Economics 8102: Problem Set I

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Due November 8

1. Consider a pure exchange economy with two goods: left shoe (good 1) and right shoe (good 2). These goods are not divisible (For this question, regard X_i as the set of all grid points in $\Re^2_+$). Every consumer has the same preference: $u_i(x) = \min\{x_{1,i}, x_{2,i}\}$. Suppose that there are m consumers endowed with only one left shoe (and no right shoe) and $m+1$ consumers endowed with only one right shoe (and no left shoe).

   (a) Find all Pareto efficient allocations.

   (b) Find all competitive equilibria given the above endowments.

   (c) Is every competitive equilibrium Pareto-efficient? If not, explain why FFTWE fails.

2. Suppose that u_i is concave and strongly monotone for every i. Show that $u(x) \in \Re^L$ is on the Pareto frontier ($u(x) \in U^P$) if and only if x solves $\max_{x \in A} \sum_{i=1}^N a_i u_i(x_i)$ for some $a \in \Re^L_+ / \{\mathbf{0}\}$.

3. Consider the following problem:

$$\max_{x \in A} u_1(x_1)$$
$$\text{s.t. } u_i(x_i) \geq \underline{u}_i, \ i = 2, ..., N$$

   Suppose that $u_i, i = 1, ..., N$ are locally nonsatiated (remember that continuity is always assumed). In addition, assume that u_1 is strongly monotone. Prove that $x' \in X$ is a Pareto efficient allocation if and only if $x' \in X$ solves this problem for some $\underline{u}_i \in \Re$, $i = 2, ..., N$.

4. A **coalition** S, a subset of consumers, improves upon $x \in A$ if $\exists x' \in A$ such that $u_i(x_i') > u_i(x_i)$ for every $i \in S$ and $\sum_{i \in S} x_i' \leq \sum_{i \in S} e_i$. A feasible allocation $x \in A$ has the **core property** if there is no coalition that can improve upon x. The **core** of an economy E is the set of feasible allocations that have the core property.

   (a) Suppose that consumers' preferences are strongly monotone. Show that $x \in A$ is Pareto efficient when it has the core property. Also show that this may not be true when the preferences are just monotone.

   (b) (a) implies that the following statement is a refinement of the first fundamental theorem of welfare economics:

   If $(x^*, p^*) \in X \times \Re_+^L$ *is a CE for* E^{pure}*, then* x *is in the core of* E^{pure}.

   Prove this statement.

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Solutions to Problem Set 1
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Question 1

My convention will be to let the first dimension of consumption to be left shoes, and the second to be right shoes. We also assume that there is no free disposal technology.

Definition 1 (Feasibile Allocation)**.** *An allocation, x, is* feasible *if and only if*

$$\sum_{i=1}^{2m+1} x_1^i = m, \text{ and } \sum_{i=1}^{2m+1} x_2^i = m+1.$$

We will denote the set of feasible allocations by A.

1A: Find all Pareto Efficient Allocations

To answer this question, we find a necessary and sufficient condition for pareto efficiency (note that a sufficient condition alone does not characterize all PE allocations).

Claim 2. *$x \in A$ is pareto efficient if and only if $x \in A$ satisfies the following property:*

- *$\nexists$ two agents i, j such that $x_1^i > x_2^i$ and $x_2^j > x_1^j$*

(An alternative statement could be, $x \in A$ is pareto efficient if and only if $x \in A$ satisfies the following property:

- *$\exists$ some $i \in \{1, \ldots, 2m+1\}$ s.t. $x_2^i - x_1^i = 1$ and for all $j \neq i, x_1^j = x_2^j$.*

I will prove the first statement.)

Comment 3. First, observe that any feasible allocation x satisfying this property cannot have $x_1^k > x_2^k$ for any k. If this were so, then the property would imply that for all $i \neq k$, $x_1^i \geq x_2^i$. Since x is feasible, though, this implies that

$$m+1 = \sum_{i=1}^{2m+1} x_2^i < \sum_{i=1}^{2m+1} x_1^i = m$$

which is a contradiction.

Proof. ($\Rightarrow$, by contradiction) Suppose that x is pareto efficient. Assume for contradiction that $\exists$ two agents i, j such that $x_1^i > x_2^i$ and $x_2^j > x_1^j$. Then, consider the following modified allocation, $\hat{x}$ defined by:

- $\left(\hat{x}_1^i, \hat{x}_2^i\right) = \left(x_1^i - 1, \hat{x}_2^i\right)$
- $\left(\hat{x}_1^j, \hat{x}_2^j\right) = \left(x_1^j + 1, \hat{x}_2^j\right)$
- $\left(\hat{x}_1^k, \hat{x}_2^k\right) = \left(x_1^k, x_2^k\right)\ \forall\, k \neq i, k \neq j$

Notice, as $x_1^i > x_2^i$ and X_i is the set of grid points in $\mathbb{R}_+^2$, it is immediate that $x_1^i - 1 \geq x_2^i$. Similarly we have that $x_2^j \geq x_1^j + 1 > x_1^j$. Certainly, $\hat{x} \in A$ given that $x \in A$. Thus, $u_k(\hat{x}^k) = u_k(x^k)$ for all $k \neq i, j$, $u_i(\hat{x}^i) \geq u_i(x^i)$ and $u_j(\hat{x}^j) > u_j(x^j)$, which contradicts the assumption that x is pareto efficient. Thus, it must be that $\nexists$ two agents i, j such that $x_1^i > x_2^i$ and $x_2^j > x_1^j$.

($\Leftarrow$, also by contradiction). Let x satisfy the given property, and suppose for contradiction that x is not pareto efficient. Then, there exists $\hat{x} \in A$ such that $u(\hat{x}) > u(x)$. Let $k \in \{1, \ldots, 2m+1\}$ such that $u_k(\hat{x}^k) > u_k(x^k)$. In order to make k better off at $\hat{x}$ than at x, we must have one of three cases:

1. At x, k has more right shoes than left, in which case k has at least 1 more left shoe at $\hat{x}$ than at x.
2. At x, k has more left shoes than right, in which case k has at least 1 more right shoe at $\hat{x}$ than at x.
3. At x, k has equal left and right shoes, in which case k has at least 1 more of both shoes at $\hat{x}$ than at x.

Thus, under x, we had one of three cases:

Case 1: $u_k(x^k) = x_1^k < x_2^k$. In this case, by our assumption, it must be that for all $i \neq k$, $x_2^i \geq x_1^i$. Hence, for all $i \neq k, u_i(x^i) = x_1^i$. Then, if $\hat{x}_1^k > x_1^k$, it must be that for some $i \neq k$, $\hat{x}_1^i < x_1^i$. But then $u_i(\hat{x}_1^i) < u_i(x^i)$. Thus, $\hat{x}$ is not a pareto improvement from x.

Case 2: $u_k(x^k) = x_2^k < x_1^k$. Observe that this is not possible under the given property as discussed in the above comment.

Case 3: $u_k(x^k) = x_1^k = x_2^k$. Under this condition, $u_k(\hat{x}^k) > u_k(x^k)$ implies $\hat{x}_1^k > x_1^k$ and $\hat{x}_2^k > x_2^k$. Feasibility, then, implies that there exists agents $i, j \neq k$ such that $\hat{x}_1^i < x_1^i$ and $\hat{x}_2^j < x_2^j$. Obviously, if $i = j$, then $\hat{x}$ is not a pareto improvement from x. If $i \neq j$, pareto dominance requires $u_i(\hat{x}^i) \geq u_i(x^i)$ and $u_j(\hat{x}^j) \geq u_j(x^j)$. However, this implies that $x_1^i > x_2^i$ and $x_2^j > x_1^j$, which violates our original assumption (there exists no two agents such that...). Hence, we have a contradiction.

In each case, pareto dominance is not possible. Thus, x must be pareto efficient. $\square$

1B: Find all competitive equilibria given the above endowments

Note, m agents are endowed with only one left shoe and $m+1$ are endowed with only one right shoe. There are $2m+1$ agents, and we order them from 1 to $2m+1$ starting with those endowed with one left shoe followed by those endowed with one right shoe.

Claim 4. *With $p_1 > 0, p_2 = 0$, there are two qualitative CE allocations. With $p_1 = 0, p_2 > 0$, there are no competitive equilibria. With $p_1 > 0, p_2 > 0$, the allocation*

$$x_i^* = \begin{cases} (1,0) & \text{if} \quad i \leq m \\ (0,1) & \text{if} \quad i > m \end{cases}$$

constitutes an equilibria. Moreover, with $p_1 = p_2$, any allocation where m agents consume $(1,0)$ and $m+1$ agents consume $(0,1)$ regardless of whether they were endowed with one left shoe or one right shoe is an equilibrium.

Proof. **Case 1:** $p_1 > 0, p_2 = 0$. Normalize $p_1 = 1$.

Then, for $i \leq m$, the budget constraint is $x_1^i = 1$. So budget feasible bundles are $\{(0,k),(1,k)\}$ for $k \in \mathbb{N}$. At these prices, $(0,k)_{k\in\mathbb{N}}, (1,0)$ cannot be optimal. Hence, for all $i \leq m$, we have that $\min x_2^{i*} = 1$.

For $i > m$, the budget constraint is $x_1^i = 0$. Budget feasible bundles for these agents are $\{(0,k)\}$ for $k \in \mathbb{N}$ and in fact each of these bundles are in the demand.

Hence, at these prices, CE allocations are of the following form:

$$x_i^* = \begin{cases} (1,1) & \text{if} \quad i \leq m \\ (0,1) & \text{if} \quad i = k, \text{ for some } k \in \{m+1,\ldots,2m+1\} \\ (0,0) & \text{if} \quad i > m, i \neq k \end{cases}$$

$$x_i^* = \begin{cases} (1,2) & \text{if} \quad i = k, \text{ for some } k \in \{1,\ldots,m\} \\ (1,1) & \text{if} \quad i \leq m, i \neq k \\ (0,0) & \text{if} \quad i > m \end{cases}$$

Case 2: $p_1 = 0, p_2 > 0$. Normalize $p_2 = 1$.

For $i \leq m$, the budget constraint is $x_2^i = 0$. Budget feasible bundles for these agents are $\{(k,0)\}$ for $k \in \mathbb{N}$ and in fact each of these bundles are in the demand.

For $i > m$, the budget constraint is $x_2^i = 1$. So budget feasible bundles are $\{(k,0),(k,1)\}$ for $k \in \mathbb{N}$. At these prices, $(k,0)_{k\in\mathbb{N}}, (0,1)$ cannot be optimal. Hence, for all $i > m$, we have that $\min x_1^{i*} = 1$. This implies, however, that $\sum_{i=1}^{2m+1} x_1^{i*} \geq m+1 > m$ which violates a market clearing condition. So there can be no equilibrium at these prices.

