Economics 8102 (Microeconomic Theory) Suggested Answer for the Final Exam

Ichiro Obara

January 24, 2008

1. (20 pts.) Consider an economy with one consumer, whose preference is given by $u(x) = \ln x_1 + \ln x_2$ ($X = \Re^2_{++}$).

   (a) (6 pts.) Show that there exists a unique competitive equilibrium for any initial allocation $e >> 0$ (after normalizing prices) and derive the competitive equilibrium for each $e >> 0$.

   Answer. This follows from the market clearing conditions almost immediately. The market clearing conditions imply that $(x_1^*, x_2^*) = (e_1, e_2)$ is the only candidate for the competitive equilibrium. Then the unique equilibrium price (with $p_2^* = 1$) must be $p^* = \left(\frac{e_2}{e_1}, 1\right)$.

   (b) (14 pts.) Let $(x^*(e), p^*(e))$ be the unique competitive equilibrium for each e. Show that $(x^*(e), p^*(e))$ is a differentiable function at $e = (1, 1)$. Make sure to write down the statement of any theorem you use.

   Answer. The competitive equilibrium at $e = (1, 1)$ is clearly $(x^* p^*) = ((1, 1), (1, 1))$. For each $e >> 0$, the unique competitive equilibrium is characterized by the following equations:

$$\begin{aligned} \frac{1}{x_1} - \lambda p_1 &= 0 \\ \frac{1}{x_2} - \lambda &= 0 \\ p_1(e_1 - x_1) + p_2(e_2 - x_2) &= 0 \\ e_1 - x_1 &= 0 \end{aligned}$$

where p_2 is normalized to 1 and the market clearing condition for good 2 is dropped (as a consequence of Walras' law).

By Implicit function theorem (see the slides for the definition), we just need to check if the derivative of the left hand side at (x^*p^*) with respect to (x, λ, p_1) is a regular 4×4 matrix. This matrix at $(x^*p^*) = ((1,1), (1,1))$ is

$$\begin{pmatrix} -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

This is a full rank matrix because it is a triangular matrix with nonzero diagonals (you can change this matrix into a diagonal matrix by applying transformations explained in the class).

2. (18 pts.) Prove or disprove each of the following statements.

(a) (6 pts.) Consider an economy with two consumers, who have a nonsatiated, convex preference. If both (x_1^*, x_2^*) and (x_1^{**}, x_2^{**}) are Pareto-efficient allocations, then $(\alpha x_1^* + (1-\alpha)x_1^{**}, \alpha x_2^* + (1-\alpha)x_2^{**})$ is Pareto-efficient for any $\alpha \in [0,1]$.

Answer. This is not correct. Example: $N = L = 2$, $u_1(x_1) = \min\{2x_{11}, x_{12}\}$, $u_2(x_2) = \min\{x_{21}, 2x_{22}\}$, and $r = (1,1)$. $((1,1), (0,0))$ and $((0,0), (1,1))$ are Pareto-efficient, but $\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$ is not.

(b) (6 pts.)Consider a two-period economy with $S = 3$ (i.e. three possible states at date 1). If there are five financial assets in this economy, at least two of them are redundant.

Answer. This is correct. The return of these assets can be represented by 3×5 matrix. Since the rank of this matrix is at most three, at least two columns are not independent from the other columns. This means that at least two assets are redundant.

(c) (6 pts.) There exists a competitive equilibrium in $E^{pure} = \left\{(X_i \preceq_i, e_i)_{i=1}^N\right\}$ if $X_i = \Re_+^N$, $\preceq_i$ is continuous and locally nonsatiated, and $e_i >> 0$ for $i = 1, 2, ..., N$.

Answer. This is not correct. Example: $N = 1$, $L = 2$, $e_i = (1,1)$, and $u(x) = \max\{x_1, x_2\}$.

3. (12 pts.) Write down the definition of Radner equilibrium.

Answer. See the slides.

4. (25 pts.) For each $\lambda \in \Re_{++}^N$, let $v(\lambda) = \max_{x \in A} \sum_{i=1}^N \lambda_i u_i(x_i)$, where u_i is consumer i's continuous utility function and A is the feasible consumption set of this economy $\left(0 \leq \sum_{i=1}^N x_i \leq r\right)$. Answer the following questions.

   (a) (7 pts.) Show that there exists $x \in A$ to maximize $\sum_{i=1}^N \lambda_i u_i(x_i)$ for every $\lambda \in \Re_+^N$.

