s^t) + k_{t+1}(s^t) = (1-\tau_t(s^t))A\, k_t(s^{t-1})$$

(by MP. cond.)

Thus, this is the same as PP's feasibility.

Therefore, TDCE allocation solves a PP (#).

Note :

PP w/ stochastic discount factor : $\tilde{\beta}(s^t) \equiv \beta(1-\tau_t(s^t))$

does NOT work unlike S06 Q II.2.

Why ?

↳ gov't expenditure is not zero.

→ does not satisfy the feasibility.

8/05

(b) <u>Mean preserving spread increases the mean growth rate</u>

$u(c) = c_t^{1-\sigma}$ is str. concave. $\Rightarrow$ sol'n is unique.

$\Rightarrow$ TDCE can be analysed by the sol'n of PP (#). (by (a))

Reproduce PP :

$$P(k_0, s_0) \equiv \begin{cases} \max\limits_{c_t, k_{t+1}} & E_0 \sum_{t=0}^{\infty} \beta^t \dfrac{c_t(s^t)^{1-\sigma}}{1-\sigma} \\ \text{s.t} & c_t(s^t) + k_{t+1}(s^t) \leq (1-\tau_t(s^t)) A k_t(s^{t-1}) \quad \forall t, s^t \\ & \text{nonnegativity}, \quad k_0(s^{-1}) = k_0 \text{ given} \end{cases}$$

- When the solution exists? Weirstrass $\rightarrow$ $\begin{cases} \text{objective fcn is continuous.} \\ \text{Choice set is compact.} \end{cases}$

For this, we need to say that $\exists$ maximum capital $\hat{k}$ s.t. $\hat{k}' \leq \hat{k}$.

By the feasibility, $\hat{k}'(s^t) \leq (1-\tau_t(s^t)) A \hat{k}$

$\leq A\hat{k}$

Thus $A\hat{k} \leq \hat{k} \Leftrightarrow A \leq 1$

?

Then a solution exists

Let $V(k_0, s_0)$ be the maximized value of $P(k_0, s_0)$,

and $\{c_t^*(s^t; k_0, s_0), k_{t+1}^*(s^t; k_0, s_0)\}_{t=0, s^t}^{\infty}$ be the solution.

Let $\Gamma(k_0, s_0)$ be the feasibility set :

$$\Gamma(k_0, s_0) = \{\{c_t(s^t), k_{t+1}(s^t)\}_{t=0, s^t}^{\infty} \mid \forall t, \forall s^t \quad c_t(s^t) + k_{t+1}(s^t) \leq [1-\tau_t(s^t)] A k_t(s^{t-1})\}$$

Then,

$\boxed{\text{Proposition}}$ (LN 4. p.42 / Lecture 10/16)

(i) $\forall k_0, s_0$, $V(\lambda k_0, s_0) = \lambda^{1-\sigma} V(k_0, s_0)$ HD$(1-\sigma)$

(ii) $\{c_t^*(s^t; \lambda k_0, s_0), k_{t+1}^*(s^t; \lambda k_0, s_0)\}_{t=0, s^t}^{\infty}$ HD 1

$= \{\lambda c_t^*(s^t; k_0, s_0), \lambda k_{t+1}^*(s^t; k_0, s_0)\}$

(pf)

- If $\{c_t, k_{t+1}\} \in \Gamma(k_0, s_0)$, then $\{\lambda c_t, \lambda k_{t+1}\} \in \Gamma(\lambda k_0, s_0)$. (linear production)
- $U(\{\lambda c_t(s^t)\}) = \lambda^{1-\sigma} U(\{c_t(s^t)\})$ (Homothetic utility)
- If $\{c_t, k_{t+1}\}$ solves $P(k_0, s_0)$, then $\{\lambda c_t, \lambda k_{t+1}\}$ solves $P(\lambda k_0, s_0)$.

choice correspondence Γ is nonempty valued, the graph is measurable, and has measurable selection.
Return fcn F is measurable
v measurable, and so on ...

Assumption for V to satisfy the Bellman eqn. (SL ch. 9)

Bellman eqn:

$$v(k,\tau) = \sup_{c,k'} \frac{c^{1-\sigma}}{1-\sigma} + \beta E\, v(k',\tau') \quad \leftarrow \text{by iid assumption, unconditional expectation}$$
$$\text{s.t.} \quad c + k' \leq (1-\tau)Ak$$

By the proposition, we have

$$E\, v(k',\tau') = E\, k'^{1-\sigma} v(1,\tau') = k'^{1-\sigma} E\, v(1,\tau') \equiv k'^{1-\sigma} D$$

Then, we can rewrite the Bellman as

$$v(k,\tau) = \sup_{c,k'} \frac{c^{1-\sigma}}{1-\sigma} + \beta k'^{1-\sigma} D$$
$$\text{s.t.} \quad c + k' \leq (1-\tau)Ak.$$

homogeneous and homothetic problem.

$\Rightarrow$ Objective fcn is homothetic in (c, k')

★ → 10/16 $\Rightarrow \exists\, \varphi \in [0,1]$ s.t.

policy fcn's
$$c(k,\tau) = \varphi(1-\tau)Ak$$
$$k'(k,\tau) = (1-\varphi)(1-\tau)Ak$$

Note that since τ is iid, D does not depend on τ and hence φ $\forall t$.

Euler for $P(k_0, s_0)$: (drop s^t)

$$c_t^{-\sigma} = \beta E_t (1-\tau_{t+1}) A c_{t+1}^{-\sigma}$$
$$\Leftrightarrow 1 = \beta E (1-\tau_{t+1}) A \left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}$$
$$= \beta E (1-\tau_{t+1}) A [(1-\tau_{t+1})A(1-\varphi)]^{-\sigma} \quad (\because \text{below})$$
$$= \beta A^{1-\sigma} (1-\varphi)^{-\sigma} E[(1-\tau_{t+1})^{1-\sigma}]$$

Assume $\sigma > 1$ (risk averse)

$\Rightarrow [1-\tau_{t+1}(s^{t+1})]^{1-\sigma}$ is convex function. (nondecreasing in τ)

$\Rightarrow$ If $G >_{MPS} F$, then $E_G(1-\tau_{t+1})^{1-\sigma} \geq E_F(1-\tau_{t+1})^{1-\sigma}$
$\Rightarrow (1-\varphi_G)^{\sigma} \geq (1-\varphi_F)^{\sigma} \Rightarrow \varphi_G \leq \varphi_F.$

Mean growth of consumption:

$$E[c_{t+1}(s^{t+1})/c_t(s^t)] = E\,\frac{\varphi(1-\tau_{t+1})Ak_{t+1}}{\varphi(1-\tau_t)Ak_t} = \frac{(1-\tau_{t+1})(1-\varphi)(1-\tau_t)Ak_t}{(1-\tau_t)k_t}$$
$$= A(1-\varphi)E[1-\tau_{t+1}].$$

Thus

If $G >_{MPS} F$, then $\varphi_G \leq \varphi_F$ and hence $\gamma_G \geq \gamma_F$. □

Macro 2005 Fall QI.3 Ramsey Taxation

8/05

- Ramsey Problem / Finance only by τ_c
- 1 consumer, leisure in utility: $\frac{c_t^{1-\sigma}}{1-\sigma} + v(l_t)$
- Optimal consumption tax is constant $\forall\ t \geq 1$.