Case 3: $p_1, p_2 > 0$. Normalize $p_1 = 1$. Then budget constraints are

$$\begin{aligned} \text{for } i \leq m: \quad & x_1^i + \bar{p}_2 x_2^i \leq 1 \\ \text{for } i > m: \quad & x_1^i + \bar{p}_2 x_2^i \leq \bar{p}_2 \end{aligned}$$

Thus, budget feasible bundles are of the form

$$\begin{aligned} &\text{for } i \leq m \quad B^i = \{(0,0),(1,0),(0,k) | k \in \mathbb{N}, k \leq \tfrac{1}{\bar{p}_2}\} \\ &\text{for } i > m \quad B^i = \{(0,0),(0,1),(k,0) | k \in \mathbb{N}, k \leq \bar{p}_2\} \end{aligned}$$

Now, suppose $\bar{p}_2 > 1$ (that is, $p_2 > p_1$). Then budget feasible bundles become

$$\begin{aligned} &\text{for } i \leq m \qquad\qquad B^i = \{(0,0),(1,0)\} \\ &\text{for } i > m \quad B^i = \{(0,0),(0,1),(1,0),\ldots,(\lfloor \bar{p}_2 \rfloor, 0)\}. \end{aligned}$$

No agents $j \leq m$ can afford to consume x_2, and market clearing implies $\sum_{i=1}^{2m+1} x_2^i = m+1$. So, if any $i > m$ chose demand other than $(0,1)$, demand for x_2 would fall short of supply. Thus, market clearing requires that all agents $i > m$ choose to consume bundle $(0,1)$. Thus, we must also have all agents $j \leq m$ choosing $(1,0)$. Hence, the CE allocation is:

$$x_i^* = \begin{cases} (1,0) & \text{if} \quad i \leq m \\ (0,1) & \text{if} \quad i > m \end{cases}$$

Suppose $\bar{p}_2 < 1$. Then, budget feasible bundles are

$$\begin{aligned} &\text{for } i \leq m \quad B^i = \{(0,0),(1,0),(0,1),\ldots,(0,\left\lfloor \tfrac{1}{\bar{p}_2} \right\rfloor)\} \\ &\text{for } i > m \qquad\qquad B^i = \{(0,0),(0,1)\}. \end{aligned}$$

Similar reasoning as above based on the market clearing condition implies that in equilibrium, all agents $i \leq m$ must choose $(1,0)$, and then all agents $j > m$ must choose $(0,1)$. Hence we again have that the equilibrium allocation is

$$x_i^* = \begin{cases} (1,0) & \text{if} \quad i \leq m \\ (0,1) & \text{if} \quad i > m \end{cases}$$

Suppose $\bar{p}_2 = 1$. Then, budget feasible bundles are

$$\begin{aligned} &\text{for } i \leq m \quad B^i = \{(0,0),(1,0),(0,1)\} \\ &\text{for } i > m \quad B^i = \{(0,0),(0,1),(1,0)\}. \end{aligned}$$

Then, the equilibrium allocation is any allocation such that m agents choose $(1,0)$ and $m+1$ agents choose $(0,1)$ regardless of whether they were endowed with one left shoe or one right shoe. Note, of course, that this includes

$$x_i^* = \begin{cases} (1,0) & \text{if} \quad i \leq m \\ (0,1) & \text{if} \quad i > m \end{cases}$$

□

1C: Is every CE allocation PE?

Given strictly positive prices, in Part B we showed that each agent consuming her own endowment is a competitive equilibrium allocation. However, this immediately violates our condition for pareto optimality in this environment. So the First Welfare Theorem does not apply.

Why does the First Welfare Theorem fail? One should note that local nonsatiation does not have meaning as it is usually defined when there are indivisibilities in the consumption set. Moreover, even though leontief preferences are locally nonsatiated when the consumption space is $\mathbb{R}^L_+$, it is clear from our example that agents do not need to be on the frontier of their budget sets in order to be maximizing utility. Hence, the step in the proof of the FWT that argues "because $u_i(x_i') \geq u_i(x_i)$, it must be that $p \cdot x_i' \geq p \cdot x_i$," need not hold. That is, you can leave agents just as well off by spending less of their budget and this can actually free up resources to allow for pareto improvements.

Question 2

Claim 5. *Suppose u_i is concave and strongly monotone for every i. $u(x) \in U^P$ if and only if x solves*

$$\textit{(P1)} \quad \max_{x \in A} \sum_{i=1}^{N} a_i u_i(x_i)$$

for some $a \in \mathbb{R}^L_+ / \{0\}$.

Proof. ($\Rightarrow$) First, note that concavity of u_i implies the set U is convex:
Let $u, u' \in U$ and $\alpha \in [0,1]$. Then, $\exists\, x, x' \in A$ such that $u_i \leq u_i(x_i)$ and $u_i' \leq u_i(x_i')$ for all i. Thus,

$$\alpha u_i + (1-\alpha) u_i' \leq \alpha u_i(x_i) + (1-\alpha) u_i(x_i') \leq u_i(\alpha x_i + (1-\alpha) x_i')$$

where the last inequality follows from concavity. Hence, as long as A is convex, that is, we require $\alpha x_i + (1-\alpha) x_i' \in A$ given $x, x' \in A, \alpha \in [0,1]$, U is convex.

Now, let $\tilde{u} \in U^P$. It is clear that $U \neq \emptyset$ and U is convex by above. Then, define the set $Z_{\tilde{u}}$ as

$$Z_{\tilde{u}} = \{u' \in \mathbb{R}^L | u' >> \tilde{u}\}.$$

Clearly, $Z_{\tilde{u}}$ is nonempty and convex and $U \cap Z_{\tilde{u}} = \emptyset$ (which follows from the definition of U^P). Thus, by the Minkowsky Separating Hyperplane Theorem, $\exists a^* \neq 0$ such that for all $\hat{u} \in Z_{\tilde{u}}, u \in U$ we have $a^* \cdot \hat{u} \geq a^* \cdot u$. For any $j = 1, \ldots, N$, we can rewrite the last inequality as

$$\sum_{\substack{i=1 \\ i \neq j}}^{N} a_i^* \left(\hat{u}_i - u_i\right) \geq -a_j^* \left(\hat{u}_j - u_j\right).$$

If for any j we have that $a_j^* < 0$, then for fixed $\hat{u}$, we can vary u by keeping u_i fixed for $i \neq j$ and making u_j arbitrarily negative until we violate the above condition. Thus, it must be that $a_i^* \geq 0$

for all i. Now, let ϵ^N denote the unit vector in $\mathbb{R}^N$ scaled by $\epsilon > 0$. Then, clearly $\tilde{u} + \epsilon^N \in Z_{\tilde{u}}$. Thus, for all $\epsilon > 0$,

$$a^* \cdot \left(\tilde{u} + \epsilon^N\right) \geq a^* \cdot u \text{ for all } u \in U.$$

Letting $\epsilon \to 0$ implies that $a^* \cdot \tilde{u} \geq a^* \cdot u$ for all $u \in U$.

Finally, since $\tilde{u} \in U^P$, we have $\tilde{u} \in U$. So there exists $\tilde{x} \in A$ such that $\tilde{u} \leq u(\tilde{x})$. Note that $\tilde{u} \in U^P$ implies that the previous inequality must hold with equality. Moreover, take arbitrary $x \in A$. Then we know that $u(x) \in U$. Thus, we in fact have that $a^* \cdot u(\tilde{x}) \geq a^* \cdot u(x)$ for all $x \in A$. So $\tilde{x}$ solves $(P1)$ with $a = a^*$.

($\Leftarrow$) Let x solve $(P1)$ for some $a \in \mathbb{R}^L_+/\{0\}$. Suppose $u(x) \notin U^P$. Then $\exists\, \tilde{x} \in A$ such that $u(\tilde{x}) > u(x)$. If the strict inequality occurs for an agent i with a strictly positive weight a_i in $(P1)$, then we immediately have that $a \cdot u(\tilde{x}) > a \cdot u(x)$, which is a contradiction. So it must be the case that for all j such that $a_i > 0$, we have $u_i(\tilde{x}_i) = u_i(x_i)$, and for some k with $a_k = 0$, we have $u_k(\tilde{x}_k) > u_k(x_k)$ (obviously, $\tilde{x}_k > 0$).

Let $j \in \{1, ..., L\}$ such that $\tilde{x}_{k,j} \neq 0$. Consider a new allocation where $\hat{x}_{k,j} = \tilde{x}_{k,j} - \epsilon$ where ϵ satisfies $\tilde{x}_{k,j} - \epsilon \geq 0$, and $\hat{x}_{k,l} = \tilde{x}_{k,l}$ for $l \neq j$. Then, fix $i \in \{1, \ldots, N\}$ such that $a_i > 0$. Let $\hat{x}_{i,j} = \tilde{x}_{i,j} + \epsilon$ and $\hat{x}_{i,l} = \tilde{x}_{i,l}$ for $l \neq j$. Let $\hat{x}_m = \tilde{x}_m$ for all $m \neq i, k$. Strong monotonicity implies that $u_i(\hat{x}_i) > u_i(\tilde{x}_i) = u_i(x_i)$, and we have that for all $m \neq i$ such that $a_m > 0$, $u_m(\hat{x}_m) = u_m(\tilde{x}_m) = u_m(x_m)$ Then $\hat{x}$ is feasible and $a \cdot u(\hat{x}) > a \cdot u(x)$, which contradicts our assumption that x solves $(P1)$. Thus, $u(x) \in U^P$. □

Question 3

Claim 6. *Suppose u_i is locally nonsatiated and continuous and that u_1 is strongly monotone. Then x' is Pareto Efficient if and only if x' solves*

$$\begin{aligned} &(P2) \qquad \max_{x \in A} u_1(x_1) \\ &\text{subject to} \\ &u_i(x_i) \geq \underline{u}_i, i = 2, \ldots, I \end{aligned}$$

for some $\underline{u}_i \in \mathbb{R}, i = 2, \ldots, I$.

Proof. ($\Rightarrow$) Suppose x' is Pareto Efficient. Then, for $i = 2, \ldots, I$, set $\underline{u}_i = u_i(x'_i)$. Then, by definition of Pareto Efficiency, there exists no feasible allocation $\hat{x}$ such that $u_1(\hat{x}_1) > u_1(x'_1)$ and $u_i(\hat{x}_i) \geq u_i(x'_i)$ for each $i = 2, \ldots, I$. In other words, for all feasible allocations $x \in A$ satisfying $u_i(x_i) \geq u_i(x'_i)$ for each $i = 2, \ldots, I$, it must be that $u_1(x'_1) \geq u_1(x_1)$. Thus, x' solves $(P2)$ given $\underline{u}_i = u_i(x'_i)$.

