Brian Adams

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Economics 8101
Fall Semester 2007
Jan Werner

Problem Set 1
Due: Thursday, September 13

1. (10 points) Prove that the properties of constant, increasing, and decreasing returns to scale for a production function $f : \Re^n_+ \to \Re_+$ with single output and n inputs imply that the corresponding production set Y_f, as defined by (2) in Course Handouts, exhibits constant, nondecreasing, and nonincreasing returns to scale, respectively.

2. (10 points) Consider production set given by

$$Y = \{(y_1, y_2): y_1 \leq 0, \quad y_2 \leq \max\{\ln(\frac{-y_1+1}{3}), 0)\}.$$

Find the profit function π^* and the supply function (or correspondence) s^* associated with Y. Clearly specify the sets of price vectors for which π^* and s^* are well defined.

3. (10 points) Consider production function of two inputs given by

$$f(x_1, x_2) = x_1 + \ln(x_2 + 1),$$

for $x_1 \geq 0$ and $x_2 \geq 0$.

   (a) Verify whether this production function exhibits decreasing returns to scale, increasing returns to scale, or neither one.

   (b) What is the range of prices $p > 0$, $w_1 > 0$, and $w_2 > 0$ for which there exist profit-maximizing input quantities? Find these quantities as functions of prices.

4. (10 points) Let Y be a production set in $\Re^L$. Assume that Y is closed and $0 \in Y$. Production set $Y \subset \Re^L$ is said to be *additive* if $y + y' \in Y$ for every $y, y' \in Y$.
For each of the following two statements A and B, if it is true, then prove it; if it false, then give a counterexample.

   (A) If Y is convex and additive, then Y exhibits constant returns to scale.

   (B) If Y exhibits nondecreasing returns to scale, then Y is convex.

Brian Adams
8 Sep 2007

PROBLEM SET 1

1. CLAIM $F: \mathbb{R}^n_+ \to \mathbb{R}_+$ exhibiting constant returns to scale $\Rightarrow Y = \{(x,z) \in \mathbb{R}^{n+1} \mid x \in \mathbb{R}^n_-,\ 0 \le z \le F(-x)\}$ exhibits constant returns to scale.

Proof Let $(x,z) \in Y$ (WTS: $\forall \lambda \ge 0\ (\lambda x, \lambda z) \in Y$) $0 \le z \le F(-x) \Rightarrow 0 \le \lambda z \le \lambda F(-x)\ \forall \lambda \ge 0$ F exhibits constant returns to scale $\Rightarrow \lambda F(-x) = F(-\lambda x) \overset{\forall \lambda \ge 0}{\Rightarrow} 0 \le \lambda z \le F(-\lambda x)\ \forall \lambda \ge 0 \Rightarrow$ $(\lambda x, \lambda z) \in Y\ \forall \lambda \ge 0$. ■

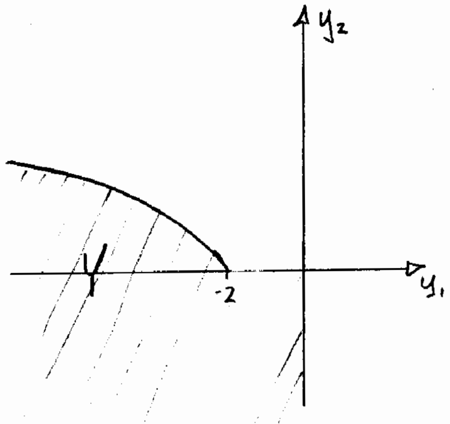
CLAIM $F: \mathbb{R}^n_+ \to \mathbb{R}_+$ exhibiting increasing returns to scale $\Rightarrow Y = \{(x,z) \in \mathbb{R}^{n+1} \mid x \in \mathbb{R}^n_-,\ 0 \le z \le F(-x)\}$ exhibits nondecreasing returns to scale.

Proof Let $(x,z) \in Y$ (WTS: $\forall \lambda \ge 1\ (\lambda x, \lambda z) \in Y$) $0 \le z \le F(-x) \Rightarrow 0 \le \lambda z \le \lambda F(-x)\ \forall \lambda \ge 0$ F exhibits increasing returns to scale $\Rightarrow F(-\lambda x) \ge \lambda F(-x)\ \forall \lambda \ge 1 \Rightarrow 0 \le \lambda z \le F(-\lambda x)\ \forall \lambda \ge 1 \Rightarrow$ $(\lambda x, \lambda z) \in Y\ \forall \lambda \ge 1$ ■

CLAIM $F: \mathbb{R}^n_+ \to \mathbb{R}_+$ exhibiting decreasing returns to scale $\Rightarrow Y = \{(x,z) \in \mathbb{R}^{n+1} \mid x \in \mathbb{R}^n_-,\ 0 \le z \le F(-x)\}$ exhibits nonincreasing returns to scale.

Proof Let $(x,z) \in Y$ (WTS: $\forall \lambda \in [0,1]\ (\lambda x, \lambda z) \in Y$) $0 \le z \le F(-x) \Rightarrow 0 \le \lambda z \le \lambda F(-x)\ \forall \lambda \ge 0$ F exhibits decreasing returns to scale $F(-\mu t) \le \mu F(-t)\ \forall \mu \ge 1$ Let $\mu = \frac{1}{\lambda}\ \forall \lambda \in (0,1]$ Let $t = \lambda x$ $F(-\frac{1}{\lambda}\lambda x) \le \frac{1}{\lambda} F(-\lambda x)\ \forall \lambda \in (0,1] \Rightarrow F(-x) \le \frac{1}{\lambda} F(-\lambda x) \Rightarrow \lambda F(-x) \le F(-\lambda x)$. Thus $0 < \lambda z \le F(-\lambda x)\ \forall \lambda \in (0,1] \Rightarrow$ $(\lambda x, \lambda z) \in Y\ \forall \lambda \in (0,1]$. $(0,0) \in Y$, so $(\lambda x, \lambda z) \in Y\ \forall \lambda \in [0,1]$. ■

2. profit function $\pi^*(p) = \max\limits_{y \in Y} p \cdot y$



$$Y_1 = \{(y_1, y_2) : y_1 \leq 0,\ y_2 \leq \ln(\tfrac{-y_1+1}{3})\}$$

$$Y_2 = \{(y_1, y_2) : y_1 \leq 0,\ y_2 \leq 0\}$$

$$Y_1 \cup Y_2 = Y$$

CASE 1 $y^* = Y_1$ $\max p_1 y_1 + p_2 y_2$ such that $-y_1 \geq 0$, $\ln(\frac{-y_1+1}{3}) - y_2 \geq 0$

$$\mathcal{L}: p_1 y_1 + p_2 y_2 - \lambda_1 y_1 + \lambda_2 (\ln(\tfrac{-y_1+1}{3}) - y_2)$$

$$0 = \frac{\partial \mathcal{L}}{\partial y_1} \quad p_1 = -\lambda_1 + \lambda_2 \frac{1}{y_1 - 1} \qquad p_1 = -\lambda_1 - p_2 \cdot \frac{1}{y_1 - 1}$$

$$0 = \frac{\partial \mathcal{L}}{\partial y_2} \quad p_2 = -\lambda_2$$

CASE 1a $y_1^* < 0 \Rightarrow \lambda_1 = 0$

$$p_1 = -p_2 \frac{1}{y_1 - 1}$$

$$y_1 - 1 = -\frac{p_2}{p_1}$$

$$y_1^* = -\frac{p_2}{p_1} + 1 \qquad y_2^* = \ln\left(\frac{p_2}{3p_1}\right)$$

$$\pi(p) = -p_2 + p_1 + p_2 \ln\left(\frac{p_2}{3p_1}\right)$$

CASE 1b $y_1^* = 0 \Rightarrow \lambda_1 > 0$

$$p_1 = -\lambda_1 - 3p_2$$

$$p_1 + 3p_2 = -\lambda_1$$

CASE 2 $y^* = Y_2$ $\max p_1 y_1 + p_2 y_2$ such that $-y_1 \geq 0$ $-y_2 \geq 0$