   Answer. This follows from continuity of the objective function and compactness of A.

   (b) (8 pts.) Show that $x^* \in A$ is Pareto-efficient if x^* solves $\max_{x \in A} \sum_{i=1}^N \lambda_i u_i(x_i)$ for some $\lambda \in \Re_{++}^N$.

   Answer. See the slides.

   (c) (10 pts.) Prove that $v(\lambda)$ is continuous in $\Re_+^N$ without using the theorem of Maximum.

   Answer. Suppose that $v(\lambda)$ is not continuous at $\lambda^* \in \Re_+^N$. Then there exists a sequence $\lambda_n, n = 1, 2, ...$ that converges to λ^* such that $v(\lambda_n)$ does not converge to $v(\lambda^*)$. Then we can find $\varepsilon > 0$ and a subsequence such that $|v(\lambda^*) - v(\lambda_n)| \geq \varepsilon$ for every n. By taking a subsequence again, we can assume that either (1) $v(\lambda_n) \geq v(\lambda^*) + \varepsilon$ for every n or (2) $v(\lambda_n) \leq v(\lambda^*) - \varepsilon$ for every n. Suppose that (1) is the case. We can take a convergent subsequence so that the solution of the maximization problem x_n

given λ_n converges to x'. Then clearly $v(\lambda_n) \to \sum_{i=1}^N \lambda_i^* u_i(x_i') \geq v(\lambda^*) + \varepsilon$. This is a contradiction. Next suppose that (2) is the case. Let x^* be the solution given λ^*. Then $\sum_{i=1}^N \lambda_{i,n} u_i(x_i^*) > v(\lambda_n)$ for a large n because $\sum_{i=1}^N \lambda_{i,n} u_i(x_i^*)$ converges to $v(\lambda^*)$. Again this is a contradiction. So $v(\lambda)$ is continuous in $\Re_+^N$.

5. (25 pts.)Consider the following economy: $N = S = 2$, $L = 1$, $u_1(x_1) = 2\ln x_{1,1} + \ln x_{1,2}$, $u_2(x_2) = \ln x_{2,1} + 2\ln x_{2,2}$, and $e_1 = (0,1,0), e_2 = (0,0,1)$. Answer the following questions.

   (a) (12 pts.) Find all Arrow-Debreu equilibria (normalize p_2 to 1).

   Answer. Since utility functions are strictly concave, the optimal consumption vectors are characterized by the first order (Kuhn-Tucker) conditions. Hence Arrow-Debreu equilibrium is characterized by the following equations:
   First order conditions:

$$
\begin{aligned}
\frac{2}{x_{11}} - \lambda_1 p_1 &= 0 \\
\frac{1}{x_{12}} - \lambda_1 p_2 &= 0 \\
\frac{1}{x_{21}} - \lambda_2 p_1 &= 0 \\
\frac{2}{x_{22}} - \lambda_2 p_2 &= 0
\end{aligned}
$$

   Budget constraints:

$$
\begin{aligned}
p_1 x_{11} + p_2 x_{12} &= p_1 \\
p_1 x_{21} + p_2 x_{22} &= p_2
\end{aligned}
$$

   Market clearing conditions:

$$
\begin{aligned}
x_{11} + x_{21} &= 1 \\
x_{12} + x_{22} &= 1
\end{aligned}
$$

   The unique solution of this system of equations is $(x^*, p^*, \lambda^*) = \left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right), (1,1), (3,3)\right)$. Therefore the unique Arrow-Debreu equilibrium is $(x^*, p^*) = \left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right), (1,1)\right)$.

(b) (13 pts.) Suppose that there is only one asset in this economy (so the market is not complete) with return $r_1 = r_2 = 1$. Is there any Radner equilibrium? If so, find one.

Answer. The budget constraints at date 0 are

$$\begin{aligned} -qb_1 &\leq 0 \\ -qb_2 &\leq 0. \end{aligned}$$

And the market clearing condition for the asset is

$$b_1 + b_2 = 0$$

For these conditions to be satisfied, either $q = 0$ or $b_1 = b_2 = 0$. Neither is compatible with the optimal consumption of consumers (If $q = 0$, then consumers buy an infinite amount of assets. $b_1 = b_2 = 0$ cannot be true, either. The demand for the asset is always strictly positive however high the prices are, since the marginal utilities are ∞ with 0 consumption). Therefore there is no Radner equilibrium.