($\Leftarrow$) Suppose x' solves $(P2)$. Then, suppose for contradiction that x' is not Pareto Efficient. Then there exists a feasible allocation x such that $u(x) > u(x')$. Trivially, x satisfies the constraint of the programming problem. Since x' is optimal, though, it must be that $u_1(x_1) = u_1(x'_1)$. Then, for some $i \in \{2, \ldots, I\}$ we have $u_i(x_i) > u_i(x'_i) \geq \underline{u}_i$. Then continuity of u_i implies that there

exists $x_i'' \geq 0$ such that $u_i(x_i'') > \underline{u}_i$ and $x_i'' < x_i$. Hence, there exists feasible x'' with $x_1'' > x_1$ satisfying $u_i(x_i'') \geq \underline{u}_i$ and $u_1(x_1'') > u_1(x_1) = u_1(x_1')$ contradicting optimality of x'. Thus, x' is Pareto Efficent. □

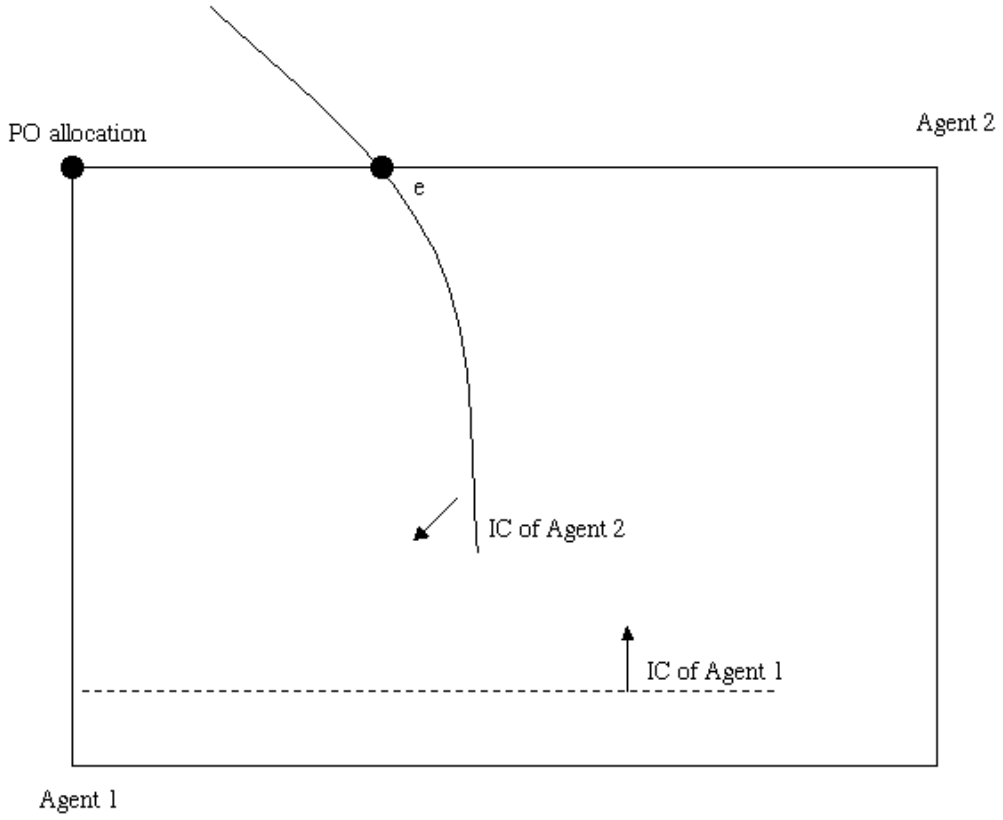
Question 4

4A: Core ⊂ P.O.

Claim 7. *Suppose consumers' preferences are strongly monotone. If $x \in A$ satisfies the Core property, then x is Pareto Efficent.*

Proof. Let $x \in Core$, and suppose for contradiction that x is not Pareto Efficient. Then there exists $x' \in A$ with $u(x') > u(x)$ (notice here that we know that $x' > 0$, but not that $x' >> 0$, which is why we should need strong monotonicity for this proof). Let $j \in \{1, \ldots, N\}$ such that $u_j(x_j') > u_j(x_j)$ and let $k \in \{1, \ldots, L\}$ such that $x_{j,k}' > 0$ Then by continuity of u_j there exists $\epsilon > 0$ such that $u_j(x_j'') > u_j(x_j)$ where $x_{j,k}'' = x_{j,k}' - \epsilon$ and $x_{j,l}'' = x_{j,l}'$ for $l \neq k$. Then, for all $i \neq j$, let $x_{i,k}'' = x_{i,k}' + \frac{\epsilon}{N-1}$ and $x_{i,l} = x_{i,l}'$ for $l \neq k$ so that by strong monotonicity, $u_i(x_i'') > u_i(x_i') \geq u_i(x_i)$. But then x'' improves upon x where the coalition consists of all agents in the economy. This contradicts $x \in Core$. Hence, it must be that x is Pareto Efficient. □

Comment 8. To see why strong monotonicity is required, consider the following edgeworth box:



Notice, Agent 1's preferences are not strongly monotone. Given the endowment e, we see immediately that e is in the core of this economy. There are three potential coalitions in this economy: $S \in \{(1), (2), (1, 2)\}$. It is clear that there is no feasible allocation that makes Agent 1 strictly better off. Thus, $S \in \{(1), (1, 2)\}$ cannot block e. Moreover, Agent 2 cannot be made strictly better off using only her endowment. So $S = (2)$ cannot block. Thus, there is no coalition that can strictly improve all members in the coalition (or $\nexists S$ that can block). But obviously, e is pareto dominated by the corner of the edgeworth box, so labeled pareto efficient.

4B: CE $\subset$ Core

Claim 9. *Suppose consumer preferences are strongly monotone. If $(x, p) \in A \times \mathbb{R}^L_+$ is a CE for E^{pure}, then x is in the Core.*

Proof. Let (x, p) be a CE. Suppose that x does not satisfy the Core property. Then there exists an allocation $x' \in A$ and a coalition $S \subset I$ such that $u_i(x'_i) > u_i(x_i)$ for each $i \in S$ and

$$\sum_{i \in S} x'_i \leq \sum_{i \in S} e_i.$$

Then, since $p > 0$, we have

$$p \cdot \sum_{i \in S} e_i \geq p \cdot \sum_{i \in S} x'_i.$$

For each $i \in S$, it must be that $p \cdot x'_i > p \cdot x_i = p \cdot e_i$ by local nonsatiation of preferences (which is implied by strong monotonicity). Hence, by summing up these inequalities, we have

$$p \cdot \sum_{i \in S} x'_i > \sum_{i \in S} p \cdot x_i = \sum_{i \in S} p \cdot e_i.$$

Combining inequalities we have

$$p \cdot \sum_{i \in S} e_i \geq p \cdot \sum_{i \in S} x'_i > p \cdot \sum_{i \in S} x_i = p \cdot \sum_{i \in S} e_i$$

which is a contradiction. Thus, x must satisfy the core property. □

Economics 8102: Problem Set II

Ichiro Obara

November 6, 2007
(corrected on November 12)
Due November 15

1. We know that, under appropriate assumptions, a feasible allocation $x >> 0$ is Pareto efficient if and only if it maximizes the weighted sum of utilities for some weight vector $(a_1, ..., a_N) \in \Re^N_{++}$. Answer the following questions regarding this result.

   (a) This result can be extended to the case with consumption externality, i.e. $u_i : X \to \Re$ (not $u_i : X_i \to \Re$) under some appropriate assumptions on u_i. List such assumptions, justify your answer, and state the theorem. (Note: A variety of assumptions would do this, but try to make sensible assumptions. For example, you may like to allow both positive and negative externality).

   (b) Consider the following two-good pure exchange economy: $i = 1, 2$, $u_1(x_1) = \ln x_{1,1} + \ln x_{1,2} - x_{2,1}$, $u_2(x_2) = \ln x_{2,1} + \ln x_{2,2}$, and $e_1 = e_2 = (1/2, 1/2)$. Characterize the (strictly positive) Pareto efficient allocations in this economy and show that the competitive equilibrium (which is appropriately defined with the presence of externality) is not Pareto efficient. You may use the result from (a)

2. (M.W.G. 16.D.4.) Consider a two-good pure exchange economy with N identical consumers and $e_i = e >> 0$. Preferences are strongly monotone, but not necessarily convex. Show that either (1) $(x_1^*, ..., x_N^*) = (e, ..., e)$ is a competitive equilibrium allocation for some price $p^* >> 0$ or (2) $(e, ..., e)$ is not Pareto efficient when N is large enough (Hint: Remember the proof of SFTWE. You may convexify V).

3. Consider the following utility maximization problem given $p_n >> 0$ and $e_i = (1, ..., 1)$:

$$\max_{x_i \in \Re_+^L} \sum_{l=1}^{L} \sqrt{x_{i,l}}$$
$$\text{s.t. } p_n \cdot x_i \leq p_n \cdot e_i$$

   (a) Suppose that $L = 2$. Consider a sequence of prices $\{p_n\}_{n=1}^{\infty}$ that converges to some $p^* (\neq 0) \in \Re_+^L$ such that $p_1^* = 0$. Either prove that $\lim_{n\to\infty} x_{i,1}(p_n, e_i) = \infty$ holds for any such sequence or find a counterexample where $\lim_{n\to\infty} x_{i,1}(p_n, e_i) < \infty$.

   (b) Suppose that $L = 3$. Answer the same question as (a).

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Solutions to Problem Set 2
November 20, 2007

Question 1

Part A

Claim 1. *Assume that each u_i is concave and $(u_i(\cdot))_{i\in I}$ satisfy the following property:*

- *for any strictly positive feasible allocation (ie. $x \in A, x >> 0$) and any agent $i \in I$, there exists $x' \in A$ such that for all $j \neq i, u_j(x'_j) > u_j(x_j)$.*

Then, a strictly positive feasible allocation is pareto efficient if and only if it maximizes the weighted sum of utilities for some weight vector $(a_1, \ldots, a_N) \in \mathbb{R}^N_{++}$.

Notice that this property is rather general and is a limited restriction on the set of agents preferences (obviously, environments without externalities satisfy this property).

Proof. ($\Leftarrow$). Fix $a \in \mathbb{R}^N_{++}$. Suppose $x \in \arg\max_{x\in A} \sum_{i=1}^N a_i u_i(x)$. Suppose for contradiction that x is not pareto efficient. Then, there exists a feasible x' such that $u(x') > u(x)$. As $a >> 0$, we immediately have that $a \cdot u(x') > a \cdot u(x)$, a contradiction. Thus, x is pareto efficient.

($\Rightarrow$). Suppose $x >> 0$ is pareto efficient. Then, as usual, define $U = \{u \in \mathbb{R} | \exists x \in A \text{ s.t. } u(x) \geq u\}$.

Step 1. $u(x) \in \partial U$. Suppose $u(x) \in intU$. Then, $\exists \epsilon > 0$ s.t. $N_\epsilon(u(x)) \in U$. Then, $\exists u' \in U$ s.t. $u > u(x)$, and thus $\exists x' \in A$ s.t. $u(x') \geq u > u(x)$ which contradicts pareto optimality of x. Hence, since $u(x) \in U$ and $u(x) \notin intU$, it must be that $u(x) \in \partial U$.

Step 2. U is convex. Let $u, u' \in U$ and $\alpha \in [0, 1]$. Then, $\exists\ x, x' \in A$ such that $u_i \leq u_i(x)$ and $u'_i \leq u_i(x')$ for all i. Thus,

$$\alpha u_i + (1-\alpha)u'_i \leq \alpha u_i(x) + (1-\alpha)u_i(x') \leq u_i(\alpha x + (1-\alpha)x')$$

where the last inequality follows from concavity of each u_i. Hence, as long as A is convex, that is, we require $\alpha x + (1-\alpha)x' \in A$ given $x, x' \in A, \alpha \in [0, 1]$, U is convex.