$p_1 > 0 \Rightarrow y_1^* = 0$ $p_1 < 0 \Rightarrow y_1^* \to \infty$

$p_2 > 0 \Rightarrow y_2^* = 0$ $p_2 < 0 \Rightarrow y_2^* \to \infty$

} no solution for $p_1 < 0$, $p_2 < 0$

$\pi(p) = 0$ for $p_1, p_2 > 0$

$$0 = -p_2 + p_1 + p_2 \ln\left(\frac{p_2}{3p_1}\right)$$

$$0 = -\frac{p_2}{p_1} + 1 + \frac{p_2}{p_1} \ln\left(\frac{p_2}{3p_1}\right)$$

$$1 = p - p \ln\left(\frac{p}{3}\right) \quad \text{where } p = \frac{p_2}{p_1}$$

$$p = 7.0808,\ .3040$$

✓

$$\pi^*(p) = \begin{cases} -p_2 + p_1 + p_2 \ln\left(\frac{p_2}{p_1}\right) & \text{for } 0 < .3040 p_1 < p_2 < 7.0808 p_1 \\ 0 & \text{For } 0 < p_2 < .3040 p_1 \text{ and } 0 < 7.0808 p_1 < p_2 \end{cases}$$

$$y_1^*(p) = \begin{cases} 1 - \frac{p_2}{p_1} & \text{For } 0 < 3040 p_1 < p_2 < 7.0808 p_1 \\ 0 & \text{For } 0 < p_2 < .3040 p_1 \text{ and } 0 < 7.0808 p_1 < p_2 \end{cases}$$

$$y_2^*(p) = \begin{cases} \ln\left(\frac{p_2}{3p_1}\right) & \text{For } 0 < .3040 p_1 < p_2 < 7.0808 p_1 \\ 0 & \text{For } 0 < p_2 < .3040 p_1 \text{ and } 0 < 7.0808 p_1 < p_2 \end{cases}$$

3. (a) $F(x_1, x_2) = x_1 + \ln(x_2 + 1)$

For $(x_1, x_2) = (1, 0)$ $\quad F(\lambda x) = \lambda + \ln(1) = \lambda$

$\lambda F(x) = \lambda \cdot 1 + \ln(1) = \lambda$

For any $\lambda > 1$ $\quad \lambda F(x) \not> F(\lambda x) \Rightarrow$ returns to scale are not decreasing

$\lambda F(x) \not< F(\lambda x) \Rightarrow$ returns to scale are not increasing

For $(x_1, x_2) = (0, 1)$ $\quad F(\lambda x) = \ln(\lambda + 1)$

$\lambda F(x) = \lambda \ln 2$

For $\lambda = 2$ $\quad 2 \ln 2 \neq \ln 3 \Rightarrow$ returns to scale are not constant

(b) $F(x_1, x_2) = x_1 + \ln(x_2 + 1)$

$\max\ p \cdot F(x_1, x_2) - w_1 x_1 - w_2 x_2$, subject to $x_1 \geq 0$, $x_2 \geq 0$

$$p \frac{1}{x^2 + 1} - w_2 = 0$$

$$\frac{p}{w_2} - 1 = x_2^* \qquad x_2^* \Rightarrow w_2 < p$$

$$x_1^* = \begin{cases} 0 & \text{if} \quad w_1 > p \\ \text{undefined} & \quad w_1 < p \end{cases}$$

Profit maximizing input quantities exist for $w_1 > p$ and $0 < w_2 < p$

Namely $(x_1^*, x_2^*) = \begin{cases} (0, \frac{p}{w_2} - 1), & \text{for } w_1 > p,\ 0 < w_2 < p \\ (0, 0) & \text{for } w_1 > p,\ w_2 > p \\ \text{undefined} & \text{for } w_1 < p \end{cases}$

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what if $w_1 = p$?

4. A) <u>CLAIM</u> If $Y \subset \mathbb{R}^L$ is closed, convex, additive and $0 \in Y$, then Y exhibits constant returns to scale.

<u>Proof</u> Let $y \in Y$. Want to show: $\lambda y \in Y$ for every $\lambda \geq 0$)

CASE 0 Let $\lambda = 0$. $0y = 0 \in Y$ by assumption.

CASE 1 Let $0 \leq \lambda < 1$. Y convex $\Rightarrow \forall x, z \in Y$ and $\forall \theta \in (0,1)$ $\theta x + (1-\theta) z \in Y$.
$y \in Y$ and $0 \in Y$ and $\lambda \in (0,1)$ so $\lambda y + (1-\lambda) 0 = \lambda y \in y$.

CASE 2 Let $\lambda = 1$. $\lambda y = y$. $y \in Y$.

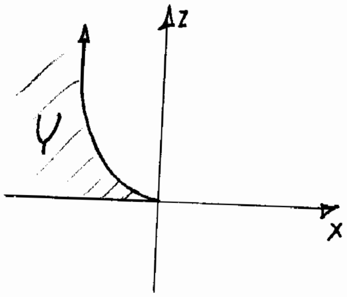
CASE 3 Let $\lambda > 1$. Then $\lambda = a + b$ where $a < 1$ and $b \in \mathbb{Z}$. Y convex, $0, y \in Y \Rightarrow$
$ay \in Y$. Y additive $\Rightarrow \left(\sum_{i=1}^{b} y\right) + ay \in y$.

Thus $\lambda y \in Y$ $\forall \lambda \geq 0 \Rightarrow$ constant returns to scale.

B) CLAIM If Y exhibits nondecreasing returns to scale, then Y is convex.

<u>COUNTEREXAMPLE</u> - Let $Y = \{(x,z) \in \mathbb{R}^2 \mid x \leq 0,\ 0 \leq z \leq x^2\}$

Let $\lambda \geq 1$
$(x,z) \in Y \Rightarrow 0 < z \leq x^2 \Rightarrow 0 \leq \lambda z \leq \lambda x^2$.
$0 < \lambda z \leq \lambda x^2 \leq (\lambda x)^2 \Rightarrow (\lambda x, \lambda z) \in Y \Rightarrow Y$ exhibits nondecreasing returns to scale.



$(0,0) \in Y$, $(1,1) \in Y$ $\frac{1}{2}(0,0) + (1-\frac{1}{2})(1,1) = (\frac{1}{2}, \frac{1}{2}) \notin Y$ since $\frac{1}{2} \nleq \left(\frac{1}{2}\right)^2$

Therefore y is not convex.

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Problem Set 2
Due: Thursday, September 20

1. (10 points) Consider the following supply function of a producer

$$\psi(p_1, p_2) = \left(-\frac{p_2^2}{p_1^2}, \frac{p_2}{p_1}\right) \quad \text{for} \quad p_1 > 0, p_2 > 0.$$

The producer produces good 2 using good 1 as input.

Can this supply function result from profit maximization on a production set? Justify your answer.

2. (15 points) Consider two production sets $Y_1, Y_2 \subset \Re^L$. Assume that both sets are closed and convex and such that $0 \in Y_1$ and $0 \in Y_2$. Let π_1^* and π_2^* denote the profit functions associated with Y_1 and Y_2.