Derive the implementability condition.

Consumer's problem: given tax and prices,

$$\left\{\begin{array}{l} \max\limits_{\{c_t, x_t, k_{t+1}, n_t, l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + v(l_t)\right] \\ \text{s.t.} \quad \sum_{t=0}^{\infty} p_t \left[(1+\tau_{ct}) c_t + x_t\right] \leq \sum_{t=0}^{\infty} \left[r_t k_t + w_t n_t\right] \\ \qquad l_t + n_t \leq 1 \\ \qquad k_{t+1} \leq (1-\delta) k_t + x_t \\ \qquad \text{nonnegativity}, \ k_0 \text{ given} \end{array}\right.$$

FOC (sufficient):

wrt c_t: $\beta^t c_t^{-\sigma} - \lambda p_t (1+\tau_{ct}) = 0 \Rightarrow \frac{p_t(1+\tau_{ct})}{1+\tau_{c0}} = \beta^t \left(\frac{c_t}{c_0}\right)^{-\sigma}$

k_{t+1}: $-p_t + r_{t+1} + (1-\delta) p_{t+1} = 0$

n_t: $-\beta^t v'(1-n_t) + \lambda w_t = 0 \Rightarrow \frac{w_t}{1+\tau_{c0}} = \beta^t \frac{v'(1-n_t)}{c_0^{-\sigma}}$

By firm's problem:

$$r_t / p_t = F_k(k_t, n_t)$$
$$w_t / p_t = F_n(k_t, n_t)$$

Then HHBC becomes

$$\sum p_t \left[(1+\tau_t) c_t + k_{t+1} - (1-\delta) k_t\right] = \sum \left[r_t k_t + w_t n_t\right]$$

$$\Leftrightarrow \sum \left[p_t (1+\tau_t) c_t - w_t n_t\right] = \sum \left[(r_t + (1-\delta) p_t) k_t - p_t k_{t+1}\right]$$

$$\text{LHS} = \sum \left[\beta^t \frac{1+\tau_{c0}}{c_0^{-\sigma}} c_t^{1-\sigma} - \beta^t \frac{1+\tau_{c0}}{c_0^{-\sigma}} v'(1-n_t) n_t\right]$$

$$= \frac{1+\tau_{c0}}{c_0^{-\sigma}} \sum_{t=0}^{\infty} \beta^t \left[c_t^{1-\sigma} - v'(1-n_t) n_t\right]$$

$$\text{RHS} = \left[\underset{F_k(k_0,n_0)}{\underset{\parallel}{r_0}} + (1-\delta) p_0\right] k_0 + \sum_{t=1}^{\infty} \left[\underbrace{r_{t+1} + (1-\delta) p_{t+1} - p_t}_{0}\right] k_t - \underbrace{\lim_{t\to\infty} p_t k_{t+1}}_{0}$$

TVC

$\lim \beta^t F_k k \to 0$?

Thus

$$(\text{IC}) \Leftrightarrow \sum_{t=0}^{\infty} \beta^t \left[c_t^{1-\sigma} - v'(1-n_t) n_t\right] = \frac{c_0^{-\sigma}}{1+\tau_{c0}} \left[F_k(k_0, n_0) + 1 - \delta\right] k_0$$

Ramsey Problem $\begin{cases} \max \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + v(1-n_t) \right] \\ \text{s.t.} \quad (IC) \qquad \leftarrow \text{put multiplier } \lambda \\ \quad c_t + k_{t+1} - (1-\delta)k_t + g_t \leq F(k_t, n_t) \\ \quad \text{nonnegativity}, \ k_0 \text{ given.} \end{cases}$

$\star \to$ $t \geq 1$ $\Rightarrow$ $\begin{cases} \max \sum_{t=1}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + v(1-n_t) - \lambda\left(c_t^{1-\sigma} - v'(1-n_t)n_t\right) \right] + \frac{c_0^{1-\sigma}}{1-\sigma} + v(1-n_0) + \lambda \frac{c_0^{-\sigma}}{1+\tau_{c0}} [F_k(0) + 1 - \delta] k_0 \\ \text{s.t.} \quad c_t + k_{t+1} - (1-\delta)k_t + g_t \leq F(k_t, n_t) \quad , \text{ nonneg. } k_0 \text{ given.} \end{cases}$

For RP, FOC is always from $t = 1$.

FOC wrt c_t : $\beta^t [1 - \lambda(1-\sigma)] c_t^{-\sigma} - \mu_t = 0 \qquad \forall t \geq 1$

k_{t+1} : $-\mu_t + \mu_{t+1} (F_k(t+1) + 1 - \delta) = 0$

$\Rightarrow$ Euler :

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} [F_k(k_{t+1}, n_{t+1}) + 1 - \delta] , \quad \forall t \geq 1$$

On the other hand, for TDCE allocation

Euler : $\frac{c_t^{-\sigma}}{1+\tau_{ct}} = \beta \frac{c_{t+1}^{-\sigma}}{1+\tau_{ct+1}} [F_k(k_{t+1}, n_{t+1}) + 1 - \delta]$,

Thus optimal tax is

$$\tau_{ct} = \tau_{ct+1} \quad \forall t \geq 1.$$

Macro 2005 Fall Q II. 2 Temporary vs. Permanent Wage Changes

8105

- 1 consumer / leisure in utility / log
- r is fixed.
- Reduction of BC (a) eliminate $\{k_t\}_{t\geq 1}$, (b) " $\{p_t\}$
- closed form expression for eqm labor supply.
- Comparative statics on the eqm labor supply in period 0 : temporary vs permanent wage changes.

