Ariel Zetlin-Jones
ECON 8103
Note on IESDS and NE
February 7, 2008

Recall that our induction hypothesis was that $supp(s) \subseteq E_k$. The subclaim, which I did not prove in recitation, that if $b^{-i} \notin E_k^{-i}$, then $s^{-i}(b^{-i}) = 0$ can be proved as follows.

First, assume $b^{-i} \notin E_k^{-i}$ and suppose (for contradiction) that $s^{-i}(b^{-i}) > 0$. Then, since for sure we have $a^i \in E_k^i$ such that $s^i(a^i) > 0$, we must have that $s^i(a^i)s^{-i}(b^{-i}) > 0$. That is, $(a^i, b^{-i}) \in supp(s)$. But by the induction hypothesis, $supp(s) \subseteq E_k$, which in turn implies that $b^{-i} \in E_k^{-i}$, a contradiction. Hence, $\forall\, b^{-i} \notin E_k^{-i}, s^{-i}(b^{-i}) = 0$.

Ariel Zetlin-Jones
ECON 8103
Note on Perfect Equlibria and Weakly Dominated Strategies
February 7, 2008

In what follows, I state a definition of a weakly dominated action and then prove that in any perfect equlibrium, no weakly dominated actions can be played.

Definition 1. *Weakly Dominated Action.* An action $a^i \in A^i$ is said to be *weakly dominated* if there exists a (possibly mixed) strategy, $s^i \in S^i$, such that

$$\forall\, a^{-i} \in A^{-i}, u^i(a^{-i}, s^i) \geq u^i(a^{-i}, a^i)$$

and for at least one $a^{-i} \in A^{-i}$,

$$u^i(a^{-i}, s^i) > u^i(a^{-i}, a^i).$$

Proposition 2. *Let $\hat{s}$ be a perfect equilibrium. Then, for any i and any $a^i_k \in A^i$ such that a^i_k is weakly dominated, $\hat{s}^i(a^i_k) = 0$. That is, no player can play a weakly dominated action in a perfect equilibrium.*

Proof. Given $\hat{s}$ is a perfect equlibrium, there exists a sequence of perturbations, $\epsilon_n \to 0$ and a sequence of strategy profiles $s_{\epsilon_n} \to \hat{s}$ such that for every $n, s_{\epsilon_n} \in NE(G^{\epsilon_n})$.

Now, fix $i \in I$ and suppose for contradiction that $a^i_k \in A^i$ is a weakly dominated strategy and $\hat{s}^i(a^i_k) > 0$. Let $\tau \in S^i$ denote the strategy that weakly dominates a^i_k.

Choose $\delta \in \mathbb{R}$ such that $\hat{s}^i(a^i_k) > \delta > 0$. Then, since $s^i_{\epsilon_n}(a^i_k) \to \hat{s}^i(a^i_k)$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1, s^i_{\epsilon_n}(a^i_k) > \delta$. Similarly, since $\epsilon^i_n(a^i_k) \to 0$, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2, \epsilon^i_n(a^i_k) < \delta$. Hence, for sufficiently large n (that is, for $n \geq \max\{N_1, N_2\}$, we have that $s^i_{\epsilon_n}(a^i_k) > \epsilon^i_n(a^i_k)$). Let $\left(s^i_{\epsilon_n}(a^i_k) - \epsilon^i_n(a^i_k)\right) = \gamma > 0$.

We now want to construct a strategy which will strictly improve player $i's$ expected payoffs to show that in fact $s_{\epsilon_n} \notin NE(G^{\epsilon_n})$. First, construct σ as follows (recall that a^i_k is the action that is weakly dominated):

$$\forall a^i_j \in A^i, \sigma(a^i_j) = \gamma\tau(a^i_j).$$

Then, consider an alternative strategy for player i in G^{ϵ_n}, t^i defined as

$$\begin{aligned} t^i(a^i_j) &= s^i_{\epsilon_n}(a^i_j) + \sigma(a^i_j), \text{ for } a^i_j \neq a^i_k \\ t^i(a^i_k) &= \epsilon^i_n(a^i_k) + \sigma(a^i_k) \end{aligned}$$

Observe that $\sum_{a^i \in A^i} t^i(a^i) = 1$.

Then, we must prove that player i is strictly better off playing t^i. First, for $a^{-i} \in A^{-i}$, we define

$$P_{s^{-i}}(a^{-i}) = \prod_{\substack{j=1 \\ j \neq i}}^{N} s^j(a^j).$$

Then, we have that

$$
\begin{aligned}
& u^i(s_{\epsilon_n}^{-i}, t^i) - u^i(s_{\epsilon_n}) \\
= \ & \sum_{a \in A} P_{s_{\epsilon_n}^{-i}}(a^{-i}) t^i(a^i) u^i(a) - \sum_{a \in A} P_{s_{\epsilon_n}^{-i}}(a^{-i}) s_{\epsilon_n}^i(a^i) u^i(a) \\
= \ & \sum_{a \in A} P_{s_{\epsilon_n}^{-i}}(a^{-i}) \left(t^i(a^i) - s_{\epsilon_n}^i(a^i)\right) u^i(a) \\
= \ & \sum_{a^{-i} \in A^{-i}} P_{s_{\epsilon_n}^{-i}}(a^{-i}) \left[\left(\sum_{a_j^i \in A^i, a_j^i \neq a_k^i} \sigma(a_j^i) u^i(a^{-i}, a_j^i) \right) + \left(\epsilon_n^i(a_k^i) + \sigma(a_k^i) - s_{\epsilon_n}^i(a_k^i)\right) u^i(a^{-i}, a_k^i) \right] \\
= \ & \sum_{a^{-i} \in A^{-i}} P_{s_{\epsilon_n}^{-i}}(a^{-i}) \left[\gamma \left(\sum_{a_j^i \in A^i, a_j^i \neq a_k^i} \tau(a_j^i) u^i(a^{-i}, a_j^i) \right) + \gamma(\tau(a_k^i) - 1) u^i(a^{-i}, a_k^i) \right] \\
= \ & \gamma \sum_{a^{-i} \in A^{-i}} P_{s_{\epsilon_n}^{-i}}(a^{-i}) \left[\left(\sum_{a^i \in A^i} \tau(a^i) u^i(a^{-i}, a_j^i) \right) - u^i(a^{-i}, a_k^i) \right] \\
= \ & \gamma \sum_{a^{-i} \in A^{-i}} P_{s_{\epsilon_n}^{-i}}(a^{-i}) \left(u^i(a^{-i}, \tau) - u^i(a^{-i}, a_k^i)\right)
\end{aligned}
$$

Finally, $\gamma > 0$, since s_{ϵ_n} is fully mixed, $P_{s_{\epsilon_n}^{-i}}(a^{-i}) > 0$ for all $a^{-i} \in A^{-i}$ and since τ weakly dominates a_k^i, we have that the right hand side must be strictly positive. That is,

$$u^i(s_{\epsilon_n}^{-i}, t^i) - u^i(s_{\epsilon_n}) > 0.$$

However, this contradicts the fact that $s_{\epsilon_n} \in NE(G^{\epsilon_n})$; in particular, $s_{\epsilon_n}^i \notin BR^i(s_{\epsilon_n})$. Hence, it must be that if a_k^i is weakly dominated, then $\hat{s}^i(a_k^i) = 0$. □