   (i) Prove that

$$Y_1 \supset Y_2 \quad \text{if and only if } \pi_1^*(p) \geq \pi_2^*(p) \ \text{ for every } p \in \Re^L.$$

   [$\pi_1^*(p)$ and $\pi_2^*(p)$ can take value $+\infty$ in this inequality.]

   (ii) Does the equivalence from (i) hold when sets Y_1, Y_2 are not necessarily convex? Justify your answer.

3. (15 points) Consider the following lexicographic preferences on the consumption set $\Re^2_+$: the value of $x_1 + x_2$ has the first priority; the value of x_2 has the second priority.

   (i) Prove that this lexicographic preference has no utility representation on $\Re^2_+$.

   (b) Does this lexicographic preference have a utility representation on the set N^2 of (ordered) pairs of natural numbers? Justify your answer.

   (c) Derive demand functions $x_i^*(p_1, p_2, w)$, $i = 1, 2$ for $p_1 > 0$, $p_2 > 0$ and $w > 0$ for this lexicographic preference on $\Re^2_+$.

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1. CLAIM A production set $Y \subset \mathbb{R}^2$ such that $\psi(p) = \underset{y \in Y}{\arg\max}\ p \cdot y \quad \forall p \in \mathbb{R}^2_{++}$, where $\psi(p_1, p_2) = \left(\frac{-p_2^2}{p_1^2}, \frac{p_2}{p_1}\right)$ does not exist.

Proof Assume $\exists Y$ so defined.

$$\psi(1,1) = (-1,1) \Rightarrow (-1,1) \in Y.$$
$$\psi(1,2) = (-4,2) \Rightarrow (-4,2) \in Y.$$

$$(-4,2) = \psi(1,2) = \underset{y \in Y}{\arg\max} \begin{pmatrix}1\\2\end{pmatrix} \cdot y, \text{ but } \begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = 1 > 0 = \begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}-4\\2\end{pmatrix}$$

So $\psi(1,2) \neq \underset{y \in Y}{\arg\max} \begin{pmatrix}1\\2\end{pmatrix} \cdot y$. contradiction, so $\nexists Y$ so defined. ∎

2. (1) Let $Y_1, Y_2 \subset \mathbb{R}^L$ with Y_1, Y_2 closed and convex and with $0 \in Y_1$ and $0 \in Y_2$. Let $\pi_1^*(p) = \max_{y \in Y_1} p \cdot y$ and $\pi_2^*(p) = \max_{y \in Y_2} p \cdot y$. $Y_1 \supset Y_2 \Leftrightarrow \pi_1^*(p) \geq \pi_2^*(p) \ \forall p \in \mathbb{R}^L$.

Proof ($\Rightarrow$) Let p be given. Let $Y_1 \supset Y_2$. Y_1, Y_2 closed, convex $\Rightarrow \max_{y \in Y_i} p \cdot y$ exists $i = 1,2$. why? boundedness not assumed!

$Y_1 \supset Y_2 \Rightarrow \forall y_1 \in Y_1,\ y_1 \in Y_2$. Let $q \in \underset{y_2 \in Y_2}{\arg\max}\ p \cdot y_2 \Rightarrow q \in Y_2 \Rightarrow q \in Y_1$

$\pi_1^*(p) = \max_{y_1 \in Y} p \cdot y_1 \Rightarrow \pi_1^*(p) \geq p \cdot y_1 \ \forall y_1 \in Y_1$. Since $q \in Y_1$,

$\pi_1^*(p) \geq p \cdot q = \max_{y_2 \in Y} p \cdot y_2 = \pi_2^*(p)$. $\Rightarrow \pi_1^*(p) \geq \pi_2^*(p)$.

($\Leftarrow$) (WTS: $\pi_1^*(p) \geq \pi_2^*(p) \Rightarrow Y_1 \supset Y_2$. Prove the contrapositive, namely $Y_1 \not\supset Y_2 \Rightarrow \exists \hat{p}$ such that $\pi_1^*(\hat{p}) < \pi_2^*(\hat{p})$.) Let Y_1, Y_2 be convex.

$Y_2 \not\subset Y_1 \Rightarrow \exists q \in Y_2$ such that $q \notin Y_1$. $\pi_1^*(p)$ exists $\Rightarrow Y_1$ nonempty.

Ok. which one?

Y_1 nonempty, convex implies by a Separating Hyperplane Theorem that $\exists \alpha \in \mathbb{R}$ and $\exists \hat{p} \in \mathbb{R}^n$ with $p \neq 0$ such that $\hat{p}x \leq \alpha \ \forall x \in Y_1$ and $\hat{p}q > \alpha$. $\hat{p}x \leq \alpha \ \forall x \in Y_1 \Rightarrow \pi_1^*(\hat{p}) \leq \alpha$. $\alpha < \hat{p}q \leq \pi_2^*(\hat{p})$ since $q \in Y_2$.

Thus $\pi_1^*(\hat{p}) < \pi_2^*(\hat{p})$. So, $Y_1 \not\supset Y_2 \Rightarrow \exists \hat{p}$ such that $\pi_1^*(\hat{p}) < \pi_2(\hat{p})$.

The contrapositive must also be true, namely $\pi_1^*(p) \geq \pi_2^*(p) \ \forall p \in \mathbb{R}^L \Rightarrow Y_1 \supset Y_2$. ∎

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2. (ii) The equivalence from (i) does not necessarily hold if Y_1 and Y_2 are not necessarily convex. Consider the example:

$$Y_2 = \{(y_1, y_2) \mid -1 \leq y_1 \leq 0,\ 0 \leq y_2 \leq -y_1\}$$

$$Y_1 = \{(y_1, y_2) \mid -2 \leq y_2 \leq 0,\ 0 \leq y_2 \leq y_1^2\}$$

$$\pi_2(p) = \begin{cases} 0 & \text{for } p_1 \geq p_2 \geq 0 \\ p_2 - p_1 & \text{for } 0 < p_1 < p_2 \text{ or } p_1 < 0 < p_2 \\ -p_1 & \text{for } p_1 < 0,\ p_2 < 0 \end{cases}$$

$$\pi_1(p) = \begin{cases} 0 & \text{for } p_1 \geq 2p_2 \geq 0 \\ 4p_2 - 2p_1 & \text{for } 0 < p_1 < 2p_2 \text{ or } p_1 < 0 < p_2 \\ -2p_1 & \text{for } p_1 < 0,\ p_2 < 0 \end{cases}$$

$\pi_1(p) \geq \pi_2(p)$ $\forall p \in \mathbb{R}^2$ but $(-\frac{1}{2}, \frac{1}{2}) \in Y_2$ but $(-\frac{1}{2}, \frac{1}{2}) \notin Y_1$.

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3. (i) Assume $u(x_1, x_2)$ represents the preferences. Let $m = x_1 + x_2$.

$\forall m$ $\exists$ a rational number $r(m)$ such that $u(m-2,2) > r(m) > u(m-1,1)$

For $m > m'$ $r(m) > u(m-1,1) > u(m'-2,2) > r(m')$. Thus $m > m'$ implies $r(m) > r(m')$. Therefore, $r(\,)$ provides a one-to-one function from the set of real numbers to the set of rational numbers. This cannot be, so no $u(\,)$ exists.