Firm: $\max\ p_t F - \tilde{r}_t k_t - \tilde{w}_t n_t$
$\Rightarrow p_t r = \underbrace{\tilde{r}_t}_{\text{payment}}$

(a) Consumer's problem :

$$\begin{cases} \max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} \frac{1}{1-\beta} \sum_{t=0}^{\infty} \beta^t [\alpha \log c_t + (1-\alpha) \log(1-n_t)] \\ \text{s.t.} \quad \sum_{t=0}^{\infty} p_t [c_t + k_{t+1}] \leq \sum_{t=0}^{\infty} [p_t r k_t + p_t w_t n_t] \\ \qquad \text{nonnegativity}, \ k_0 \text{ given.} \end{cases}$$

p_t in RHS by def. of r_t

Assume an interior eqm.

FOC wrt k_{t+1} : $-p_t + p_{t+1} r = 0$ ... $(*)$

Then BC

$$\Rightarrow \sum_{t=0}^{\infty} p_t c_t \leq p_0 r k_0 + \sum_{t=0}^{\infty} p_t w_t n_t$$

(by TVC $\lim_{t\to\infty} p_t k_{t+1} = 0$)

$$\Rightarrow \sum_{t=0}^{\infty} p_t (c_t - w_t n_t) \leq p_0 r k_0$$

(b) By $(*)$,

$$p_t = \frac{1}{r} p_{t-1} = \beta p_{t-1} = \beta^t p_0$$

Thus BC $\Rightarrow \sum \beta^t p_0 (c_t - w_t n_t) \leq p_0 r k_0$

$$\Rightarrow \sum \beta^t (c_t - w_t n_t) \leq r k_0$$

(In eqm, $p_0 > 0$)

(c) closed form solution

FOC wrt c_t : $\frac{\beta^t}{1-\beta} \frac{\alpha}{c_t} - \lambda p_t = 0$ ... (1)

" n_t : $-\frac{\beta^t}{1-\beta} \frac{1-\alpha}{1-n_t} + \lambda p_t w_t = 0$ ... (2)

$$(1) \Rightarrow \frac{\frac{\beta^t \alpha}{c_t}}{\frac{\alpha}{c_0}} = \frac{\lambda p_t}{\lambda p_0} = \frac{\beta^t p_0}{p_0} \Leftrightarrow c_t = c_0 \equiv c \quad \forall t$$

$$(2) \Rightarrow \frac{\frac{\alpha}{c_t}}{\frac{1-\alpha}{1-n_t}} = \frac{\lambda p_t}{\lambda p_t w_t} \Leftrightarrow w_t n_t = w_t - \frac{1-\alpha}{\alpha} c_t \quad \forall t$$

Thus

$$\underset{\text{(binding)}}{BC} \Rightarrow \sum \beta^t \left(c - w_t + \frac{1-\alpha}{\alpha} c \right) = r k_0$$

$$\Rightarrow \quad c = \alpha(1-\beta)\left[\sum \beta^t w_t + r k_0 \right]$$

Thus

$$n_t = 1 - \frac{1-\alpha}{\alpha} \frac{c}{w_t}$$

$$= 1 - (1-\alpha)(1-\beta) \frac{1}{w_t} \left(\sum_{T=0}^{\infty} \beta^T w_t + r k_0 \right)$$

(d) S1 : $n_t = \alpha - \dfrac{(1-\alpha)(1-\beta) r k_0}{\bar{w}}$, $\forall t \geq 0$

S2 : $n_0 = \alpha + (1-\alpha)\beta - \left(1 - \dfrac{\bar{w}}{w^*}\right) - \dfrac{(1-\alpha)(1-\beta) r k_0}{w^*}$

$n_t = 1 - (1-\alpha)\beta - (1-\alpha)(1-\beta) \dfrac{w^*}{\bar{w}} - \dfrac{(1-\alpha)(1-\beta) r k_0}{\bar{w}}$ $\quad \forall t \geq 1$

S3 : $n_t = \alpha - \dfrac{(1-\alpha)(1-\beta) r k_0}{w^*}$ $\quad \forall t \geq 0$

(e) $\alpha = 1 \iff$ no utility from leisure $\Rightarrow n_0 = 1$.

$k_0 = 0 \iff$ no initial capital (Income / substitution effect ?)

(f) S1 $\Rightarrow$ S2 : temporary increase of wage

S1 $\Rightarrow$ S3 : permanent 〃

$\boxed{\text{Claim}}$ Temporary change is bigger than permanent change

pf) Obviously $n_1 < n_3$, $n_1 < n_3$

Since

$$1 - \frac{\bar{w}}{w^*} > 1 ,$$

we have

$$n_2 > n_3 . \qquad \square$$

Why? People work more during the higher wage period to take the advantage

- Simple growth model
- Canonical Form / Functional Equation
- Conditions on u, f, β, δ

★ 1a Canonical Form

→ Assume u is str. increasing ⇒ constraints hold w/ equality.

str. concave ⇒ } unique sol'n ⇒ $n(k, k')$ is a function

continuous ⇒ }

Then the sequential problem is :

$$\max_{\{k_{t+1}, n_t\}_{t=0}^{\infty}} \sum \beta^t \, u\left[f(k_t, n_t) + k_t(1-\delta) - k_{t+1} ,\ \bar{n} - n_t \right]$$

$$\text{s.t.} \quad 0 \le k_{t+1} \le f(k_t, n_t) + k_t(1-\delta), \quad n_t \le \bar{n} \quad \forall t$$

$$k_0 \text{ given.}$$

Define the optimal labor supply given k, k' as

$$n^*(k, k') \equiv \underset{0 \le n \le \bar{n}}{\operatorname{argmax}} \ u\left[f(k, n) + k(1-\delta) - k' ,\ \bar{n} - n \right]$$

Since u is continuous and the choice set is compact, (f cont)
this has a solution. And it is unique since u is str. concave.

Then, DP in sequence form :

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \, F(k_t, k_{t+1})$$

$$\text{s.t.} \quad k_{t+1} \in \Gamma(k_t)$$

$$k_0 \text{ given}$$

where

$$F(k_t, k_{t+1}) := u\left[f(k_t, n^*(k_t, k_{t+1})) + k_t(1-\delta) - k_{t+1} ,\ \bar{n} - n^*(k_t, k_{t+1}) \right]$$

Time-stationary return fcn

$$\Gamma(k_t) = \{ k_{t+1} \in \mathbb{R} \mid 0 \le k_{t+1} \le f(k_t, n^*(k_t, k_{t+1})) + k_t(1-\delta) \}$$

Time-stationary constraint set

(b) <u>Functional Equation</u>

$$v(k) = \max_{k' \in T(k)} F(k, k') + \beta v(k')$$

Without extra assumptions on u.