Step 3. Application of Supporting Hyperplane Theorem. Since U is convex and nonempty, we may apply the Supporting Hyperplane theorem. That is, $\exists a^* \neq 0, a^* \in \mathbb{R}^N$ s.t. $a^* \cdot u(x) \geq a^* \cdot u'$ for all $u' \in U$.

Step 4. $a^* > 0$. For any $j = 1, \ldots, N$, we can rewrite $a^* \cdot u(x) \geq a^* \cdot u'$ as

$$\sum_{\substack{i=1 \\ i \neq j}}^{N} a_i^* \left(u_i(x) - u_i'\right) \geq -a_j^* \left(u_j(x) - u_j'\right).$$

If for any j we have that $a_j^* < 0$, then we can vary u' by keeping u_i' fixed for $i \neq j$ and making u_j' arbitrarily negative until we violate the above condition. Thus, it must be that $a_i^* \geq 0$ for all i.

Step 5. $a^* >> 0$. Suppose for any $j, a_j^* = 0$. Then we can rewrite $a^* \cdot u(x) \geq a^* \cdot u'$ as

$$\sum_{\substack{i=1 \\ i \neq j}}^{N} a_i^* \left(u_i(x) - u_i'\right) \geq 0.$$

By assumption, $\exists x' \in A$ s.t. for all $i \neq j, u_i(x') > u_i(x)$ and obviously $u(x') \in U$. Hence, the above inequality will be violated. Thus, it must be that $a_i^* > 0$ for each i.

Step 6. x maximizes the weighted sum of utilities for $a = a^* \in \mathbb{R}_{++}^N$. Notice, for any $x' \in A, u(x') \in U$. Hence, $a \cdot u(x) \geq a \cdot u(x')$ for all $x' \in A$.

□

Part B: Example

To be updated. Basically, demonstrate that the previous claim is applicable, and then use K-T conditions to show that CE is not PO.

Question 2

Consider a two good pure exchange economy with N identical consumers each with strongly monotone preferences and $e_i = e >> 0$.

Claim 2. *$(x_1^*, \ldots, x_N^*) = (e, \ldots, e)$ is a competitive equilibirium for some price $p^* >> 0$ (for each N) if and only if $\forall N_0 \in \mathbb{N}$, $\exists\, n \geq N_0$ s.t. $(x_1^*, \ldots, x_n^*) = (e, \ldots, e)$ is pareto efficient.*

In what follows, I prove first that if $(e, \ldots, e)$ is a CE, then $\forall N_0 \in \mathbb{N}$, $\exists\, n \geq N_0$ s.t. $(x_1^*, \ldots, x_n^*) = (e, \ldots, e)$ is pareto efficient. And for the reverse direction, I prove the contrapositive. That is, if $(e, \ldots, e)$ is not a CE, then $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N, (e, \ldots, e)$ is not pareto efficient.

Proof. Let

$$\begin{aligned} V &= \left\{x \in \mathbb{R}^2_+ | x \succeq e\right\} \\ CoV &= \left\{x \in \mathbb{R}^2_+ | x = \sum_{m=1}^{M} \alpha_m x_m \text{ for some } M, (\alpha_m)_{m=1}^{M} \text{ with } \alpha_i \geq 0, \sum_{m=1}^{M} \alpha_m = 1, x_m \in V\right\}. \end{aligned}$$

Let $\bar{e}$ denote the unit vector in $\mathbb{R}^2_+ (\bar{e} = (1,1))$ and fix $\lambda > 0$. For any λ, either $e - \lambda\bar{e} \in CoV$ or $e - \lambda\bar{e} \notin CoV$.

($\Rightarrow$). Suppose for every $\lambda > 0$, $e - \lambda\bar{e} \notin CoV$. Then we'll show that there exists a $p \in \mathbb{R}^2_+$ such that $(e, \ldots, e)$ is a C.E. (which in turn implies that $(e, \ldots, e)$ is pareto efficient for each N by the First Welfare theorem).

Since $e - \lambda\bar{e} \notin CoV$ by assumption and CoV is convex and nonempty, we can separate $e - \lambda\bar{e}$ from CoV by a separating hyperplane theorem. Hence, for each $\lambda > 0$, $\exists p_\lambda \in \mathbb{R}^2 (p \not\equiv 0)$ such that for each $x \in CoV$,

$$p_\lambda \cdot (e - \lambda\bar{e}) \leq p_\lambda \cdot x.$$

Strongly monotone preferences yields the (by now) usual argument that $p_\lambda \in \mathbb{R}^2_+$. Hence, $p_{\lambda,1} + p_{\lambda,2} = p_\lambda \cdot \bar{e} > 0$, so we can normalize the above inequality as

$$\hat{p}_\lambda \cdot (e - \lambda\bar{e}) \leq \hat{p}_\lambda \cdot x \text{ for } \hat{p}_{\lambda,i} = \frac{p_{\lambda,i}}{p_\lambda \cdot \bar{e}},\ i = 1, 2.$$

Now, for each λ, $\hat{p}_\lambda$ lies in a compact set (the simplex in $\mathbb{R}^2$), and hence as $\lambda \to 0$, $\hat{p}_\lambda$ has a convergent subsequence. That is, $\hat{p}_\lambda \to p \in \mathbb{R}^2_+$ with $p_1 + p_2 = 1$. Thus, taking limits in the previous inequality as $\lambda \to 0$ yields

$$p \cdot e \leq p \cdot x \text{ for all } x \in CoV.$$

Hence, e is cost minimizing for each consumer (that is e is the solution to $\min_{x \succeq e} p \cdot x$). Since $e >> 0$, we know that cost minimization implies utility maximization (with continuity). Thus, at p, $(e, \ldots, e)$ is a competitive equilibrium allocation. Moreover, $(e, \ldots, e)$ is Pareto Efficient by the first welfare theorem (valid by local nonsatiation of preferences implied by strong monotonicity of preferences) for each N.

($\Leftarrow$). Now, suppose for every $p \in \mathbb{R}^2_+$, $(e, \ldots, e)$ is not a competitive equilibrium allocation. Then by first part of this proof, we know that for some $\lambda > 0$, $e - \lambda\bar{e} \in CoV$. Hence,

$$e - \lambda\bar{e} = \sum_{m=1}^{M} \alpha_m x_m$$

for some $M \in \mathbb{N}, (\alpha_m)_{m=1}^{M}$ with $\alpha_m \geq 0, \sum_{m=1}^{M} \alpha_m = 1, x_m \in V$. So we can write

$$e = \sum_{m=1}^{M} \alpha_m (x_m + \lambda\bar{e}) = \sum_{m=1}^{M} \alpha_m x'_m$$

where $x'_m = x_m + \lambda\bar{e}$. Notice, $x'_m \succ x_m$ by strong monotonicity, and thus by transitivity, we have $x'_m \succ e$. For N sufficiently large, there exists integers N_m with $\sum_{m=1}^{M} N_m = N$ such that $\frac{N_m}{N}$

is arbitrarily close to α_m. Hence, as $N \to \infty$, we can choose N_m s.t. $\alpha_m \frac{N}{N_m} \to 1$ for each m. Hence, by continuity, $\alpha_m \frac{N}{N_m} x'_m \succ e$. Moreover, assigning $\alpha_m \frac{N}{N_m} x'_m$ to each of the N_m agents in the economy is feasible because

$$\sum_{m=1}^{M} N_m \alpha_m \frac{N}{N_m} x'_m = N \sum_{m=1}^{M} \alpha_m x'_m = Ne.$$

Hence, if $(e, \ldots, e)$ is not a competitive equilibrium allocation for any $p \in \mathbb{R}^2_+$, then for sufficiently large N, it is not a pareto optimal allocation. □

Question 3

Consider the following utility maximization problem given $(p_n, e) >> 0$ and $e = (1, \ldots, 1)$:

$$\max_{x \in \mathbb{R}^L_+} \sum_{i=1}^{L} \sqrt{x_l}$$
$$\text{s.t. } p_n \cdot x \leq p_n \cdot e$$

3A: $L = 2$

Claim 3. *Suppose $L = 2$. Then for any sequence of prices $\{p_n\}_{n=1}^{\infty}$ with $p_n \to p^* \in \mathbb{R}^L_+ (p^* \neq 0)$ such that $p_1^* = 0$, $\lim x_1(p_n, e) = \infty$.*

Proof. Since the objective function is concave and the constraint set is linear, first order conditions are necessary and sufficient. (Find the FOCs, divide them by each other to get rid of the multiplier, plug them into the budget constraint, and you should find the following):

$$x_1(p_n, e) = \frac{p_{n,1} + p_{n,2}}{p_{n,1} + \frac{p_{n,1}^2}{p_{n,2}}} \underset{p_{n,1} \to 0}{\to} \infty$$

since we must have $p_{n,2} \to p_2^* < \infty$. □

3B: $L = 3$

Claim 4. *Suppose $L = 3$. Then $\exists$ sequences of prices $\{p_n\}_{n=1}^{\infty}$ with $p_n \to p^* \in \mathbb{R}^L_+ (p^* \neq 0), p_1^* = 0$ such that $\lim x_1(p_n, e) \neq \infty$.*

Proof. Again, the FOCs are sufficient, and you should be able to solve for $x_1(p_n, e)$:

$$x_1(p_n, e) = \frac{p_{n,2}p_{n,3}(p_{n,1} + p_{n,2} + p_{n,3})}{p_{n,1}(p_{n,2}p_{n,3} + p_{n,1}p_{n,3} + p_{n,1}p_{n,2})}.$$

Consider the sequence $p_n = (\frac{1}{n}, \frac{1}{n^2}, 1)$. Then

$$\begin{aligned} x_1(p_n, e) &= \frac{\frac{1}{n^2}(\frac{1}{n} + \frac{1}{n^2} + 1)}{\frac{1}{n}(\frac{1}{n^2} + \frac{1}{n} + \frac{1}{n^3})} \\ &= \frac{\frac{1}{n}(\frac{1}{n} + \frac{1}{n^2} + 1)}{\frac{1}{n^2} + \frac{1}{n} + \frac{1}{n^3}} \\ &= 1 \end{aligned}$$

Hence, $\lim p_{n,1} = 0$, but $\lim x_1(p_n, e) = 1$. □

Comment 5. It is worth considering an insight from this problem. That is, the standard boundary condition we use to prove existence of competitive equilibria that $p_n \to p \in \partial\Delta$ implies $||Z(p_n)|| \to \infty$ is **not** equivalent to the statement, "if the price of a commoditiy goes to 0, then excess demand for that commodity goes to ∞.