(b) Define $u(\cdot): N^2 \to R$ by

$$u(x_1, x_2) = \begin{cases} x_1 + x_2 + \left(1 - \frac{1}{2x_2}\right). & \text{for } x_2 \neq 0 \\ x_1 + x_2 & \text{For } x_2 = 0 \end{cases}$$

$x \succsim x' \Rightarrow x_1 + x_2 > x_1' + x_2'$ or $(x_1 + x_2 = x_1' + x_2'$ and $x_2 \geq x_2')$

$x_1 + x_2 > x_1' + x_2' \Rightarrow x_1 + x_2 > x_1' + x_2' + 1$

$$u(x_1', x_2') < x_1' + x_2' + 1 \leq x_1 + x_2 < u(x_1, x_2)$$

So $x_1 + x_2 > x_1' + x_2' \Rightarrow u(x_1, x_2) \geq u(x_1', x_2')$

$$x_1 + x_2 = x_1' + x_2' \text{ and } x_2 \geq x_2' \Rightarrow 1 - \frac{1}{2x_2} \geq \begin{cases} 1 - \frac{1}{2x_2'} & x_2' \neq 0 \\ 0 & x_2 = 0 \end{cases} \Rightarrow u(x_1, x_2) \geq u(x_1', x_2')$$

$u(x): N^2 \to R$ represents the lexigraphical preferences, since $x \succsim x' \Rightarrow u(x) \geq u(x')$.

-1

need to show $u(x) \geq u(x') \Rightarrow x \succsim x'$

(c) The consumer first will maximize $x_1 + x_2$.

max $x_1 + x_2$ subject to $p_1 x_1 + p_2 x_2 \leq w$ is solved by $(w/p_1, 0)$ for $0 < p_1 < p_2$, by $(0, w/p_2)$ for $p_1 > p_2 > 0$ and by the set $\{(x_1, x_2) : p_1 x_1 + p_2 x_2 = w\}$. for $p_1 = p_2 > 0$. why?

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max x_2 subject to $p_1 x_1 + p_2 x_2 = w$ is solved by $(0, \frac{w}{p_2})$. So,

$$x_1^*(p_1, p_2, w) = \begin{cases} w/p_1 & \text{for } 0 < p_1 < p_2 \\ 0 & \text{for } p_1 \geq p_2 > 0 \end{cases}$$

$$x_2^*(p_1, p_2, w) = \begin{cases} w/p_2 & \text{for } 0 < p_2 \leq p_1 \\ 0 & \text{for} \end{cases}$$

Economics 8101 **Problem Set 3**
Fall Semester 2007 **Due: Thursday, September 27**
Jan Werner

1. (15 points) Let $u \colon \Re^L_+ \to \Re$ be a continuous and locally non-satiated utility function.
   (a) Show that the indirect utility function u^* is quasi-convex in p.
   (b) Suppose that u is quasi-linear of the form
   $$u(x_1, x_2, \ldots, x_L) = x_1 + v(x_2, \ldots, x_L),$$
   for some strictly increasing function $v \colon \Re^{L-1}_+ \to \Re$. Show that the indirect utility u^* is a linear function of income w on the domain of price-income pairs (p, w) where $x^*(p, w) >> 0$.

2. (15 points) There are two conditions often used to define continuity of preference relation $\succeq$ on consumption set $X = \Re^L_+$:
   (a) for every sequences $\{x^n\}$ and $\{y^n\}$ in X such that $\lim_n x^n = x$, $\lim_n y^n = y$, and $x^n \succeq y^n$, it holds $x \succeq y$.
   (b) For every $x \in X$, the preferred-to-x set $\{y \in X : y \succeq x\}$, and the lower contour set $\{y \in X : x \succeq y\}$ are closed.

   Assuming that $\succeq$ is transitive and complete, prove that conditions (a) and (b) are equivalent.

3. (10 points) Consider the following utility function of prices and income:
   $$v(p_1, p_2, w) = \frac{(w + p_1)^2}{4 p_1 p_2},$$
   where w is income and the p_i represent prices for two goods purchased by this consumer. Consider only prices and income such that $w > p_1$ and $p_1 > 0, p_2 > 0$.

   Could this function be the indirect utility function of a consumer who maximizes a utility function of consumption subject to the budget constraint? Justify your answer.

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27 Sep 2007

Econ 8101 Homework #3

1 (a) Let $u : \mathbb{R}^L_+ \rightarrow \mathbb{R}$ be continuous and a locally non-satiated utility function. Then the indirect utility function $v(p,w)$ is quasi-convex in p.

Proof (WTS: th[illegible] $w) : v(p,w) \leq \bar{v}\}$ is convex $\forall \bar{v}$)

Suppose $v(p,w) \leq \bar{v}$ and $v(\hat{p},\hat{w}) \leq \bar{v}$. Let $\alpha \in [0,1]$ and define $(\tilde{p},\tilde{w}) = (\alpha p + (1-\alpha)\hat{p},\ \alpha w + (1-\alpha)\hat{w})$, so $(\tilde{p},\tilde{w})$ is a convex combination of (p,w) and $(\hat{p},\hat{w})$.

(WTS: $(\tilde{p},\tilde{w}) \in \{(p,w) : v(p,w) \leq \bar{v}\}$)

If $\tilde{p} \cdot x \leq \tilde{w}$, then $\alpha p \cdot x + (1-\alpha)\hat{p} \cdot x \leq \alpha w + (1-\alpha)\hat{w}$.

This implies either $\alpha p \cdot x \leq \alpha w$ or $(1-\alpha)\hat{p} \cdot x \leq (1-\alpha)\hat{w}$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$p \cdot x \leq w \qquad\qquad \hat{p} \cdot x \leq \hat{w}$$

$p \cdot x \leq w \Rightarrow u(x) \leq v(p,w)$ since $v(p,w)$ is max u subject to $p \cdot x \leq w$

$\hat{p} \cdot x \leq \hat{w} \Rightarrow u(x) \leq v(\hat{p},\hat{w})$ since $v(\hat{p},\hat{w})$ is max u subject to $\hat{p} \cdot x \leq \hat{w}$.

$v(p,w) \leq \bar{v}$ and $v(\hat{p},\hat{w}) \leq \bar{v}$ from above, so $u(x) \leq \bar{v}$.

Thus $\forall x \in \mathbb{R}^L_+$ such that $\tilde{p} \cdot x \leq \tilde{w}$ $\quad u(x) \leq \bar{v} \Rightarrow v(\tilde{p},\tilde{w}) \leq \bar{v} \Rightarrow$

$\{(p,w) : v(p,w) \leq \bar{v}\}$ is convex $\Rightarrow$ $v(p,w)$ is quasi-convex. ∎

(approach from MWG)

1. (b) $u^*(p_1, p_2, \ldots, p_L, w) = \max_{x_1, \ldots, x_L} u(x_1, x_2, \ldots, x_L)$ subject to $\sum_{i=1}^{L} p_i x_i = w$

(approach from Wyatt Brooks)

$u^*(p_1, p_2, \ldots, p_L, w) = \max_{x_1, \ldots, x_L} x_1 + V(x_2, x_3, \ldots, x_L)$ subject to $p_1 x_1 + \sum_{i=2}^{L} p_i x_i = w$

$$x_1 = \frac{w}{p_1} - \frac{1}{p_1} \sum_{i=2}^{L} p_i x_i$$

$$u^*(p_1, p_2, \ldots, p_L, w) = \max_{x_1, \ldots, x_L} \frac{w}{p_1} - \frac{1}{p_1} \sum_{i=2}^{L} p_i x_i + V(x_2, x_3, \ldots, x_L)$$

$$u^*(p_1, p_2, \ldots, p_L, w) = \frac{w}{p_1} + \underbrace{\max_{x_1, \ldots, x_L} \left(-\frac{1}{p} \sum_{i=2}^{L} p_i x_i + V(x_2, x_3, \ldots, x_L) \right)}_{\text{not dependent on } w}$$

-2

So... ?

What if $\frac{w}{p_1} - \frac{1}{p_1} \sum_{i=2}^{L} p_i x_i^* < 0$?

what if $x_n \in \{q \in X \mid r \sim q\}$?