(a)'

$$\max_{\{c_t, l_t, n_t, x_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F\left[(c_{t-1}, l_{t-1}, n_{t-1}, x_{t-1}, k_t), (c_t, l_t, n_t, x_t, k_{t+1})\right]$$

$$\text{s.t.} \quad (c_t, l_t, n_t, x_t, k_{t+1}) \in T(c_{t-1}, l_{t-1}, n_{t-1}, x_{t-1}, k_t), \forall t$$

$$k_0 \text{ given}, \quad (c_{-1}, l_{-1}, n_{-1}, x_{-1}) \equiv 0$$

where

$$F[\sim] = u(c_t, l_t)$$

$$T(\sim) = \{(c_t, l_t, n_t, x_t, k_{t+1}) \in \mathbb{R}^5 \mid c_t + x_t \le f(k_t, n_t),\ k_{t+1} \le (1-\delta)k_t + x_t,\ n_t + l_t \le \bar{n},\ c_t \ge 0,\ k_{t+1} \ge 0\}$$

(b)'

$$v(c_-, l_-, n_-, x_-, k) = \max F[(c_-, l_-, n_-, x_-, k), (c, l, n, x, k')] + \beta v(c, l, n, x, k')$$

$$\text{s.t.} \quad (c, l, n, x, k') \in T(c_-, l_-, n_-, x_-, k)$$

$$k_0 \text{ given}, \quad (c_{-1}, l_{-1}, n_{-1}, x_{-1}) \equiv 0$$

(c) <u>Cond's that guarantee $\exists!$ bdd & cont. v</u>

$A4.3 + A4.4 \Rightarrow \boxed{\text{Thm } 4.6}$

A4.3 : X is a convex subset of $\mathbb{R}$ ①

$\Gamma : X \rightrightarrows X$ is nonempty ②, compact-valued ③ and continuous ④.

A4.4 : $F : A \to \mathbb{R}$ is bdd ⑤ and continuous ⑥ and $\beta \in (0,1)$ ⑦,

where $A = \{ (x, y) \in X \times X \mid y \in \Gamma(x) \}$: the graph of Γ.

- by def ⇒ ①
- $\delta \in [0,1]$, $f \geq 0$ ⇒ ②
- 〃 + f bdd ⇒ ③
- f cont. ⇒ ④
- u bdd ⇒ ⑤
- u cont. ⇒ ⑥
- $\beta \in (0,1)$ ⇒ ⑦

(d) Labor supply is inelastic ⇒ $n_t = \bar{n}$, $\ell_t = 0$.

FE in (b) :

$$v(k) = \max_{k'} \; u(f(k, \bar{n}) + k(1-\delta) - k') + \beta v(k')$$

$$\text{s.t.} \quad 0 \leq k' \leq f(k, \bar{n}) + k(1-\delta)$$

(u str inc, Inada, diff)

Assume u satisfies the Inada cond. (+ already assumed u str inc) (Also assume u, v diff)

FE :

$$v(k) = \max_{k'} \; u[f(k, \bar{n}) + k(1-\delta) - k'] + \beta v(k')$$

Policy fcn :

$g^*(\cdot)$ that solves $u'[f(k) + (1-\delta)k - g^*(k)] = \beta u'[f[g^*(k)] + (1-\delta) g^*(k) - g^* \circ g^*(k)] [f'(g^*(k)) + 1 - \delta]$.

(FOC : $u'[f(k, \bar{n}) + k(1-\delta) - k'] = \beta v'(k')$

Envelope : $v'(k) = u'[f(k, \bar{n}) + k(1-\delta) - k'] (f_k(k, \bar{n}) + 1 - \delta]$)

(e) Policy function is str. inc.

How to prove? use str. concavity of v.

$\Rightarrow$ $\boxed{\text{Thm 4.8}}$ need A4.3, 4.4, 4.7, 4.8

$\underbrace{\text{A4.3, 4.4}}_{\text{in (c)}}$

A 4.7 : F is str concave $\leftarrow$ u is str. concave.

A 4.8 : T is convex $\leftarrow$ f is concave.

($k_1' \in T(k_1)$, $k_2' \in T(k_2)$. Then $\lambda k_1' + (1-\lambda) k_2' \in T(\lambda k_1 + (1-\lambda) k_2)$ since

$\lambda k_1' + (1-\lambda) k_2' \le \lambda f(k_1, \bar{n}) + (1-\lambda) f(k_2, \bar{n}) + \lambda k_1 (1-\delta) + (1-\lambda) k_2 (1-\delta)$

$\le f(\lambda k_1 + (1-\lambda) k_2, \bar{n}) + \sim$.)

Assumptions.

in (c)

+

(u str concave

f concave

increasing)

Then v is str. concave.

Sps not, i.e., policy g is decreasing : $k < \hat{k} \Rightarrow g(k) \ge g(\hat{k})$.

FOC :

$$u'(f(k, \bar{n}) + k(1-\delta) - g(k)) = \beta v'(g(k))$$

$$u'(f(\hat{k}, \bar{n}) + \hat{k}(1-\delta) - g(\hat{k})) = \beta v'(g(\hat{k}))$$

$\vee$

since u is str. concave (Assume f is increasing in k)

$\Rightarrow$ $v'(g(k)) > v'(g(\hat{k}))$

$\Rightarrow$ $g(k) < g(\hat{k})$

contradiction.

Thus g is str. increasing.

(f) A steady-state is a value of capital k^* s.t. $k^* = g(k^*)$.

(In SS, $1 = \beta [f'(k^*) + 1 - \delta]$

$\Rightarrow$ f str. concave + Inada) $\Rightarrow \exists$ ss.

$\boxed{\text{Claim}}$ The positive SS is unique.

pf Sps $k^*, k^{**} > 0$ are SS.

Since these are positive, FOC and Envelope cond. hold.

EE is SS : $v'(k) = \beta v'(k)(f_k(k, \bar{n}) + 1 - \delta)$

Thus $\begin{cases} v'(k) = 0 & \text{or} \\ \beta(f_k(k, \bar{n}) + 1 - \delta) = 1 \end{cases}$ holds.

Assume u str inc and f inc. Then v is str. inc by Thm 4.7.

$v'(k) > 0 \Rightarrow \beta(f_k(k, \bar{n}) + 1 - \delta) = 1$.

Assume f is str. concave. Then this has a unique solution $\rightarrow\leftarrow$ □

f cont. diff

for Envelope be well-defined.