Economics 8102: Problem Set III

Ichiro Obara

November 14, 2007
(corrected on November 21)
Due November 27

1. Let $X \subset \Re^n$ be a convex set and $u : X \to \Re$ be a differentiable, concave function.

   (a) Show that $u(x') > u(x)$ implies $u'(x)(x' - x) > 0$ (pseudo concavity).[1]

   (b) How is pseudo concavity related to strict quasi-concavity? Prove or disprove the following two statements:
   (1) If $u : X \to \Re$ is pseudo concave, then it is also strictly quasi-concave.
   (2) If $u : X \to \Re$ is strictly quasi-concave, then it is also pseudo concave.

2. (MWG, 17.D.1.) Consider a pure exchange economy with two goods and two consumers $i = 1, 2$. The preference of consumers is given by

$$\begin{aligned} u_1(x_{1,1}, x_{2,1}) &= (2x_{1,1}^\rho + x_{2,1}^\rho)^{\frac{1}{\rho}} \\ u_2(x_{1,2}, x_{2,2}) &= (x_{1,2}^\rho + 2x_{2,2}^\rho)^{\frac{1}{\rho}} \end{aligned}$$

   where $\rho = -4$. The endowments are $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Compute the excess demand functions of this economy and verify that there are multiple equilibria.

[1] This is a weaker condition for the sufficiency of K-T conditions (cf. MWG M.K.3).

3. Show that $P = \{p \in \triangle | p >> 0,\ \max_{l=1,...L} \{z_l(p)\} \leq \overline{g}\}$ is nonempty and compact for a large enough $\overline{g} > 0$ (Assume that z satisfies (I) - (V)).

4. Let f, $g : \Re^n_+ \to \Re$ be continuous functions and $X \subset \Re^n_+$ be a convex set. Also assume that f is quasi concave and does not have a local maxima in $\Re^n_+$. Show that if (1) $x^* \in \Re^n_{++}$ solves $\max_x f(x)$ s.t. $x \in X$ and $g(x) \geq 0$ and (2) $g(x^*) > 0$, then $x^* \in \Re^n_+$ solves $\max_x f(x)$ s.t. $x \in X$. Discuss why both quasi-concavity and no local maxima is needed for this result. (what would happen if one of them is dropped?).[2]

5. Consider a pure exchange economy with externality. Consumer i's preference is given by a continous function $u_i : X \to \Re$. Also assume that, given any x_{-i}, u_i is locally nonsatiated and strictly quasi-concave in $x_i \in X_i \left(= \Re^L_+\right)$. Consumer i solves the following problem given x_{-i} :

$$\max_{x_i \in X_i} u_i(x)$$
$$s.t.\ p \cdot x_i \leq p \cdot e_i.$$

Then i's demand depends on $p \in \Re^L_{++}$ and x_{-i}. Thus the excess demand function $z : \Re^L_{++} \times X \to \Re^L$ is given by:

$$z(p, x) = \sum_{i=1}^{N} x_i(p, x_{-i}) - \sum_{i=1}^{N} e_i$$

(x^*, p^*) is a CE if (1) $x^*_i \in X_i$ solves the above problem given x^*_{-i} for $i = 1, ..., N$ and $z(p^*, x^*) \leq 0$ ($z_l(p^*, x^*) = 0$ if $p^*_l > 0$).

(a) Show that $z(p, x)$ is continuous in $\Re^L_{++} \times X$.

(b) Assume that $e_i >> 0$ for $i = 1, ..., N$. Prove that there exists a competitive equilibrium in this economy with externality. (Hint. Look for a fixed point of (x, p), not just p).

[2]Alternative assumption on f would be (1) f is stirictly quasi-concave or (2) f is pseudo-concave. (1) was discussed in the class. You can find a discussion on (2) in p. 963 of MWG.

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Solutions to Problem Set 3
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Question 1

Part A: u Concave implies u pseudo-concave

Claim 1. *Let $X \subset \mathbb{R}^n$ be convex and $u : X \to \mathbb{R}$ be a differentiable and concave function. If $u(x') > u(x)$, then $u'(x)(x' - x) > 0$.*

Proof. Concavity of $u(\cdot)$ implies that for all $\alpha \in [0, 1]$, and all $x, x' \in X$, we have

$$u(\alpha x' + (1 - \alpha)x) \geq \alpha u(x') + (1 - \alpha)u(x).$$

We can rewrite this inequality as

$$(x' - x)\frac{u(x + \alpha(x' - x)) - u(x)}{\alpha(x' - x)} \geq u(x') - u(x).$$

Since $u(\cdot)$ is differentiable, we have that

$$\lim_{\alpha \to 0} \frac{u(x + \alpha(x' - x)) - u(x)}{\alpha(x' - x)} = u'(x).$$

Hence,

$$u'(x)(x' - x) \geq u(x') - u(x).$$

That is, if $u(x') > u(x)$, then $u'(x)(x' - x) > 0$. □

Part B: u Pseudo concave implies u explictly quasiconcave, but not conversely

Claim 2. *If $u : X \to \mathbb{R}$ is pseudo concave, then it is also "explicitly" quasi concave. The converse is not true.*

I modify the definition of strict quasiconcavity to make for a somewhat more interesting problem.

Definition 3. *A function f is said to be* explicitly quasiconcave *if it is quasiconcave and for all $x, x' \in X, x \neq x', \alpha \in (0, 1)$,*

$$f(x) > f(x') \Rightarrow f(\alpha x + (1 - \alpha)x') > f(x').$$

Notice that explicit quasiconcavity is a weaker assumption than strict quasiconcavity. With the standard definition of strict quasi-concavity, you can derive a counterexample using a linear (or constant) function.

Proof. ($\Rightarrow$)Let u be pseudo concave and suppose for contradiction that u is not explicitly quasi-concave. Then, there exists $x, x' \in X$ and $\alpha \in (0,1)$ with $x \neq x'$ such that $u(x') > u(x)$ and

$$u(\alpha x' + (1-\alpha)x) \leq u(x) = \min\{u(x), u(x')\}.$$

Hence, there exists $\bar{x} \in L$, where

$$L = \{x_\alpha | x_\alpha = \alpha x' + (1-\alpha)x, \alpha \in (0,1)\}$$

such that

$$u(\bar{x}) = \min_{x_\alpha \in L \cup \{x, x'\}} u(x_\alpha).$$

Now, define the function $f : [0,1] \to \mathbb{R}$ by

$$f(\alpha) = u(\alpha x' + (1-\alpha)x).$$

Then, f attains its minimum at $\bar{\alpha}$ where

$$\bar{x} = \bar{\alpha} x' + (1-\bar{\alpha})x$$

and $\alpha \in (0,1)$. That is, $f(\bar{\alpha}) = u(\bar{x})$. Then, differentiating f with respect to α, we have that

$$0 = \frac{df(\alpha)}{d\alpha} = u'(\bar{x})(x' - x)$$

since f is differentiable and the FOC is necessary. Now, observe that

$$x' - \bar{x} = x' - \alpha x' - (1-\alpha)x = (1-\alpha)(x' - x).$$

Hence, we have that

$$0 = \frac{df(\alpha)}{d\alpha} = u'(\bar{x})(x' - x) = u'(\bar{x})(x' - \bar{x})(1-\alpha)^{-1}.$$

Since $\alpha < 1$, $u'(\bar{x})(x' - \bar{x}) = 0$. Hence, by pseudo concavity of u (using the contrapositive of the definition), we must have that $u(x') \leq u(\bar{x})$. Moreover, by assumption, we have that $u(x) < u(x')$, and thus $u(x) < u(\bar{x})$, which contradicts the fact that $\bar{x}$ minimizes $u(\cdot)$ over $L \cup \{x, x'\}$. Notice, this is precisely where regular strict quasiconcavity cannot be guaranteed (we need $u(x) < u(x')$ to be strict inequality in order to get a contradiction). Thus, for all $x, x' \in X, x \neq x', \alpha \in (0,1)$

$$u(x') > u(x) \Rightarrow u(\alpha x' + (1-\alpha)x) > u(x).$$

($\Leftarrow$) It is straightforward to check that $u(x) = x^3$ is strictly quasiconcave (in fact, it is also explicitly quasiconcave), but not pseudo concave. $\square$

Question 2

You should be able to set up each agent's maximixation problem and find the following from the (sufficient) FOCs (subscripts denote agents, and instead of goods 1 and 2, I use x and y):

$$x_1^5 = 2\frac{p_y}{p_x}y_1^5, \qquad y_1 = \frac{p_x}{p_y}(1 - x_1)$$
$$y_2^5 = 2\frac{p_x}{p_y}x_2^5, \qquad x_2 = \frac{p_y}{p_x}(1 - y_2)$$

These equations let you solve for $Z(p)$. Normalize prices to the simplex so that $(p_x + p_y = 1)$. Then, $\frac{p_x}{p_y} = \frac{p_x}{1-p_x} \equiv a$ (for $p >> 0$). Then, after substituting out y_1, the first equation becomes

$$x_1^5 = \frac{2}{a}\left[a * (1 - x_1)\right]^5$$

which implies

$$x_1 = \frac{(2a^4)^{\frac{1}{5}}}{1 + (2a^4)^{\frac{1}{5}}}$$

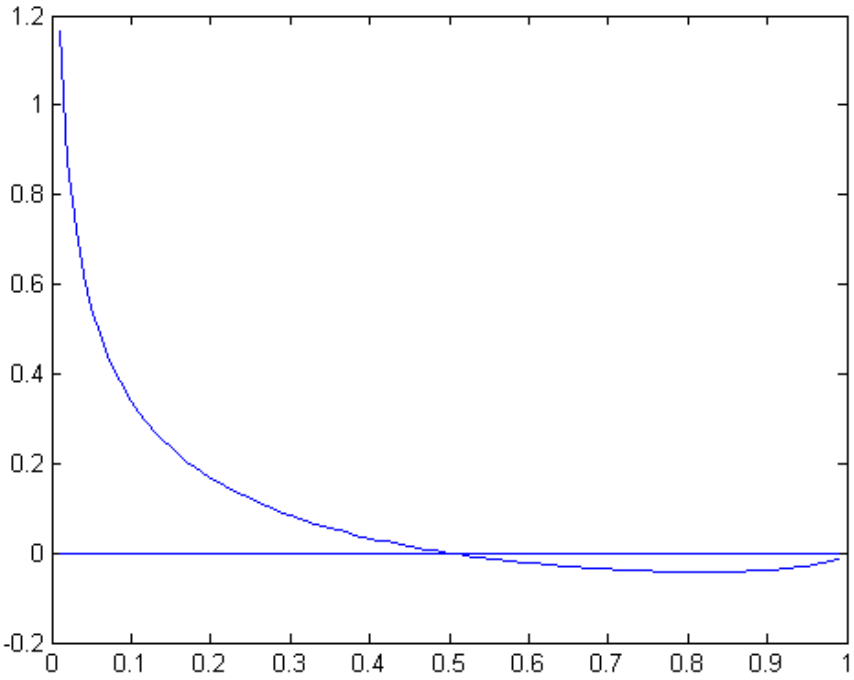
Similarly, you can find that

$$x_2 = a^{-1}\left(1 - \frac{(\frac{2}{a^4})^{\frac{1}{5}}}{1 + (\frac{2}{a^4})^{\frac{1}{5}}}\right)$$