2. <u>$a \Rightarrow b$</u>

Let $\{x_n\}$ be any sequence such that $\{x_n\} \subset \{q \in X \mid r \succ q\}$ and $\lim_{n\to\infty} x^n = x$. Define $\{r^n\} = (r, r, r, \dots)$. Note $\lim_{n\to\infty} r^n = r$, and $x_n \prec r_n$. Condition (a) $\Rightarrow x \prec r \Rightarrow x \in \{q \in X \mid r \succ q\}$. So $\{q \in X \mid r \succ q\}$ contains all its limit points $\Rightarrow$ $\underline{\{q \in X \mid r \succ q\}}$ is closed.

not the lower contour set!

-3

Let $\{v_n\}$ be any sequence such that $\{v_n\} \subset \{q \in X \mid r \prec q\}$ and $\lim_{n\to\infty} v^n = v$. Again define $\{r^n\} = (r, r, \dots)$. Note $\lim_{n\to\infty} r^n = r$ and $v_n \succ r_n$. Condition (a) $\Rightarrow v \succ r \Rightarrow v \in \{q \in X \mid r \prec q\}$. So $\{q \in X \mid r \prec q\}$ contains all its limit points $\Rightarrow$ $\{q \in X \mid r \prec q\}$ is closed.

<u>$b \Rightarrow a$</u> (heavily influenced by Hitoshi Tsujiyama)

For every $q \in X$, $\{p \in X : p \succeq q\}$ is closed and $\{p \in X : p \preceq q\}$ is closed. Let $\{x^n\}$ and $\{y^n\}$ be sequences in X such that $\lim_{n\to\infty} x^n = x$, $\lim y^n = y$ and $x^n \succeq y^n$. Suppose $x \prec y$.

Subclaim $\exists z \in X$ such that $x \prec z \prec y$.

Suppose $\nexists z$ such that $x \prec z \prec y$. Then $\forall z \in X$ $x \gtrsim z$ or $y \lesssim z$.

So $z \in \{p \in X : p \gtrsim y\} \cup \{p \in X : p \lesssim x\} = X = \mathbb{R}^L_+$

$x \prec y \Rightarrow \{p \in X : p \gtrsim y\} \cap \{p \in X\ p \lesssim x\} = \phi$

$\{p \in X : p \gtrsim x\}$ and $\{p \in X : p \lesssim y\}$ are closed. $\mathbb{R}^L_+$ contains in two closed, disjoint sets $\Rightarrow \mathbb{R}^L_+$ not connected. $\rightarrow\leftarrow$ so $\exists z \in X$ such that $x \prec z \prec y$.

$\{p \in X : p \succeq z\}$ closed $\Rightarrow \{p \in X : p \prec z\}$ open

$\{p \in X : p \lesssim z\}$ closed $\Rightarrow \{p \in X : p \succ z\}$ open

2. (cont'd)

$y \succ z \Rightarrow y \in \{p \in X : p \succ z\}$ which is open. $\exists\, \varepsilon_1 > 0$ $B(y, \varepsilon_1) \subset \{p \in X : p \succ z\}$

$x \prec z \Rightarrow x \in \{p \in X : p \prec z\}$ which is open. $\exists\, \varepsilon_2 > 0$ $B(x, \varepsilon_2) \subset \{p \in X : p \prec z\}$

Since $\lim_{n \to \infty} y_n = y$, $\exists N_1$ such that $\forall m > n$ $y_m \in B(y, \varepsilon_1) \subset \{p \in X : p \succ z\}$

Since $\lim_{n \to \infty} x_n = x$ $\exists N_2$ such that $\forall m > n$ $x_m \in B(x, \varepsilon_2) \subset \{p \in X : p \prec z\}$

Let $k > \max(N_1, N_2)$. $y_k \in \{p \in X : p \succ z\} \Rightarrow y_k \succ z$.

$x_k \in \{p \in X . p \prec z\} \Rightarrow x_k \prec z$.

By transitivity $x_k \prec y_k$. CONTRADICTION. So $y \succ x$.

3. $$V(p_1, p_2, w) = \frac{(w+p_1)^2}{4p_1p_2}$$

By Roy's Identity

$$q_1(p_1,p_2,w) = \frac{-\partial v/\partial p_1}{\partial v/\partial w} = \frac{\dfrac{-4p_1p_2\cdot 2(w+p_1)-(w+p_1)^2 4p_2}{(4p_1p_2)^2}}{\dfrac{2(w+p_1)}{4p_1p_2}} = \frac{-8p_1p_2-(w+p_1)4p_2}{8p_1p_2} = \frac{2p_1-(w+p_1)}{-2p_1} = \frac{w-p_1}{2p_1}$$

$$q_2(p_1,p_2,w) = \frac{-\partial v/\partial p_2}{\partial v/\partial w} = \frac{-\dfrac{(w+p_1)^2}{4p_1}\cdot\left(-\dfrac{1}{p_2^2}\right)}{\dfrac{2(w+p_1)}{4p_1p_2}} = \frac{w+p_1}{2p_2}$$

---

q_1, q_2 continuously differentiable

$$q_1(\lambda p_1, \lambda p_2, \lambda w) = \frac{\lambda w-\lambda p_1}{\lambda p_2} = \frac{\lambda(w-p_1)}{\lambda p_2} = \frac{w-p_1}{p_2} = q_1(p_1,p_2,w) \Rightarrow q_1 \text{ homogeneous of deg } 0$$

$$q_2(\lambda p_1, \lambda p_2, \lambda w) = \frac{\lambda w+\lambda p_1}{2\lambda p_2} = \frac{\lambda(w+p_1)}{\lambda(2p_2)} = \frac{w+p_1}{2p_2} = q_2(p_1,p_2,w) \Rightarrow q_2 \text{ homogeneous of deg } 0$$

$$p_1 \cdot q_1(p_1,p_2,w) + p_2 \cdot q_2(p_1,p_2,w) = p_1 \cdot \frac{w-p_1}{2p_1} + p_2\frac{w+p_1}{2p_2} = w$$

$$S = \begin{bmatrix} -\dfrac{w}{2p_1^2} + \dfrac{1}{2p_1}\cdot\dfrac{w-p_1}{2p_1} & \dfrac{1}{2p_2} + \dfrac{1}{2p_2}\,\dfrac{w-p_1}{2p_1} \\[2ex] 0 + \dfrac{1}{2p_1}\cdot\dfrac{w+p_1}{2p_2} & -\dfrac{w+p_1}{2p_2^2} + \dfrac{1}{2p_2}\,\dfrac{w+p_1}{2p_2} \end{bmatrix} = \begin{bmatrix} \dfrac{-w-p_1}{4p_1^2} & \dfrac{w+p_1}{4p_1p_2} \\[2ex] \dfrac{w+p_1}{4p_1p_2} & \dfrac{-w-p_1}{4p_2^2} \end{bmatrix} \Rightarrow S \text{ symmetric}$$

$$\det S = \left(\frac{-w-p_1}{4p_1^2}\right)\left(\frac{-w-p_1}{4p_2^2}\right) - \left(\frac{w+p_1}{4p_1p_2}\right)\left(\frac{w+p_1}{4p_1p_2}\right) = 0$$

$$\det\left[\frac{-w-p_1}{4p_1^2}\right] < 0$$

$\Rightarrow S$ negative semidefinite

✓

$\therefore$ by Theorem 12.1, there exists $\exists$ strictly increasing, strictly quasi-concave utility function such that q_1, q_2 are demand functions and $V(p_1,p_2,w)$ are its indirect utility function. ■

Ariel Zetlin-Jones
October 2, 2007
Notes on Problem Set 3

One note about your problem sets. In general, I encourage you to be more clear in your exposition of your proofs. Your understanding of a proof or a problem should be demonstrated by the precision of your solution. In general, even if your solutions are correct, if your methods are not clear, points may be deducted.