(u str. inc

f inc

f str concave

Inada)

Macro 2005 Spring Q I.3 Taxation

8/05

- 1 consumer, 1 sector
- gov't finance $\{g_t\}$ by τ_{ct} and τ_{nt}.
- 〃 $\hat{\tau}_{kt}$ and $\hat{\tau}_{nt}$.
- $\tau_{nt} = \hat{\tau}_{nt}$?

(a) A CE is a set of quantities $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty} \equiv \Omega_{HH}$ for HH,
〃 $\{n_t^f, k_t^f\}_{t=0}^{\infty} \equiv \Omega_f$ for firm,
a price system $\{p_t, r_t, w_t\}_{t=0}^{\infty} \equiv \mathbb{P}$, and
a policy $\{g_t, \tau_{ct}, \tau_{nt}\}_{t=0}^{\infty}$

such that

(i) given $\mathbb{P}$, Ω_{HH} solves

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t)$$

s.t.

$$\sum_{t=0}^{\infty} p_t[(1+\tau_{ct})c_t + k_{t+1} - (1-\delta)k_t] \le \sum_{t=0}^{\infty}[r_t k_t + w_t(1-\tau_{nt})n_t]$$

nonnegativity, k_0 given.

(ii) given $\mathbb{P}$, Ω_f solves

$$\max \sum_{t=0}^{\infty} \{p_t F(k_t^f, n_t^f) - r_t k_t^f - w_t n_t^f\}$$

s.t. nonnegativity.

(iii) GBC:

$$\sum_{t=0}^{\infty} p_t g_t = \sum_{t=0}^{\infty} (p_t \tau_{ct} c_t + w_t \tau_{nt} n_t)$$

(iv) Mkt clear:

$$k_t = k_t^f, \quad n_t = n_t^f$$

$$c_t + k_{t+1} + g_t = F(k_t, n_t) - (1-\delta)k_t \quad \forall t$$

(b) Allocation w/ τ_{ct} & τ_{nt} and w/ $\hat{\tau}_{kt}$ & $\hat{\tau}_{nt}$

Bad way ⇐ (Assume u is cont., str inc., str concave and satisfies the Inada cond.
F is cont, CRS)

FOC (sufficient):

$$\begin{cases} \text{wrt } c_t: & \beta^t u_{ct} - \lambda p_t(1+\tau_{ct}) = 0 \\ \text{wrt } k_{t+1}: & -p_t + r_{t+1} + (1-\delta)p_{t+1} = 0 \\ \text{wrt } n_t: & -\beta^t u_{nt} + \lambda w_t(1-\tau_{nt}) = 0 \end{cases}$$

$$\rightarrow \begin{cases} \text{Intertemporal EE}: & u_{ct}/(1+\tau_{ct}) = \beta u_{ct+1}(F_{kt+1} + 1 - \delta)/(1+\tau_{ct+1}) \\ \text{Intratemporal EE}: & u_{nt}/u_{ct} = F_{nt}\cdot(1-\tau_{nt})/(1+\tau_{ct}) \end{cases}$$

Note that TDCE is characterized by

$$\begin{cases} \text{EE} & \\ \text{TVC} & \Rightarrow \text{easy to check} \\ \text{Feasibility} & \Rightarrow \text{same} \end{cases}$$

* different tax system
$\Downarrow$
different price

EE's of TDCE w/ $\hat{\tau}_{kt}$ and $\hat{\tau}_{nt}$:

$$\begin{cases} \text{FOC wrt } c_t : & \beta^t U_{ct} - \lambda \hat{p}_t = 0 \\ \phantom{\text{FOC wrt }} k_{t+1} : & -\hat{p}_t + \hat{r}_{t+1}(1-\hat{\tau}_{kt+1}) + \hat{p}_{t+1}(1-\delta) = 0 \\ \phantom{\text{FOC wrt }} n_t : & -\beta^t U_{nt} + \lambda \hat{w}_t (1-\hat{\tau}_{nt}) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \text{Intertemporal EE} : & U_{ct} = \beta U_{ct+1}[F_{kt+1}(1-\hat{\tau}_{kt+1}) + 1-\delta] \\ \text{Intratemporal EE} : & U_{nt}/U_{ct} = F_{nt}(1-\hat{\tau}_{nt}) \end{cases}$$

Thus $\{\hat{\tau}_{kt}, \hat{\tau}_{nt}\}_{t=0}^{\infty}$ has to satisfy

$$\begin{cases} F_{kt+1}(1-\hat{\tau}_{kt+1}) + 1-\delta = [F_{kt+1} + 1-\delta]\dfrac{1+\tau_{ct}}{1+\tau_{ct+1}} & \forall t \\ 1-\hat{\tau}_{nt} = \dfrac{1-\tau_{nt}}{1+\tau_{ct}} & \forall t \end{cases}$$

where F_{kt+1} is calculated by the eqm allocation, i.e., $F_1(k_{t+1}^*, n_{t+1}^*)$.

We also need to specify $\hat{\tau}_{k0}$.

Since the eqm allocation satisfies the Implementability condition (necessary),

$$\frac{U_{c0}}{1+\tau_{c0}}[F_{k0} + 1-\delta]k_0 = U_{c0}[F_{k0}(1-\hat{\tau}_{k0}) + 1-\delta]k_0$$

$$\Leftrightarrow \quad F_{k0}(1-\hat{\tau}_{k0}) + 1-\delta = [F_{k0} + 1-\delta]\frac{1}{1+\tau_{c0}}$$

given the eqm allocation and $\{\tau_{ct}, \tau_{nt}\}$, we can define $\{\hat{\tau}_{kt}, \hat{\tau}_{nt}\}$ and it attains the same allocation

(c) By the result in (b), if $\tau_{ct} = 0$, we get

$$\tau_{nt} = \hat{\tau}_{nt}, \quad \forall t.$$

Also this implies

$$\tau_{kt} = 0, \quad \forall t.$$

In words, if the gov't finance the expenditure only by the labor tax, then the tax rate has to be equal (obvious).

Macro 2005 Spring Q I. 3 Taxation

8/05

(b)' Good approach for the equivalence of TDCE

$\hookrightarrow$ Check equiv. of BC's

* Assume F is diff'. (only)

Claim Given TDCE allocation for econ. 1, define

$$\hat{p} = \frac{1+\tau_{ct}}{1+\tau_{c0}} p_t, \quad \hat{r}_t = \frac{1+\tau_{ct}}{1+\tau_{c0}} r_t, \quad \hat{w}_t = \frac{1+\tau_{ct}}{1+\tau_{c0}} w_t$$

$$1 - \hat{\tau}_{nt} = \frac{1-\tau_{nt}}{1+\tau_{ct}}$$

$$1 - \hat{\tau}_{kt+1} = \frac{1}{F_{kt+1}} \left\{ \frac{1+\tau_{ct}}{1+\tau_{ct+1}} [F_{kt+1} + 1 - \delta] - (1-\delta) \right\}$$

$$1 - \hat{\tau}_{k0} = \frac{1}{F_{k0}} \left\{ \frac{1}{1+\tau_{c0}} [F_{k0} + 1 - \delta] - (1-\delta) \right\}$$

$\forall t$

Then $BC_2 = BC_1$ and hence the allocation is the solution for economy 2.

