

Economics 8102

Suggested Answer for Midterm Exam

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1. Consider a pure exchange economy $E^{pure} = ((X_i, \succeq_i, e_i)_{i=1}^N)$. Define competitive equilibrium (or competitive equilibrium with transfer) formally. Then state and prove the first fundamental theorem of welfare economics.

Answer. See the slides.

2. Prove or disprove each of the following statements.

- (a) Walras' law ($\sum_{i=1}^N p \cdot (e_i - x_i(p, e_i)) = 0$) is satisfied if $X_i = \mathfrak{R}_{++}^L$, $e_i \in \mathfrak{R}_{++}^L$, and \succeq_i is strictly concave for $i = 1, \dots, N$.

Answer. This is not correct. Example: $L = 2$, $u_i(x_i) = -(x_{i,1} - 1)^2 - (x_{i,2} - 1)^2$, and $e_i = (2, 2)$ for $i = 1, \dots, N$.

- (b) If \succeq_i is monotone and $e_i \in \mathfrak{R}_{++}^L$ for $i = 1, \dots, N$, then every competitive equilibrium price p^* is strictly positive ($p^* \gg 0$).

Answer. This is not correct. Example: $N = 1$, $L = 2$, $u_i(x_i) = \min\{x_{i,1}, x_{i,2}\}$, and $e_i = (1, 1)$ for $i = 1, \dots, N$. In this economy, (p^*, e) is a competitive equilibrium for any $p^* (\neq 0) \in \mathfrak{R}_+^2$.

- (c) Suppose that $X_i \subset \mathfrak{R}_+^L$ is convex and $0 \in X_i$. Let $u_i : X_i \rightarrow \mathfrak{R}$ be a continuous utility function. If $x_i^* \in X_i$ solves $\min_{x_i \in X_i} p \cdot x_i$ s.t. $u_i(x_i) \geq u_i(x_i^*)$ and satisfies $p \cdot x_i^* > 0$ for some $p \in \mathfrak{R}^L$, then $x_i^* \in X_i$ also solves $\max_{x_i \in X_i} u_i(x_i)$ s.t. $p \cdot x_i \leq p \cdot x_i^*$.

Answer. This is correct. See the slides.

(d) Let $x^* \gg 0$ be a Pareto efficient allocation for $E^{pure} = \left((X_i, \succeq_i, e_i)_{i=1}^N \right)$.

Suppose that $X_i = \mathfrak{R}_+^L$, $e_i \in \mathfrak{R}_{++}^L$ and \succeq_i is continuous and locally nonsatiated for $i = 1, \dots, N$. Then, for any $\varepsilon > 0$, there exists another Pareto efficient allocation x' ($\neq x^*$) $\in X$ such that $\|x' - x^*\| < \varepsilon$.

Answer. This is not correct. Let $L = 2$, $N = 2$, $e_1 + e_2 = (2, 2)$. Suppose that the utility functions of the consumers are given by:

$$\begin{aligned} u_1(x_{11}, x_{12}) &= \max \left\{ 2 \min \{2x_{11}, x_{12}\}, (\min \{x_{11}, 2x_{12}\})^2 + 1 \right\} \\ u_2(x_{21}, x_{22}) &= x_{21} + x_{22} \end{aligned}$$

Consumer 1's preference is given by the maximum of two (increasing transformation of) Leontief preferences: $\min \{2x_{11}, x_{12}\}$, $\min \{x_{11}, 2x_{12}\}$. Since consumer 2's preference is linear, Pareto optimal allocations are always at a kink of one of these Leontief preferences. When $x_{21} + x_{22} = 0.5$, there are two Pareto-efficient allocations: $(0, 5, 1)$, and $(1, 0.5)$ (just the allocations for consumer 1). The former corresponds to the kink of $2 \min \{2x_{11}, x_{12}\}$ and the latter corresponds to the kink of $(\min \{x_{11}, 2x_{12}\})^2 + 1$. However, this is the only case that these two Leontief functions generate the same value on consumer 2's indifference curves. Whenever $x_{21} + x_{22}$ is constant and not 0.5, the latter dominates the former because $2u \leq u^2 + 1$ and the equality holds if and only if $u = 1$. Therefore the Pareto efficient allocations are characterized by the kink points of $(\min \{x_{11}, 2x_{12}\})^2 + 1$ except for $(0.5, 1)$, which is isolated from the other Pareto-efficient allocations.