2 Problem 2

- Many of you proved existence of a z s.t. $y \succ z \succ x$ by showing that if such a z does not exist, then you can contradict the connectedness of $\mathbb{R}$. To achieve a contradiction, though, you need to prove three statements:

  1. $P(y)$ and $L(x)$ are closed
  2. $P(y)$ and $L(x)$ are nonempty
  3. $P(y) \cup L(x) = \mathbb{R}^L_+$
  4. $P(y) \cap L(x) = \emptyset$

  Almost everyone who used this method forgot number 2!

3 Problem 3

- If you provided a guess of the utility function, points were deducted from your score because you should be able to show how you would develop a guess on your own.
- If you use specific results or theorems, you should cite them directly (eg. "Werner Lecture Notes Theorem 12.1 says that it suffices to show x* satisfies the following three properties..."). Points were not deducted this time, but they will be in the future.

Economics 8101
Fall Semester 2007
Jan Werner

Problem Set 4
Due: Thursday, October 11

1. (10 points) Consider the set of all lotteries with three possible prizes z_1, z_2, and z_3, where $z_i \in \Re$ and $0 < z_1 < z_2 < z_3$. The set of lotteries can be identified with the unit simplex Δ in $\Re^3$ so that $\pi = (\pi_1, \pi_2, \pi_3) \in \Delta$ represents a lottery for which prize z_i obtains with probability π_i. Let $U\colon \Delta \to \Re$ be a utility function on Δ defined by

$$U(\pi) = \pi_2 z_2 + \sqrt{\pi_1 \pi_3}(z_1 + z_3).$$

Does U have an expected utility representation? Justify your answer.

2. (20 points) Consider the Allais paradox as described in MWG, Section 6.B.2. One specification of preferences on lotteries proposed in order to explain the Allais paradox has been the Rank-Dependent Expected Utility of J. Quiggin (1982). For a lottery with three outcomes z_1, z_2 and z_3, where $z_1 < z_2 < z_3$, with respective probabilities π_1, π_2, π_3, the rank-dependent expected utility is

$$v(z_1)f(\pi_1) + v(z_2)[f(\pi_1 + \pi_2) - f(\pi_1)] + v(z_3)[1 - f(\pi_1 + \pi_2)],$$

where v is a utility function on outcomes and $f : [0,1] \to [0,1]$ is an increasing function such that $f(0) = 0$ and $f(1) = 1$. Function f is called probability transformation function. Note that if $f(t) \geq t, \ \forall\ t$, then the weight assigned by RDEU to the worst outcome z_1 is higher than the probability of z_1 while the weight assigned to the best outcome z_3 is lower than the respective probability. This is a pessimistic probability transformation function. If $f(t) \leq t, \ \forall\ t$, then f is an optimistic transformation function.

(i) Explain whether the rank-dependent expected utility can be consistent with the pattern of preferences in the Allais paradox. If so, would the transformation function be pessimistic or optimistic?

(ii) Show that the rank-dependent expected utility satisfies the von Neumann-Morgenstern independence axiom if and only if $f(t) = t, \forall\ t$.

3. (10 points) Verify whether the following utility functions on $\Re^L_+$ are supermodular:
   (a) additively separable function, $u(x) = \sum_{i=1}^{L} v_i(x_i)$, where each function $v_i : \Re_+ \to \Re$ is strictly increasing and continuous (but need not be differentiable),
   (b) Leontief utility function, $u(x) = \min_i x_i$.
   (c) For $L = 2$, $u(x_1, x_2) = g(x_1)h(x_2)$ for strictly increasing functions $g, h : \Re_+ \to \Re$. Do not assume that g and h are differentiable.

Micro Problem Set #4

Brian Adams
10 Oct 2007

1. $U(\pi) = \pi_2 z_2 + \sqrt{\pi_1 \pi_3}\,(z_1 + z_3)$

Let $v_i = u(z_i)$ for $i=1,2,3$

Suppose there is an expected utility representation

$$\tilde{U}(\pi) = \pi_1 v_1 + \pi_2 v_2 + \pi_3 v_3$$

$U(\frac{1}{2},0,\frac{1}{2}) = 0 + \frac{1}{2}(z_1+z_3)$ $\qquad$ $\tilde{U}(\frac{1}{2},0,\frac{1}{2}) = \frac{1}{2}v_1 + \frac{1}{2}v_3$

$U(0,0,1) = 0$ $\qquad$ $\tilde{U}(0,0,1) = v_3$

$U(1,0,0) = 0$ $\qquad$ $\tilde{U}(1,0,0) = v_1$

$$U(\tfrac{1}{2},0,\tfrac{1}{2}) > U(1,0,0) \Rightarrow (\tfrac{1}{2},0,\tfrac{1}{2}) \succ (1,0,0) \Rightarrow \tilde{U}(\tfrac{1}{2},0,\tfrac{1}{2}) > \tilde{U}(1,0,0) \Rightarrow \frac{v_1}{2}+\frac{v_3}{2} > v_1 \Rightarrow v_3 > v_1$$

$$U(\tfrac{1}{2},0,\tfrac{1}{2}) > U(0,0,1) \Rightarrow (\tfrac{1}{2},0,\tfrac{1}{2}) \succ (0,0,1) \Rightarrow \tilde{U}(\tfrac{1}{2},0,\tfrac{1}{2}) > \tilde{U}(0,0,1) \Rightarrow \frac{v_1}{2}+\frac{v_3}{2} > v_3 \Rightarrow v_1 > v_3$$

/ CONTRADICTION, so $U(\pi)$ has no expected utility form.

2. (i) To satisfy the Allias Paradox given in MWG

$L_1 \succ L_1'$ $$V_0 F(0) + V_5 F(1) + V_{25}[1 - F(1)] > V_0 F(.01) + V_5[F(.9) - F(.01)] + V_{25}[1 - F(.9)]$$

$$V_5 > V_0 F(.01) + V_5[F(.9) - F(.01)] + V_{25}[1 - F(.9)] \qquad \text{(eq. 1)}$$

and $L_2' \succ L_2$ $$V_0 F(.9) + V_5[F(.9) - F(.9)] + V_{25}[1 - F(.9)] > V_0 F(.89) + V_5[F(1) - F(.89)] + V_{25}[1 - F(1)]$$

$$V_0 F(.9) + V_{25}[1 - f(.9)] > V_0 f(.89) + V_5[1 - F(.89)]$$

(eq 1) $$V_5 > V_0 F(.01) + V_5[F(.9) - f(.01)] + V_{25}[1 - f(.9)]$$

$$V_5[1 - F(.9) + F(.01)] - V_0 F(.01) > V_{25}[1 - F(.9)]$$

[illegible] eq 2) $$V_5[1 - f(.89)] - V_0[F(.9) - F(.89)] < V_{25}[1 - F(.9)]$$

$$V_5[1 - f(.89)] - V_0[F(.9) - F(.89)] < V_{25}[1 - F(.9)] < V_5[1 - F(.9) + F(.01)] - V_0 F(.01)$$

$$V_5 - V_0 f(.9) - (V_5 - V_0) F(.89) < V_{25}[1 - F(.9)] < V_5[1 - F(.9)] + F(.01)(V_5 - V_0) \qquad \text{(eq. 3)}$$

All algebra steps have been reversible, so eq 3 is a sufficient and necessary condition for satisfying MWG's Allias Paradox.