Hence,

$$z_1(p) = \frac{(2a^4)^{\frac{1}{5}}}{1 + (2a^4)^{\frac{1}{5}}} + a^{-1}\left(1 - \frac{(\frac{2}{a^4})^{\frac{1}{5}}}{1 + (\frac{2}{a^4})^{\frac{1}{5}}}\right) - 1$$

Local nonsatiation of the utility functions imply, by Walras' Law, that the second market clearing condition is redundant. Obviously, if $a = 1$ (or $p_1 = \frac{1}{2}$), then $z_1(p) = 0$. When I plot this in matlab, I find one equilibria. In the plot below, p_x is plotted on the x-axis, and $z_1(p)$ on the y-axis. Interestingly, if you change the value inside the agent's utility function from 2 to 20, (e.g. $u_1(x, y) = (20x^4 + y^4)^{(1/4)}$, then you would find multiple equilibria (3 in fact). This is the second plot (I have truncated the space below $p_1 = .1$ and above $p_1 = .9$ to make the picture more clear).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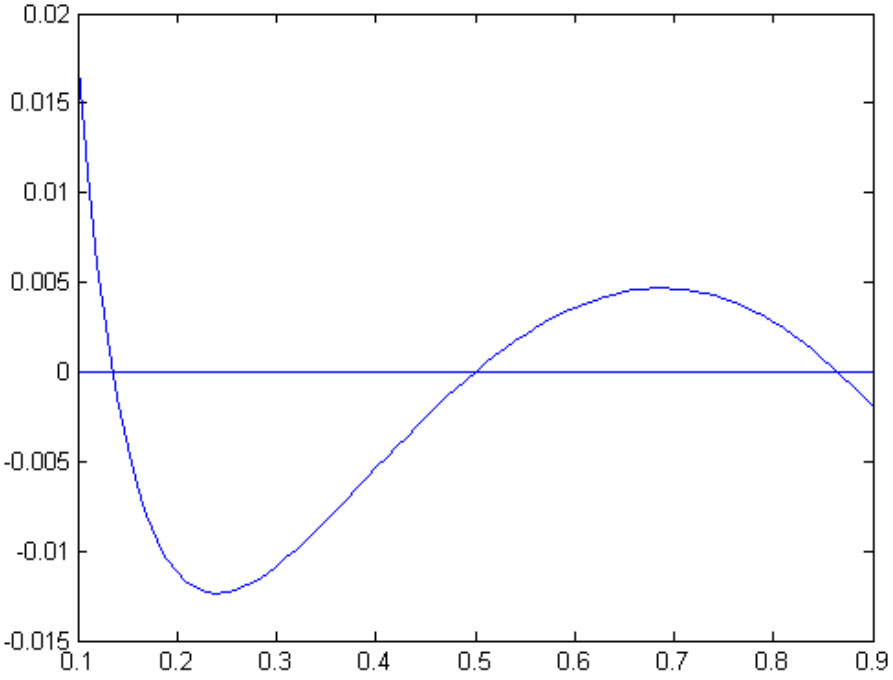
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Question 3

Claim 4. *The set* $P = \{p \in \Delta | p >> 0, \max_{l=1...,L}\{z_l(p)\} \leq \bar{g}\}$ *is nonempty and compact.*

Proof. Let $p \in \mathbb{R}^L_{++}$. Then, since price are strictly positive, agents' budget sets are bounded. Hence, demand by each consumer must bounded above. Thus, $\max_{l=1...,L}\{z_l(p)\}$ must be finite. Normalizing prices to the simplex and using the fact that z is homogenous of degree 0 implies $P \neq \emptyset$.

We now prove P is compact. By Heine-Borel, it suffices to show that P is closed and bounded. Boundedness is obvious as $P \subset \Delta$. To show closedness, take an arbitrary sequence $p^n \to p^*$ with $p^n \in P$ for all n. Then we need to show that $p^* \in P$. Since Δ is compact, we know that $p^* \in \Delta$. Moreover, $p^* \in int\Delta$. Suppose not. That is, suppose that for some l, $p_l = 0$. Then, $p^n \to p^* \in \partial\Delta$. Under assumption V, the boundary condition, this implies that $\max_{l=1...,L}\{z_l(p^n)\} \to \infty$, which contradicts $p^n \in P$ for all n for sufficiently large $\bar{g}$. Hence, we must have $p >> 0$. Given $p >> 0$, since z is continuous, $\max_{l=1...,L}\{z_l(p)\}$ is also continuous in p. Thus, since $\max_{l=1...,L}\{z_l(p^n)\} \leq \bar{g}$, continuity implies that it must be the case that $\max_{l=1...,L}\{z_l(p^*)\} \leq \bar{g}$. So P is closed, which implies that P is compact. □

Question 4

Claim 5. *Suppose* $f, g : \mathbb{R}^n_+ \to \mathbb{R}$ *are continuous, and* $X \subset \mathbb{R}^n_+$ *is convex. Also assume that* f *is quasiconcave and does not have a local maxima in* $\mathbb{R}^n_+$*. If* $x^* \in \mathbb{R}^n_{++}$ *solves* $\max_x f(x)$ *subject to* $x \in X, g(x) \geq 0$*, and* $g(x^*) > 0$*, then* x^* *solves* $\max_x f(x)$ *subject to* $x \in X$*.*

Proof. First, fix $\epsilon > 0$ s.t. for all $x \in N_\epsilon(x^*), g(x) > 0$ and $x >> 0$ (the last two inequalities are ok by continutity of g and $x^* >> 0$). Then, define $Z = \{z \in \mathbb{R}^n_+ | f(z) > f(x^*)\}$, which is convex by quasiconcavity of f and nonempty by the "no local max" assumption. Clearly, $\hat{X} = N_\epsilon(x^*) \cap X$ is also convex. Moreover, these two convex sets (Z and $\hat{X}$)are nonempty and disjoint (disjointness follows from optimality of x^*. Hence, we may apply a separating hyperplane theorem. That is $\exists\, a \neq 0, a \in \mathbb{R}^n$ such that for all $z \in Z$ and $x \in \hat{X}, a \cdot z \geq a \cdot x$.

Now, suppose for some $z \in Z$, and some $x \in \hat{X}$, $a \cdot z = a \cdot x$. Then, $f(z) > f(x)$. Since $x >> 0$, there exists some $x' >> 0$ such that $a \cdot x > a \cdot x'$. Now, consider the convex combination of z and x'. By continuity of f, for α sufficiently close to 1, we know that $f(\alpha z + (1-\alpha)x') > f(x^*)$. Hence, by the separating hyperplane, we have that $a \cdot (\alpha z + (1-\alpha)x') \geq a \cdot x$ for all $x \in \hat{X}$. But, we also have that

$$\alpha a \cdot z + (1-\alpha) a \cdot x' = \alpha a \cdot x + (1-\alpha) a \cdot x' < a \cdot x$$

which is a direct contradiction to the separating hyperplane result. Thus, for all $z \in Z, x \in \hat{X}$, we in fact have that $a \cdot z > a \cdot x$. [1]

[1] Alternatively, assuming that $intX$ is nonempty would also ensure the strict separation of these two sets. To see this, suppose again that for some $z \in Z$, and some $x \in \hat{X}$, $a \cdot z = a \cdot x$. So we have $f(z) > f(x)$. Then, consider

Now, suppose there exists $x'' \in X$ such that $f(x'') > f(x)$. Then, $x'' \in Z$ so that $a \cdot x'' > a \cdot x^*$. Consider a convex combination $x_\lambda = \lambda x'' + (1-\lambda)x^*$. For all $\lambda > 0$, we must have that $a \cdot x_\lambda > a \cdot x^*$, but for λ sufficiently close to 0, we know that $x_\lambda \in \hat{X}$, which implies a contradiction since it must be that $a \cdot x^* \geq a \cdot x$ for all $x \in \hat{X}$ since $x^* \in \partial X$.[2]

□

Question 5

Part A: Continuity of $z(p, x)$

Claim 6. *$z(p, x)$ is continuous in $\mathbb{R}^L_{++} \times X$.*

Proof. First, define $x_i(p, x_{-i}, e_i)$ as the solution to

$$\begin{gathered} max_{x_i \in X_i} u(x_i, x_{-i}) \\ \text{subject to} \\ p \cdot x_i \leq p \cdot e_i \\ x_{-i} \text{ given} \end{gathered}$$

As usual, the Budget Correspondence is nonempty, compact, and continuous [3] (for $p >> 0$), and u is a continuous function. So, by the Theorem of the Maximum, x_i is upper hemicontinous (in p and x_{-i}) and nonempty. Moreover, strict quasiconcavity of u ensures that x_i is single-valued and, therefore, a continuous function.

Then, as z is the sum of continuous functions, it is continuous. That is, take an arbitrary sequence of prices and allocations $\{(p^n, x^n)\}, (p^n, x^n) \in \mathbb{R}^L_{++} \times X$ such that $(p^n, x^n) \to (p, x) \in \mathbb{R}^L_{++} \times X$. We'd like to show that $z(p^n, x^n) \to z(p, x)$. Towards that end, observe that

$$||z(p^n, x^n) - z(p, x)|| \leq \sum_i \left|\left|x_i(p^n, x^n_{-i}) - x_i(p, x_{-i})\right|\right|.$$

Now, for each i, by continuity of x_i, we know that for each $\epsilon > 0, \exists\, N_{i\epsilon} \in \mathbb{N}$ such that for all $n \geq N_{i\epsilon}$,

$$\left|\left|x_i(p^n, x^n_{-i}) - x_i(p, x_{-i})\right|\right| < \frac{\epsilon}{N}.$$

---

an open neighborhood in X, denoted $\bar{X}$, which exists because $intX$ is nonempty. For each $x' \in \bar{X}$, for α sufficiently close to 1, we have that $f(\alpha z + (1-\alpha)x') > f(x^*)$, and thus $a \cdot (\alpha z + (1-\alpha)x') \geq a \cdot x$. Yet, we can again write.

$$\alpha a \cdot z + (1-\alpha) a \cdot x' = \alpha a \cdot x + (1-\alpha) a \cdot x',$$

and for some some $x' \in \bar{X}$, we must have $\alpha a \cdot x + (1-\alpha) a \cdot x' < a \cdot x$

[2] Why must this be so? It is because we must have $x^* \in \partial X$. Suppose $x^* \in intX$. Then, for some $\epsilon > 0$, for all $x \in N_\epsilon(x^*)$, $x \in intX$ and $g(x) > 0$. Then, there exists $\hat{x} \in N_\epsilon(x^*)$ such that $f(\hat{x}) > f(x^*)$ and $g(\hat{x}) > 0$, contradicting optimality of x^*.

[3] Proving continuity of the Budget Correspondence for either part A or part B was not technically required for this problem because it is unchanged with $u : X \to \mathbb{R}$. However, you are required to know this proof.