(The original counterexample was for $x^* > 0$. So we replaced it with this example).

3. Consider an economy with two goods $(x, y) \in \mathfrak{R}_+^2$ and two consumers: consumer 1 and 2. Consumer 1 owns all y and consumer 2 owns all x initially ($e_1 = (0, \bar{y})$, $e_2 = (\bar{x}, 0)$). Preferences are continuous, locally nonsatiated, and strictly convex for both consumer 1 and 2. In addition, assume that consumer 1 has a strict preference in x ($u_1(x', y) > u_1(x, y)$ for any $x' > x$ and y) and consumer 2 has a strict preference in y . Prove that there exists a competitive equilibrium in this pure exchange economy.

Answer. Existence is trivial if either $\bar{x} = 0$ or $\bar{y} = 0$. So assume that $\bar{x}, \bar{y} > 0$. Since preferences are continuous and strictly convex. There exists the unique optimal consumption vector for each consumer. Normalize prices so that $p_x + p_y = 1$. Then we can define the excess demand functions just as a function of p_x : $z_x(p_x) = \sum_{i=1}^2 \{x_i(p_x, e_i) - e_{i,x}\}$ and $z_y(p_x) = \sum_{i=1}^2 \{y_i(p_x, e_i) - e_{i,y}\}$. Since Walras' Law follows from local nonsatiation, we can drop one market clearing condition. We just need to find $p_x \in (0, 1)$ to satisfy $z_x(p_x) = 0$. Note that $z_x(p_x)$ is continuous in $(0, 1)$ (by the theorem of Maximum). We show that $z_x(p_x) > 0$ for some (small) p_x and $z_x(p_x) < 0$ for some (large) p_x . Then existence of a competitive equilibrium follows from the intermediate value theorem.

Consider a sequence of prices $p_{x,n} \in (0, 1)$ that converges to 0. Suppose that consumer 1's optimal consumption vector $(x_{1,n}, y_{1,n})$ is bounded. Then there exists a convergent subsequence such that $(x_{1,n}, y_{1,n}) \rightarrow (x_1^*, y_1^*)$ because $(x_{1,n}, y_{1,n})$ is contained in a compact set. Consider any consumption vector (x_1, y_1) which satisfies the budget constraint in the limit $((x_1, y_1) \in B_1((0, 1), e_1))$. For any $\alpha \in (0, 1)$, $(\alpha x_1, \alpha y_1)$ is in the budget set for large n (because $\bar{y} > 0$), so $(x_{1,n}, y_{1,n}) \succeq_1 (\alpha x_1, \alpha y_1)$. Taking $n \rightarrow \infty$ first and then $\alpha \rightarrow 1$, we have $(x_1^*, y_1^*) \succeq_1 (x_1, y_1)$ for any such (x_1, y_1) . Therefore (x_1^*, y_1^*) must be the optimal consumption vector for 1 given $p = (0, 1)$. But this is a contradiction because consumer 1 has a strict preference in x , hence there should not exist the optimal consumption vector for 1 given $p = (0, 1)$. This means that $(x_{1,n}, y_{1,n})$ is unbounded. Since the limit price of y is positive and $x_{1,n}$ is bounded below by 0, $y_{1,n}$ must be bounded above. Hence $x_{1,n}$ is unbounded. This proves that there exists $p_x \in (0, 1)$ such that $z_x(p_x) > 0$. Similarly we can find p_x such that $z_x(p_x) < 0$ by considering a sequence of prices $p_{x,n} \in (0, 1)$ that converges to 1.

(There was a typo in the original question. x and y were reversed in "Consumer 1 owns all y and consumer 2 owns all x ", although the initial allocations $e_1 = (0, \bar{y}), e_2 = (\bar{x}, 0)$ were correctly described. We took this into account when grading the exams).