Suppose $F(t) = t$. Then both end terms in eq 3 equal each other, which is a contradiction. So f cannot be both pessimistic and optomistic.

Examples can be found for all other cases (using this intuition and a spreadsheet):

Example: Pessimistic

$V_0 = 1 \quad V_5 = 2 \quad V_{25} = 5$

$$F(t) = \begin{cases} 30t & \text{for } 0 \le t \le .01 \\ \frac{15t}{22} + \frac{129}{440} & .01 < t \le .89 \\ 5t - 71/20 & .89 < t \le .9 \\ \frac{1}{2}t + \frac{1}{2} & .9 < t \le 1 \end{cases}$$

$U(L_1) = 2 > 1.85 = U(L_1')$

$U(L_2') = 1.2 > 1.1 = U(L_2)$

2(i)

Example: Optimistic

$V_0 = 1 \quad V_5 = 4 \quad V_{25} = 4.1$

$$f(t) = \begin{cases} \frac{9t}{10} & \text{For } 0 \le t \le .01 \\ t - \frac{1}{1000} & .01 < t \le .89 \\ \frac{t}{10} + \frac{4}{5} & .89 < t \le .9 \\ \frac{11t}{10} - \frac{1}{10} & .9 < t \le 1 \end{cases}$$

should not change the v's...

$u(L_1) = 4 > 3.984 = u(L_1')$

$u(L_2') = 1.341 > 1.333 = u(L_2)$

I am not sure f can be optimistic.

~~Are you sure your~~

Example: Neither optimistic nor pessimistic

$V_0 = 1 \quad V_5 = 2 \quad V_{25} = 5$

Let

$F(0) = 0$

$F(.01) = .3$

$F(.89) = .88$

$F(.9) = .95$

$F(1) = 1$

$F(t)$ everywhere increasing

($F(t)$ linearly connecting those five dots would be a more specific example)

$u(L_1) = 2 > 1.85 = u(L_1')$

$u(L_2') = 1.2 > 1.12 = u(L_2)$

Depending on V, F can be optimistic, pessimistic, or neither

2. (ii)
CLAIM RDEU satisfies the von Neumann - Morgenstern independence axiom $\Leftrightarrow$

$F(t) = t \quad \forall t.$

Proof $(\Leftarrow)$

$F(t) = t \quad \forall t$ so $U(\pi) = \pi_1 v(z_1) + \pi_2 v(z_2) + (1-\pi_1-\pi_2) v(z_3)$

WTS: $\forall L, L', L'' \in \underset{\sim}{\mathcal{L}}$ and $\forall \alpha \in (0,1)$ $\quad L \succsim L' \Leftrightarrow \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''$

$$U(L) \geq U(L') \Leftrightarrow \ell_1 v(z_1) + \ell_2 v(z_2) + (1-\ell_1-\ell_2) v(z_3) \geq \ell_1' v(z_1) + \ell_2' v(z_2) + (1-\ell_1'-\ell_2') v(z_3)$$

$$\Leftrightarrow \alpha \ell_1 v(z_1) + \alpha \ell_2 v(z_2) + \alpha(1-\ell_1-\ell_2) v(z_3) \geq \alpha \ell_1' v(z_1) + \alpha \ell_2' v(z_2) + \alpha(1-\ell_1'-\ell_2') v(z_3)$$

$$\Leftrightarrow (1-\alpha)\ell_1'' v(z_1) + (1-\alpha)\ell_2'' v(z_2) + (1-\alpha)(1-\ell_1''-\ell_2'') v(z_3) + \alpha \ell_1 v(z_1) + \alpha \ell_2 v(z_2) + \alpha(1-\ell_1-\ell_2) v(z_3)$$

$$\geq (1-\alpha)\ell_1'' v(z_1) + (1-\alpha)\ell_2'' v(z_2) + (1-\alpha)(1-\ell_1''-\ell_2'') v(z_3) + \alpha \ell_1' v(z_1) + \alpha \ell_2' v(z_2) + \alpha(1-\ell_1'-\ell_2') v_3$$

$$\Leftrightarrow [\alpha \ell_1 + (1-\alpha)\ell_1''] v(z_1) + [\alpha \ell_2 + (1-\alpha)\ell_2''] v(z_2) + [\alpha(1-\ell_1-\ell_2) + (1-\alpha)(1-\ell_1''-\ell_2'')] v(z_3) \geq$$

$$[\alpha \ell_1' + (1-\alpha)\ell_1''] v(z_1) + [\alpha \ell_2' + (1-\alpha)\ell_2''] v(z_2) + [\alpha(1-\ell_1'-\ell_2') + (1-\alpha)(1-\ell_1''-\ell_2'')] v(z_3)$$

$$\Leftrightarrow U(\alpha L + (1-\alpha)L'') \geq U(\alpha L' + (1-\alpha)L'').$$

Thus $U(\pi)$ satisfies the von Neumann - Morgenstern independence axiom.

($\Rightarrow$) RDEU satisfies the von Neumann - Morgenstern independence axiom.

Suppose $\exists\, \hat{t} \in [0,1]$ such that $F(\hat{t}) \neq \hat{t}$

Let $L, L' \in \mathcal{L}$ such that $L \sim L'$ and $L \neq L'$. Let $\alpha = \frac{1}{2}$

By independence axiom $$v(z_1) F\left(\frac{l_1 + l_1''}{2}\right) + v(z_2)\left[F\left(\frac{l_1 + l_2 + l_1'' + l_2''}{2}\right) - F\left(\frac{l_1 + l_1''}{2}\right)\right] + v(z_3)\left[1 - F\left(\frac{l_1 + l_2 + l_1'' + l_2''}{2}\right)\right] =$$

$$v(z_1) F\left(\frac{l_1' + l_1''}{2}\right) + v(z_2)\left[F\left(\frac{l_1' + l_2' + l_1'' + l_2''}{2}\right) - F\left(\frac{l_1' + l_1''}{2}\right)\right] + v(z_3)\left[1 - F\left(\frac{l_1' + l_2' + l_1'' + l_2''}{2}\right)\right]$$

3. (a) $u(x) = \sum_{i=1}^{L} V_i(x_i)$ where $V_i : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing and continuous is supermodular

Proof (WTS $u(x \vee y) - u(x) \geq u(y) - u(x \wedge y)$)

Fix $x, y \in \mathbb{R}^L_+$. Denote $A = \{ i \in \{1,2,\dots,L\} \mid y_i > x_i \}$ and $B = \{ i \in \{1,2,\dots,L\} \mid y_i \leq x_i \}$. Note $A \cap B = \emptyset$ and $A \cup B = \{1,2,\dots,L\}$

$$u(x \vee y) = \sum_A V_i(y_i) + \sum_B V_i(x_i)$$

$$u(x) = \sum_A V_i(x_i) + \sum_B V_i(x_i)$$

$$u(x \wedge y) = \sum_A V_i(x_i) + \sum_B V_i(y_i)$$

$$u(y) = \sum_A V_i(y_i) + \sum_B V_i(y_i)$$

$$u(x \vee y) - u(x) = \sum_A V_i(y_i) - V_i(x_i)$$

$$u(y) - u(x \wedge y) = \sum_A V_i(y_i) - V_i(x_i)$$

So, $u(x \vee y) - u(x) = u(y) - u(x \wedge y)$.