Note that
$\hat{p}, \hat{r}, \hat{w} \leftarrow$ FOC wrt c_t
$\hat{\tau}_{nt} \leftarrow$ Intratemporal EE
$\hat{\tau}_{kt} \leftarrow$ Intertemporal EE
$\hat{\tau}_{k0} \leftarrow$ Implementability cond.

Proof Check

$$\sum \hat{p}_t (c_t + x_t) = \sum \hat{r}_t (1 - \hat{\tau}_{kt}) k_t + \hat{w}_t (1 - \hat{\tau}_{nt}) n_t$$

$\Leftrightarrow$ ⋮

$$\Leftrightarrow \sum p_t ((1+\tau_{ct}) c_t + x_t) = \sum [r_t k_t + w_t (1-\tau_{nt}) n_t]$$

Macro 2005 Spring Q II. 1 Uncertainty

8105

- Stochastic economy
- 2 states Markov, technology shock.
- AD Eqm / SME / SPP the connection b/w them.
- 1 agent → 2 agents can we use DP ?

(a) An AD eqm is a set of quantities $\{c_t(s^t), l_t(s^t), k_{t+1}(s^t)\}_{t=0,s^t}^{\infty} \equiv \Omega_{HH}$,

$\{k_t^f(s^{t-1}), l_t^f(s^t)\}_{t=0,s^t}^{\infty} \equiv \Omega_f$

a price system $\{P_t(s^t), r_t(s^t), w_t(s^t)\}_{t=0,s^t}^{\infty} \equiv \mathbb{P}$

such that

(i) given $\mathbb{P}$, Ω_{HH} solves

$$\max \sum_t \sum_{s^t} \beta^t \pi(s^t) u[c_t(s^t), 1-l_t(s^t)]$$

s.t.

$$\sum_t \sum_{s^t} P_t(s^t)[c_t(s^t) + k_{t+1}(s^t) - (1-\delta)k_t(s^{t-1})] \le \sum_t \sum_{s^t} [r_t(s^t)k_t(s^{t-1}) + w_t(s^t)l_t(s^t)]$$

$0 \le l_t(s^t) \le 1 \quad \forall t, s^t$, nonnegativity $\quad k_0(s^{-1}) \equiv \bar{k}_0$

(ii) given $\mathbb{P}$, Ω_f solves

$$\max \sum_t \sum_{s^t} [P_t(s^t)\, \theta_t(s^t) k_t^f(s^{t-1})^{\alpha} l_t^f(s^t)^{1-\alpha} - r_t(s^t)k_t^f(s^{t-1}) - w_t(s^t) l_t^f(s^t)]$$

s.t. nonnegativity

(iii) Mkt clear : $\forall t, s^t$

$$k_t(s^{t-1}) = k_t^f(s^{t-1}), \quad l_t(s^t) = l_t^f(s^t)$$

$$c_t(s^t) + k_{t+1}(s^t) - (1-\delta)k_t(s^{t-1}) \le \theta_t(s^t) k_t^{\alpha}(s^{t-1}) l_t(s^t)^{1-\alpha}$$

(b) A SME is a set of quantities $\{\hat{c}_t(s^t), \hat{l}_t(s^t), \hat{k}_{t+1}(s^t), \hat{b}_t(s^t)\}_{t=0,s^t}^{\infty} \equiv \Omega_{HH}$ for HH

$\{\hat{k}_t^f(s^{t-1}), \hat{l}_t^f(s^t)\}_{t=0,s^t}^{\infty} \equiv \Omega_f$ for firm

a price system $\{\hat{q}_t(s^{t+1}), \hat{r}_t(s^t), \hat{w}_t(s^t)\}_{t=0,s^t}^{\infty} \equiv \mathbb{P}$

such that (i) given $\mathbb{P}$, Ω_{HH} solves

$$\max \sum_t \sum_{s^t} \beta^t \pi(s^t) u[\hat{c}_t(s^t), 1-\hat{l}_t(s^t)]$$

* → s.t. $\hat{c}_t(s^t) + \hat{k}_{t+1}(s^t) - (1-\delta)\hat{k}_t(s^{t-1}) + \sum_{s^{t+1} \succ s^t} \hat{q}_t(s^{t+1}) \hat{b}_{t+1}(s^{t+1})$

$$\le \hat{b}_t(s^t) + \hat{r}_t(s^t)\hat{k}_t(s^{t-1}) + \hat{w}_t(s^t)\hat{l}_t(s^t)$$

$0 \le \hat{l}_t(s^t) \le 1$, nonnegativity, no-ponzi, $\hat{k}_0(s^{-1}) = \bar{k}_0$

(ii) given $\mathbb{P}$, Ω_f solves $\forall t, s^t$

$$\max \theta_t(s^t) \hat{k}_t^{f\,\alpha}(s^{t-1}) \hat{l}_t^f(s^{t-1})^{1-\alpha} - \hat{r}_t(s^t)\hat{k}_t^f(s^t) - \hat{w}_t(s^t)\hat{l}_t^f(s^t) \quad \text{s.t. nonneg}$$

→ (iii) Mkt clear : goods, capital, labor (same as above), bond: $\hat{b}_t(s^t) = 0 \quad \forall t, s^t$.

Macro 2005 Spring Q II . 2 Global Stability

8105

- 2 sector growth model:

$$\begin{cases} \max_{\{c_t, k_{t+1}\}} \sum_t \beta^t u(c_t) \\ \text{s.t.} \quad c_t \le f(k_{ct}) \\ \qquad k_{t+1} \le \phi f(k_{kt}) \\ \qquad k_{ct} + k_{kt} \le k_t \\ \qquad k_0 = k^* \text{ fixed.} \end{cases}$$

- sufficient cond. on ϕ, β, u, f for the global stability

See 2005 Spring Q II . 3 (e)

(a) ϕ : $\phi > 0$, finite.