Let $N_\epsilon = \max_i N_{i\epsilon}$. Then we have that for all $n \geq N_\epsilon$,

$$\sum_i \left|\left|x_i(p^n, x^n_{-i}) - x_i(p, x_{-i})\right|\right| < \epsilon$$

and thus $z(p^n, x^n) \to z(p, x)$. □

Part B: Existence of an Equilibrium

Claim 7. *Assuming that $e_i >> 0$ for each i, there exists a CE in the economy with externality.*

Proof. First, define $\bar{x}_i(p, x_{-i}, e_i)$ as the solution to the auxiliary problem

$$\begin{gathered} max_{x_i \in X_i} u(x_i, x_{-i}) \\ \text{subject to} \\ p \cdot x_i \leq p \cdot e_i \\ x_i \leq \bar{r}(>> r) \\ x_{-i} \text{ given} \end{gathered}$$

As in the proof in class, we construct $\bar{z} : \Delta \times X \to \mathbb{R}^L$, and $e_i >> 0$ ensures that each agents' truncated budget set is continuous in $x \in X, p \in \Delta$. Then, $\bar{x}_i$ is again continuous so that $\bar{z}$ is a continuous function.

Now, let $\bar{X}_i = \{x_i \in X_i | x_i \leq \bar{r}\}, \bar{X} = \times_i \bar{X}_i$. Then, we can define $f : \Delta \times \bar{X} \to \Delta \times \bar{X}$ by

$$f(p, x) = \left(\left(\frac{p_l + \max\{0, \bar{z}_l(p, x)\}}{1 + \sum_l \max\{0, \bar{z}_l(p, x)\}} \right)_{l=1}^{L}, (\bar{x}_i(p, x_{-i}, e_i))_{i=1}^{N} \right).$$

The domain (and range) of f are nonempty, compact, and convex, f is continuous, so the conditions for Brouwer's fixed point theorem are satisfied. Thus,$\exists\, (p^*, x^*) \in \Delta \times \bar{X}$ such that $f(p^*, x^*) = (p^*, x^*)$.

We now argue that (p^*, x^*) is an equilibrium. The fixed point property tells us that for each l,

$$p^*_l = \frac{p^*_l + \max\{0, \bar{z}_l(x^*, p^*)\}}{1 + \sum_l \max\{0, \bar{z}_l(x^*, p^*)\}}.$$

This implies (by Walras' Law) that

$$\sum_l \max\{0, \bar{z}_l(x^*, p^*)\} \bar{z}_l(x^*, p^*) = 0.$$

Thus, $\bar{z}_l(p^*, x^*) \leq 0$ for each l. So $\bar{z}(p^*, x^*) \leq 0$ and using a similar argument as that in class we in fact have that $z(p^*, x^*) \leq 0$.

Hence, (p^*, x^*) is an equilibrium. □

Economics 8102: Problem Set IV

Ichiro Obara

November 26, 2007
Due December 6, 2007

1. (Based on M.W.G. 17.BB.2) Consider a pure exchange economy E^{pure} where $\sum_{i=1}^{N} e_i >> 0$. For each consumer $i = 1, ..., N$, there exists an associated (nonempty) set of commodities $C_i \subset \{1, ..., L\}$ such that u_i depends on i's consumption of good l if and only if $l \in C_i$. Let $x_{i,C_i} = (x_{i,l}, l \in C_i) \in \Re_+^{\#C_i}$. Consumer i's utility $u_i(x_{i,C_i})$ is continuous, quasi-concave, and strongly monotone in x_{i,C_i}.

   Suppose that the economy satisfies the following condition:

   Indecomposability Condition:
   There is no (nontrivial) partition of consumers $N_A,\ N_B$
   such that $l \notin \cup_{i \in N_B} C_i$ if $\sum_{i \in N_A} e_{i,l} > 0$.

   Intuitively, this means that it is impossible to decompose the consumers into two groups where the consumers in one group does not desire any of the good owned by the consumers in the other group.

   (a) Show that there exists a competitive equilibrium with transfer in this economy.

   (b) Show that there exists a competitive equilibrium in this economy when u_i is also strictly quasi-concave in x_{i,C_i}.

   Note. When you use a proof from the class, you do not need to replicate the whole proof at every step if it is identical. You may just point out where the proof must be modified.

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ECON 8102
Solutions to Problem Set 4
December 16, 2007

Question 1

Claim 1. *Consider a pure exchange economy E^{pure} with $\sum_{i=1}^{N} e_i >> 0$. Suppose for each consumer $i = 1, \ldots, N$, there exists a nonempty set of commodities $C_i \subset \{1, \ldots, L\}$ such that u_i depends on i's consumption of l if and only if $l \in C_i$. Let $x_i \in \mathbb{R}_+^{\#C_i}$ with $x_i = (x_{i,l})_{l \in C_i}$. Assume $u_i(x_i)$ is continuous, quasi-concave, and strongly monotone.*

Further assume that the economy satisfies the following Indecomposability Conditon: There is no (nontrivial) partition of consumers N_A, N_B such that if $\sum_{i \in N_A} e_{i,l} > 0$, then $l \notin \cup_{i \in N_B} C_i$.

Then, there exists a competitive equilibrium.

Proof. It is clear that, given continuous, convex preferences, for any strictly positive price vector, the aggregate excess demand correspondence, given by $\Psi(p)$ is upper hemicontinuous, convex valued, and nonempty, and that for all $z \in \Psi(p)$, we have

i. $p \cdot z = 0$

ii. z is bounded from below

Hence, if we can prove the boundary condition, that is

iii. for all $p^n \to p \in \partial\Delta$, if $z^n \in \Psi(p^n)$, for all n, then $\{||z^n||\}$ is unbounded

Then we may apply the existence theorem we proved in recitation (which I outline briefly again for convenience).

Lemma 2. *Suppose preferences are continuous, convex, and strongly monotone, and E satisfies indecomposability. If $p^n \to p \in \partial\Delta$ and $z^n \in \Psi(p^n)$, for all n, then $\{||z^n||\}$ is unbouned.*

Proof. Suppose $p^n \to p \in \partial\Delta$ and $z^n \in \Psi(p^n)$, for all n. Suppose for contradiction that $\exists M \in \mathbb{R}$ such that for all $n, ||z^n|| < M$. Since z^n is bounded, we can take a convergent subsequence for each i of $x_i^n \in x_i(p^n, e_i)$ s.t. $x_i^n \to x_i'$.

Now, there exists i such that $p \cdot e_i > 0$. As in lecture notes, we can prove that x_i' maximizes i's utility given p. Let $L^0 = \{l \in \{1, \ldots, L\} | p_l = 0\}$. If for any $l \in L^0$, $l \in C_i$, then we must have $x_i(p, e_i) = \emptyset$, a contradiction (recall, this is where we need strong monotonicity).

Now, define $W = \{i \in \{1, \ldots, N\} | p \cdot e_i > 0\}$. For any $l \in L^0$, we must have $l \notin \cup_{i \in W} C_i$. However, for all $j \in W^c$, $p \cdot e_j = 0$. Hence, if for any $j \in W^c$, and any $l \in \{1, \ldots, L\}, e_{j,l} > 0$, it must be that $l \in L^0$. That is, all agents in W^c can only be endowed with 0-price goods. But no agent in W cares about any of the 0-price goods, so this contradicts indecomposability.

Hence, the boundary condition is satisfied. □

I will outline an existence theorem with convex preferences by the following steps.

Step 1 Fix ϵ, and define $\Delta^\epsilon = \{p \in \Delta | p_l \geq \epsilon \forall l\}$.

Step 2 For all $p \in \Delta^\epsilon$, prove that $\Psi(p) \subset \left[-B, \frac{B}{\epsilon}\right]^L$ (where B is a lower bound on the aggregate excess demand).

Step 3 Define $\mu_\epsilon : \left[-B, \frac{B}{\epsilon}\right]^L \to \Delta^\epsilon$ by

$$\mu_\epsilon(z) = \{q | q \in \Delta^\epsilon, q \cdot z = \max_{q' \in \Delta^\epsilon} q' \cdot z\}.$$

Step 4 Prove that μ_ϵ is nonempty, convex valued, and upper hemicontinuous.

Step 5 Define $\Gamma_\epsilon : \Delta^\epsilon \times \left[-B, \frac{B}{\epsilon}\right]^L \to \Delta^\epsilon \times \left[-B, \frac{B}{\epsilon}\right]^L$ by

$$\Gamma_\epsilon(p, z) = \mu_\epsilon(z) \times \Psi(p).$$

Argue that Γ_ϵ satisfies the conditions of Kakutani's FPT. Then, there exists $(p_\epsilon, z_\epsilon) \in \Gamma_\epsilon(p_\epsilon, z_\epsilon))$.

Step 6 Show that z_ϵ is bounded above as $\epsilon \to 0$. For all $q \in \Delta^\epsilon, q \cdot z_\epsilon \leq 0$. Fix l. Set $q_l = 1 - (l-1)\epsilon$ and $q_j = \epsilon$ for $j \neq l$. Then show that

$$-B \leq z_{\epsilon,l} \leq \frac{(l-1)\epsilon B}{1 - (l-1)\epsilon}$$

Since the RHS of the above inequality tends to 0 as $\epsilon \to 0$, and we have the same upper bound for each l, z_ϵ is bounded above. So $z_\epsilon \to \bar{z}$.

Step 7 Obvious that $p_\epsilon \to \bar{p} \in \Delta$. Use boundary condition and boundedness in Step 6 to show $\bar{p} \notin \partial\Delta$, that is, $\bar{p} >> 0$.

Step 8 Use upperhemicontinuity of Ψ to show that $\bar{z} \in \Psi(\bar{p})$, then Walras' Law and $\bar{p} >> 0$ to show $\bar{z} = 0$.

□

Economics 8102: Problem Set V

Ichiro Obara

December 5, 2007
Due December 13, 2007

1. Show that $D^2u_i(x_i)$ ($L \times L$ matrix) is a regular (full rank) matrix when u_i is differentiably strictly quasi-concave and $Du_i(x_i) >> 0$.

2. Verify that the following conditions are equivalent to the conditions to define upper hemicontinuity of $f : A \to Y$ in M.W.G..

   For any $x_n \in A$, $y_n \in f(x_n)$ *such that* $\lim_{n\to\infty} x_n \to x^* \in A$, *there exists a convergent subsequence such that* $\lim_{k\to\infty} y_{n(k)} \in f(x^*)$.

3. Consider a security with return $(1, -2, 3)$ $(S = 3)$. Create one call option and one put option from this security to complete the market.

4. Consider an economy with no redundant asset. Consumers' preferences are strongly monotone. Show that every asset market clears ($\sum_{i=1}^{N} b_{k,i} = 0$ for $k = 1, 2, ..., K$) when all spot markets clear ($\sum_{i=1}^{N} x_{i,s} = 0$ for $s = 0, 1, ..., S$).

5. Consider the following economy: $N = S = 2$, $L = 1$, $u_i(x_i) = \sqrt{x_{i,1}} + \sqrt{x_{i,2}}$ for $i = 1, 2$, $e_1 = (0, 1, 0), e_2 = (0, 0, 1)$. Answer the following questions.

   (a) Find all Arrow-Debreu equilibria.