β : $0 < \beta < 1$

u : str increasing, bdd, cont., str. concave

f : cont. str. increasing, $f \ge 0$, bdd, str. concave

★ + $f^{-1\prime}\left(\frac{k}{\phi}\right) = \phi\beta$ has a unique sol'n k^*.

(b) u str. inc → all constraints hold w/ equality.

f cont, str. increasing → we can define $\underset{h}{f^{-1}}$ and f^{-1} is a fcn.

Then DP problem:

$$\begin{cases} V(k) = \max_{k'} u\left\{f\left[k - h\left(\frac{k'}{\phi}\right)\right]\right\} + \beta V(k') \\ \text{s.t.} \quad 0 \le k' \le \phi f(k) \end{cases}$$

Envelope : $V'(k) = u'(c) f'\left(k - h\left(\frac{k'}{\phi}\right)\right)$

FOC : $u'(c) f'\left[k - h\left(\frac{k'}{\phi}\right)\right] h'\left(\frac{k'}{\phi}\right) \cdot \frac{1}{\phi} = \beta V'(k')$

★ SS : $h'\left(\frac{k^*}{\phi}\right) = \phi\beta$ → Assume k^* is unique.

Step 1 : V is str concave

Step 2 : g, policy fcn, is str. increasing

Step 3 : Global Stability

$\boxed{\text{Step 1}}$ V is str. concave.

Thm 4.8 :

A 4.3 $f \geq 0$, $\overset{\text{finite}}{\phi \geq 0}$, f bdd, f cont.

4.4 u bdd, cont, $\beta \in (0,1)$

4.7 u str. concave

4.8 f (str) concave

$\Rightarrow$ V str increasing

□

$\boxed{\text{Step 2}}$ g is str. increasing

Sps not. Then for $k_1 < k_2$, $g(k_1) \geq g(k_2)$.

By step 1,

$$V'(g(k_1)) \leq V'(g(k_2))$$

Thus FOC implies

$$u'[f(k_1 - h(\tfrac{g(k_1)}{\phi}))]\, f'(k_1 - h(\tfrac{g(k_1)}{\phi}))\, h'(\tfrac{g(k_1)}{\phi})\, \frac{1}{\phi} \leq \sim^{k_2}$$

Since $\left\{ \begin{array}{l} u \text{ str increasing} \\ f \text{ str increasing} \\ f \text{ str concave} \\ \phi > 0 \end{array} \right.$ $\Rightarrow \left(\begin{array}{l} h \text{ str inc} \\ h \text{ str convex} \end{array} \right)$, LHS > RHS.

This is a contradiction.

Thus, g is str. increasing.

□

$\boxed{\text{Step 3}}$ Global Stability.

By step 1, we have

$$[V'(g(k)) - V'(k)]\,[g(k) - k] \leq 0$$

$$\Leftrightarrow [\tfrac{1}{\beta} u' \cdot f' \cdot h'(\tfrac{g(k)}{\phi}) \tfrac{1}{\phi} - u' \cdot f']\,[g(k) - k] \leq 0 \quad \left(\text{by FOC + Envelope}\right)$$

$$\Leftrightarrow \underbrace{[\tfrac{1}{\beta\phi} h'(\tfrac{g(k)}{\phi}) - 1]}_{H(k)}\,[g(k) - k] \leq 0 \quad \left(\text{Since } \begin{array}{l} u \text{ str inc} \\ f \text{ str inc.} \end{array}\right)$$

Equality holds only when $g(k) = k$, i.e., $k = k^*$.

Case 1 $k > k^*$. $g(k) \overset{\text{step 2}}{>} g(k^*) = k^*$. Since $H(k^*) = 0$, $H(k) > 0$. Thus $g(k) < k$.

Case 2 $k < k^*$ $g(k) < g(k^*) = k^*$. 〃 , $H(k) < 0$. Thus $g(k) > k$.

Therefore given k_0, define $k_t = g^t(k_0)$. Then $k_t \to k^*$.

Q.E.D.

Macro 2005 Spring Q II.3 Steady States and Taxation

8105

- 1 consumer / single sector
- constant tax on capital and labor income
- period-by-period GBC
- TDCE / SS of TDCE – fcn of τ
- econ. converges to SS.

(a) A TDCE is a set of quantities $\{c_t, k_{t+1}\}_{t=0}^{\infty} \equiv \Omega_{HH}$ for HH,
$\{k_t^f, n_t^f\}_{t=0}^{\infty} \equiv \Omega_f$ for firm,
gov't policy $(\{g_t\}_{t=0}^{\infty}, \tau) \equiv G$, and
a price system $\{p_t, r_t, w_t\}_{t=0}^{\infty} \equiv \mathbb{P}$

such that

(i) Given G, $\mathbb{P}$, Ω_{HH} solves

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} p_t (c_t + k_{t+1} - (1-\delta) k_t) = \sum_{t=0}^{\infty} (r_t (1-\tau) k_t + w_t (1-\tau))$$

nonnegativity.

(ii) Given $\mathbb{P}$, Ω_f solves

$$\max \sum_{t=0}^{\infty} [p_t F(k_t^f, n_t^f) - r_t k_t^f - w_t n_t^f]$$

s.t. nonnegativity.

(iii) GBC :

$$p_t g_t = r_t \tau k_t + w_t \tau \quad , \forall t.$$

(iv) Mkt clear :

$$\left.\begin{aligned} c_t + k_{t+1} - (1-\delta) k_t + g_t &= F(k_t^f, n_t^f) \\ k_t^f &= k \\ n_t^f &= 1. \end{aligned}\right\} \forall t.$$

(b) <u>Characterization of SS as a fcn of τ</u>

EE : $u_{c_t} = \beta u_{c_{t+1}} [F_{k\,t+1} (1-\tau) + 1 - \delta]$

in SS $1 = \beta [F_1(k_{ss}, 1)(1-\tau) + 1 - \delta]$

$\Leftrightarrow \quad F_1(k_{ss}, 1) = \left[\frac{1}{\beta} - (1-\delta)\right] \frac{1}{1-\tau}$

GBC + Mkt clear : $\forall t \quad p_t g_t = \tau (r_t k_t + w_t) = \tau\, p_t F(k_t, 1)$

$\therefore \quad g_{ss} = \tau F(k_{ss}, 1)$

By the feasibility :

$$c_{ss} + \delta k_{ss} + g_{ss} = F(k_{ss}, 1)$$

$$\therefore \quad c_{ss} = (1-\tau) F(k_{ss}, 1) - \delta k_{ss}$$

Then :

$$\begin{cases} F_1(k_{ss}, 1) = \frac{1}{1-\tau}\left[\frac{1}{\beta} - (1-\delta)\right] \\ g_{ss} = \tau F(k_{ss}, 1) \\ c_{ss} = (1-\tau) F(k_{ss}, 1) - \delta k_{ss} \end{cases}$$

$\tau \uparrow \rightarrow$ $k_{ss} \downarrow$, $g_{ss} \uparrow\downarrow$, $c_{ss} \uparrow\downarrow$

assume F is concave

($g_{ss}\uparrow \Rightarrow c_{ss}\downarrow$, $c_{ss}\uparrow \Rightarrow g_{ss}\downarrow$, if $\delta = 0$, $c_{ss}\downarrow$ so on)

(c) Proof econ. converges to ss / ss is globally stable.

→ Use SPP

First, this problem is equivalent to the SPP w/ production $\tilde{F} = (1-\tau) F$.
Thus, we solve the following DP instead:

$$V(k) = \max_{c, k'} \; u(c) + \beta V(k')$$

$$\text{s.t.} \quad c + k' - (1-\delta) k = (1-\tau) F(k, 1).$$

WTS : if $k > k^*$, then $g(k) \in [k^*, k)$
$0 < k < k^*$, then $g(k) \in (k, k^*]$ (*)

where $g(\cdot)$: policy fcn for k', $k^* = k_{ss}$

Step 1 : V is str. concave
Step 2 : g is str. increasing
step 3 : prove (*).

$\boxed{\text{Step 1}}$: V is str. concave.

$\boxed{\text{Thm 4.8}}$
A 4.3 : F ≥ 0, F bdd, cont
A 4.4 : u bdd, cont. $\beta \in (0,1)$
A 4.7 : u str concave
A 4.8 : F concave.

⇒ V is str. concave.

Macro 2005 Spring Q II . 3 Steady States and Taxation

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$\boxed{\text{Step 2}}$: g is str. increasing

★ Assume u is str concave.
F is str increasing.

Sps not. Then for $k_1 < k_2$, $g(k_1) \geq g(k_2)$

FOC : $u'(c) = \beta V'(k')$

Since V is str. concave,

$$\beta V'(g(k_1)) \leq \beta V'(g(k_2))$$

Thus $u'[(1-\tau)F(k_1, 1) + (1-\delta)k_1 - g(k_1)] \leq u'[\underset{\sim}{k_2}]$.

Since F is str. increasing and u is str concave,
this is a contradiction.

Thus, g is str. increasing.

$\boxed{\text{Step 3}}$: Prove the global stability.

By step 1 : V is str. concave, $\forall k$

$$[V'(g(k)) - V'(k)][g(k) - k] \leq 0 \quad \cdots (*)$$

and equality holds when $g(k) = k$, i.e., $k = k^*$.

Envelope cond. : $V'(k) = u'(c)[(1-\tau)F_k + 1 - \delta]$

Then from this and FOC,

$$(*) \Leftrightarrow \{u'(c)/\beta - u'(c)[(1-\tau)F_k + 1 - \delta]\}(g(k) - k) \leq 0$$

where $c \equiv (1-\tau)F(k, 1) + (1-\delta)k - g(k)$

★ Assume u is str. increasing.

Then we have

$$\{1/\beta - [(1-\tau)F_k + 1 - \delta]\}(g(k) - k) \leq 0 \quad \cdots (**)$$

Case 1 : $k > k^*$.

Since in SS we have

$$1/\beta - [(1-\tau)F_{k^*} + 1-\delta] = 0$$

and $k > k^*$, we get

$$1/\beta - [(1-\tau)F_k + 1-\delta] \leq 0.$$

Since $k \neq g(k)$, we have a strict inequality in (**).

Thus, we get

$$g(k) - k < 0$$

$$\Leftrightarrow \quad g(k) < k.$$

By step 2 g is str. increasing in k,

$$g(k) > g(k^*)$$

$$\Rightarrow \quad g(k) > k^* \qquad \text{since } k^* = g(k^*).$$

Thus, if $k > k^*$,

$$g(k) \in (k^*, k)$$

Case 2 : $k < k^*$

Similarly, we get

$$g(k) > k.$$

By step 2,

$$g(k) < g(k^*) = k^*$$

Thus, if $k < k^*$,

$$g(k) \in (k, k^*).$$

Therefore no matter what k_0 and τ are, $k \to k^*$ and hence $c \to c^*$, $g \to g^*$.

Q.E.D.

Micro

F 2007	Allen	~~Rustichini~~	~~Werner*~~		
S 2007	Allen	~~Rustichini*~~	~~Werner~~		
F 2006	Allen	~~Rustichini*~~	~~Werner~~		
S 2006	Allen	McLennan	Rustichini	~~Werner*~~	
F 2005	Allen	Hurwicz	McLennan	Rustichini	~~Werner*~~
S 2005	Allen*	Hurwicz	McLennan	Rustichini	~~Werner~~
F 2004	Allen*	Hurwicz	McLennan	Rustichini	~~Werner~~
S 2004	Allen	Hurwicz	Richter	Rustichini*	
F 2003	Allen	Hurwicz	Richter*	Rustichini	
S 2003	Allen	Hurwicz	Richter	Rustichini	~~Werner*~~
F 2002	Allen	Hurwicz*	Rustichini	~~Werner~~	
S 2002	Allen	Rustichini	Richter	Werner*	
~~F 2001~~	Allen	McLennan	Richter*	Werner	
~~S 2001~~	Allen	McLennan	Richter	Werner*	
~~F 2000~~	Allen	McLennan*	Richter	Werner	
~~S 2000~~	Allen	McLennan	Richter	Werner*	

Macro

F 2007	Chari	Jones*	P.Kehoe	Perri	
S 2007	Chari*	Jones	T. Kehoe	Kocherlakota	
S 2006	Chari	Jones	T. Kehoe		
F 2005	Chari	Jones*	T. Kehoe		
S 2005	Chari*	Jones	T.Kehoe		
F 2004	Chari*	Jones	T. Kehoe		
S 2004	Chari*	Jones	T.Kehoe		
F 2003	Chari*	Jones	T.Kehoe		
S 2003	Chari	Jones	Prescott*		
F 2002	Chari	Jones	Prescott*	Kocherlakota	
S 2002	Chari*	Jones	Prescott	Kocherlakot	
F 2001	Chari*	Jones	Prescott	Kocherlakota	
S 2001	Jones*	T. Kehoe	Kocherlakota	Prescott	
F 2000	T. Kehoe*	Jones	Kocherlakota	Prescott*	
S 2000	T. Kehoe*	Jones	Kocherlakota	Santos	