   (b) Suppose that there is only one asset in this economy (so the market is not complete) with return $r_1 > 0$, $r_2 < 0$. Is there any Radner equilibrium? If so, find one.

Ariel Zetlin-Jones
ECON 8102
Solutions to Problem Set 5
December 16, 2007

Question 1

Claim 1. *Suppose u_i is strictly differentiably quasi-concave and differentiably strictly monotone. Then, the bordered hessian matrix:*

$$D = \begin{pmatrix} D^2u_i(x_i) & Du_i(x_i) \\ Du_i(x_i)^T & 0 \end{pmatrix}$$

is regular.

Proof. Recall that we can check the regularity of D by checking whether $D \cdot \Delta = 0$ if and only if $\Delta = 0$. So, suppose that $D \cdot \Delta = 0$ and $\Delta \neq 0$. Then we can rewrite Δ as $\Delta = (\Delta_1, \ldots, \Delta_L, \Delta_{L+1}) = (\Delta', \Delta_{L+1})$. Them, $D \cdot \Delta = 0$ implies

$$\begin{aligned} D^2u_i(x_i)\Delta' + Du_i(x_i)\Delta_{L+1} &= 0 \quad (1) \\ Du_i(x_i)^T\Delta' &= 0 \quad (2) \end{aligned}$$

If $\Delta' = 0$, then it the monotonicity of u_i implies that $\Delta_{L+1} = 0$ and we're done. So suppose that $\Delta' \neq 0$. Then, by assumption, it must be that $\Delta'^T D^2u_i(x_i)\Delta' < 0$. However, we can premultiply equation (1) by Δ'^T so that

$$\begin{aligned} \Delta'^T D^2u_i(x_i)\Delta' + \Delta'^T Du_i(x_i)\Delta_{L+1} &= 0 \quad (3) \\ \Rightarrow \Delta'^T D^2u_i(x_i)\Delta' &= 0 \quad (4) \end{aligned}$$

since $\Delta'^T Du_i(x_i)\Delta_{L+1} = (Du_i(x_i)^T\Delta')^T\Delta_{L+1}$ and $Du_i(x_i)^T\Delta' = 0$ by equation (2). This, of course, is a contradiction. Thus, it must be that $\Delta = 0$. □

Question 2

Definition 2. *Upper Hemicontinuous Correspondence (A). Let $A \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^K$ where Y is closed. Then the correspondence $f : A \to Y$ is* upper hemicontinuous *if it has closed graph and the image of compact sets are bounded.*

Definition 3. *Upper Hemicontinuous Correspondence (B). Let $A \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^K$ where Y is closed. Then the correspondence $f : A \to Y$ is* upper hemicontinuous *if for any $x_n \in A, y_n \in f(x_n)$ such that $x_n \to x^* \in A$, there exists a convergent subsequence of y_n such that $y_{n_k} \to y^* \in f(x^*)$.*

Claim 4. *Definitions (A) and (B) are equivalent.*

Proof. ($A \Rightarrow B$) Let $x_n \in A, y_n \in f(x_n)$ such that $x_n \to x^* \in A$. Then, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N, x_n \in B_\epsilon(x^*)$, where $B_\epsilon(x^*) = \{x \in A | \ ||x - x^*|| \leq \epsilon\}$. Obviously, $B_\epsilon(x^*)$ is compact. Hence, $f(B_\epsilon(x^*))$ is bounded. Since for $n \geq N, x_n \in B_\epsilon(x^*)$, we have that $y_n \in f(B_\epsilon(x^*))$. That is, y_n is a bounded sequence. Hence, y_n has a convergent subsequence, $y_{n_k} \to y^*$ and by the closed graph property of f, $y* \in f(x^*)$.

($B \Rightarrow A$) First, we prove that f has closed graph, and then we show that the image of compact sets is bounded. So, let $x_n \in A, y_n \in f(x_n)$ and $x_n \to x^* \in A, y_n \to y^* \in Y$. By assumption, y_n has a convergent subsequence such that $y_{n_k} \to \bar{y} \in f(x^*)$. Moreover, since y_n converges, all of its subsequences must converge to the same limit. Hence, it must be that $\bar{y} = y^*$. Hence, $\lim_{n\to\infty} y_n \in f(x^*)$, and f has closed graph.

Now, let $B \subset A$ be compact. Suppose $f(B)$ is not bounded. Then, for every $N \in \mathbb{N}$, there exists $y_n \in f(B)$ such that $||y_n|| > N$. Let $\{y_n\}_{n=1}^{\infty} \in f(B)$ such that $||y_n|| > n$ for each $n \in \mathbb{N}$. Since $y_n \in f(B)$, for every n, there exsits $x_n \in B$ such that $y_n \in f(x_n)$. Since B is compact, x_n has a convergent subsequence, $x_{n_k} \to x^* \in B$. Hence, we may assume that $y_{n_k} \to y^* \in f(x^*)$ (or at least this holds for a subsequence of y_{n_k} by assumption). We know that all convergent sequences are bounded (i.e. y_{n_k} is bounded), but this contradicts our construction of y_n. Hence, $f(B)$ must be bounded. □

Question 3

Consider a security with return $(1, -2, 3)(S = 3)$. Then, consider a call (put) option to buy (sell) this security ex post at strike price 1 (1). These securities yield:

$$\begin{aligned} &\text{call:} \quad (\max\{1-1,0\}, \max\{-2-1,0\}, \max\{3-1,0\}) = (0,0,2) \\ &\text{put:} \quad (\max\{1-1,0\}, \max\{1-(-2),0\}, \max\{1-3,0\}) = (0,3,0) \end{aligned}$$

Then, the return matrix can be written as

$$R = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 3 & 0 & 2 \end{pmatrix}$$

and it is pretty straightforward that this return matrix is full rank. Hence, these call and put options will complete the market.

Question 4

Claim 5. *Consider an economy with no redundant asset ($K \leq S$). Consumers preferences are strongly monotone. If every spot market clears ($\sum_{i=1}^{N}(x_{i,s} - e_{i,s}) = 0$ for $s = 0, \ldots, S$), then every asset market clears ($\sum_{i=1}^{N} b_{k,i} = 0$ for $s = 0, \ldots, S$).*

Proof. By local nonsatiation, the budget constraint for each state s will hold with equality. Writing these equalities and then summing over agents, we have for each $s = 1, \ldots, S$,

$$\begin{aligned}\sum_{i=1}^{N}\sum_{k=1}^{K} r_{s,k}b_{i,k} &= \sum_{i=1}^{N} p_s(x_{i,s} - e_{i,s}) \\ &= p_s \sum_{i=1}^{N}(x_{i,s} - e_{i,s}) \\ &= 0\end{aligned}$$

where the final equality holds by the spot market clearing assumption. Interchanging the sum on the LHS and these equalities for each state, we have, in matrix form:

$$\begin{bmatrix} r_{1,1} & \cdots & r_{1,K} \\ \vdots & \cdots & \vdots \\ r_{S,1} & \cdots & r_{S,K} \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{N} b_{i,1} \\ \vdots \\ \sum_{i=1}^{N} b_{i,K} \end{bmatrix} = R \cdot \begin{bmatrix} \sum_{i=1}^{N} b_{i,1} \\ \vdots \\ \sum_{i=1}^{N} b_{i,K} \end{bmatrix} = 0$$

Since there are no redundant assets, R is a regular matrix. Hence, it must be that

$$(\sum_{i=1}^{N} b_{i,1}, \ldots, \sum_{i=1}^{N} b_{i,K}) = 0.$$

That is, all asset markets clear. □

Question 5

Part A: Find all AD Equilibrium

This should be a straightforward computation using FOCs. You should find the following:

$$(x_{1,0}, x_{1,1}, x_{1,2}, x_{2,0}, x_{2,1}, x_{2,2}) = (0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2})$$

and prices are $(p_1, p_2) = (1, 1)$ are the unique ADE.

Part B: Find a Radner Equilibrium

We can write the HH problem as, for $i = 1, 2$, given r_1, r_2, q, p

$$\begin{gathered}\max_{x^i, b^i} \sqrt{x_1^i} + \sqrt{x_2^i} \\ \text{subject to} \\ (\mu^i) : qb^i \leq 0 \\ (\lambda^i) : p(x^i - e^i) \leq rb^i\end{gathered}$$

where the terms in parentheses are the Lagrange multipliers. The FOCs (which are clearly sufficient here) are

$$x_1^i: \quad \tfrac{1}{2}(x_1^i)^{-\frac{1}{2}} = \lambda_1^i p_1 \tag{5}$$
$$x_2^i: \quad \tfrac{1}{2}(x_2^i)^{-\frac{1}{2}} = \lambda_2^i p_2 \tag{6}$$
$$b_i: \quad \mu^i q = \lambda_1^i r_1 - \lambda_2^i r_2 \tag{7}$$

where we have already imposed $r_2 < 0$ (e.g. let the original return be $r_2' < 0$ and $-r_2 = r_2'$ so that $r_2 > 0$. Here, we guess that in equilibrium, $q = 0$.[1] Then, equation (7) implies that $r_1\lambda_1^i = r_2\lambda_2^i$. Then, combining equations (5) and (6), we have that

$$x_2^i = \left(\frac{r_2 p_1}{r_1 p_2}\right)^2 x_1^i.$$

Substituting these results into the budget constraints, I find that

$$x_1^1 = \tfrac{r_1 p_2}{r_1 p_2 + r_2 p_1}, \qquad x_2^1 = \tfrac{(r_2 p_1)^2}{(r_1 p_2)(r_1 p_2 + r_2 p_1)}$$
$$x_1^2 = \tfrac{(r_1 p_2)^2}{(r_2 p_1)(r_1 p_2 + r_2 p_1)}, \qquad x_2^2 = \tfrac{r_2 p_1}{r_1 p_2 + r_2 p_1}$$

These will clearly satisfy market clearing ($x_j^1 + x_j^2 = 1$ for $j = 1, 2$). Now, we check the asset markets:

$$b^1 = -\left(\tfrac{r_2}{r_1}\right) p_1^2 \left(\tfrac{1}{r_1 p_2 + r_2 p_1}\right)$$
$$b^2 = \left(\tfrac{r_1}{r_2}\right) p_2^2 \left(\tfrac{1}{r_1 p_2 + r_2 p_1}\right)$$

For the asset market to clear, we must have $b^1 + b^2 = 0$, or

$$-\left(\frac{r_2}{r_1}\right) p_1^2 + \left(\frac{r_1}{r_2}\right) p_2^2 = 0.$$

That is,

$$\frac{r_2^2}{r_1^2} = \frac{p_2^2}{p_1^2}.$$

Choosing prices to satisfy this ratio, with $q = 0$, agents will optimize (and the resulting allocations will be the same as in the ADE), and all markets will clear.

[1] This guess makes some sense because both agents are borrowing (lending) constrained if $q \neq 0$. That is, suppose $q > 0$. Then both agents must satisfy $b^i \leq 0$, and hence it must be that $b^i = 0$, but presumably at least one agent would want to go short on the asset, which would violate the asset market clearing condition.