Therefore, $u(x \vee y) - u(x) \geq u(y) - u(x \wedge y)$. ∎

3 (b) $u(x) = \min x_i$ is supermodular

Proof (WTS: $u(x \vee y) - u(x) \geq u(y) - u(x \wedge y)$)

Fix $x, y \in \mathbb{R}^L_+$

$u(x \vee y) = \min_i (\max(x_i, y_i))$

$u(x) = \min_i x_i$

$u(x \wedge y) = \min_i (\min(x_i, y_i)) = \min(u(x), u(y))$

$u(y) = \min_i y_i$

CASE $u(y) \geq u(x)$

$u(x \vee y) = \min_i (\max(x_i, y_i)) \geq \min_i y_i = u(y)$

$u(x \vee y) - u(x) \geq u(y) - u(x)$

$u(y) - u(x \wedge y) = u(y) - u(x)$

Thus $u(x \vee y) - u(x) \geq u(y) - u(x \wedge y)$

CASE $u(y) < u(x)$

$u(x \vee y) = \min_i (\max(x_i, y_i)) \geq \min_i x_i = u(x)$

$u(x \vee y) - u(x) \geq u(x) - u(x) = 0$

$u(y) - u(x \wedge y) = u(y) - u(y) = 0$

$u(x \vee y) - u(x) \geq u(y) - u(x \wedge y)$ ∎

3. (c) cont'd

CASE $x_1 \geq y_1, \quad x_2 < y_2$

$$u(x) = g(x_1)h(x_2)$$
$$u(y) = g(y_1)h(y_2)$$

$$u(x \vee y) = g(x_1)h(y_2)$$
$$u(x \wedge y) = g(y_1)h(x_2)$$

$h(y_2) > h(x_2)$ since h is strictly increasing

$$[g(x_1)-g(y_1)]h(y_2) \geq [g(x_1)-g(y_1)]h(x_2)$$ $g(x_1)-g(y_1) \geq 0$ since g increasing

$$g(x_1)h(y_2) - g(y_1)h(y_2) \geq g(x_1)h(x_2) - g(y_1)h(x_2)$$
$$u(x \vee y) - u(y) \geq u(x) - u(x \wedge y)$$
$$u(x \vee y) - u(x) \geq u(y) - u(x \wedge y)$$

CASE $x_1 < y_1 \quad x_2 \geq y_2$

$$u(x) = g(x_1)h(x_2)$$
$$u(y) = g(y_1)h(y_2)$$
$$u(x \vee y) = g(y_1)h(x_2)$$
$$u(x \wedge y) = g(x_1)h(y_2)$$

$h(y_2) \leq h(x_2)$ since h is increasing

$$[g(x_1)-g(y_1)]h(y_2) \geq [g(x_1)-g(y_1)]h(x_2)$$
$$g(x_1)h(y_2) - g(y_1)h(y_2) \geq g(x_1)h(x_2) - g(y_1)h(x_2)$$
$$u(x \wedge y) - u(y) \geq u(x) - u(x \vee y)$$
$$u(x \vee y) - u(x) \geq u(y) - u(x \wedge y)$$

For $\forall x, y \in \mathbb{R}^L_+$ $u(x \vee y) - u(x) \geq u(y) - u(x \wedge y) \Rightarrow u$ is supermodular. ∎

40

Economics 8101
Fall Semester 2007
Jan Werner

Problem Set 5
Due: Thursday, October 18

1. (15 points) Consider the following utility functions on state-contingent consumption with 3 states:

   (a) $u(c_1, c_2, c_3) = \min\{c_1 + 2c_2 + 3c_3, 2c_1 + c_2 + c_3\}$

   (b) $u(c_1, c_2, c_3) = \sqrt{c_1} + \sqrt{c_2} + 2c_3$.

   Show that (a) does not have state-separable representation. Show that (b) does not have expected utility representation for any probabilities of states.

2. (15 points) Suppose that there are two states, $s = 1, 2$, with equal probabilities $\frac{1}{2}$. An agent has an expected utility function $E[v(c)] = \frac{1}{2}v(c_1) + \frac{1}{2}v(c_2)$ with $v(c) = \ln(c)$, for $c > 0$.

   (i) Suppose that the agent has deterministic wealth $w > 0$ and faces risk $z = (z_1, z_2)$ such that $E(z) = 0$. Calculate risk compensation for $w = 12.5$ and $z = (-7.5, 7.5)$, that is, for risk of losing 7.5 in state 1 and winning 7.5 in state 2.

   (ii) Suppose that there are two assets: a risk-free asset with return in two states given by $\bar{r} = (1, 1)$, and a risky asset with return in two states given by $r = (2, 0.5)$. Find the optimal investment in the risky asset for the agent's initial wealth w, as a function of w. Is the optimal investment an increasing function of wealth w?

3. (10 points) Consider a monopolistic firm that produces two output goods, 1 and 2. In the market for good 1, the firm faces an inverse demand function $P_1(x_1)$ that associates market clearing price of good 1 with the demand x_1 for good 1. Similarly, there is an inverse demand function $P_2(x_2)$ in the market for good 2. The firm's cost of producing output bundle (z_1, z_2) is described by cost function with a parameter $\alpha > 0$. It takes the form $\alpha C(z_1, z_2)$ Assume that C is a strictly increasing function.

   The firm maximizes it's profit

   $$\max_{(z_1,z_2)\geq 0} P_1(z_1)z_1 + P_2(z_2)z_2 - \alpha C(z_1, z_2).$$

   (i) let $(z_1^*(\alpha), z_2^*(\alpha))$ denote the solution, assumed unique. Under what condition on C (supermodularity, submodularity, or none) is $(z_1^*(\alpha), z_1^*(\alpha))$ non-increasing in cost parameter α.

   Remarks: Function f is submodular if $-f$ is supermodular. Question (i) can be answered without assuming differentiability of functions P_1, P_2, and C. However, you are allowed to impose differentiability. If you do so (and only then), you should state the first-order conditions for profit maximization and comment on their sufficiency.

Brian Adams
18 Oct 2007

Microeconomics Problem Set 5

1. (a) $u(c_1, c_2, c_3) = \min\{c_1 + 2c_2 + 3c_3,\ 2c_1 + c_2 + c_3\}$ does not have a state-separable representation.

<u>Proof</u> by contradiction

Assume $u(c_1, c_2, c_3)$ has a state-separable representation. Then $\exists\ v_1: \mathbb{R} \to \mathbb{R}$, $v_2: \mathbb{R} \to \mathbb{R}$, and $v_3: \mathbb{R} \to \mathbb{R}$ such that $v_1(c_1) + v_2(c_2) + v_3(c_3) \geq v_1(c_1') + v_2(c_2') + v_3(c_3') \Leftrightarrow u(c_1, c_2, c_3) \geq u(c_1', c_2', c_3')$.

$$u(2,0,0) = 2 \leq 3 = u(0,0,3) \Rightarrow v_1(2) + v_2(0) + v_3(0) \leq v_1(0) + v_2(0) + v_3(3)$$

$$\Downarrow$$

$$v_1(2) + v_3(0) \leq v_1(0) + v_3(3)$$

$$\Downarrow$$

$$u(2,2,0) \leq u(0,2,3) \Leftarrow v_1(2) + v_2(2) + v_3(0) \leq v_1(0) + v_2(2) + v_3(3)$$

$$\Downarrow$$

$6 \leq 5$ CONTRADICTION, so $u(c_1, c_2, c_3)$ does not have a state-separable representation. ∎

(b) $u(c_1, c_2, c_3) = \sqrt{c_1} + \sqrt{c_2} + 2c_3$ does not have expected utility representation for any probability of state.

<u>Proof</u> by contradiction

Assume $\exists$ probability of states π_1, π_2, π_3 for which $u(c_1, c_2, c_3)$ has expected utility representation. Then $\exists\ v: \mathbb{R} \to \mathbb{R}$ such that

$$u(c_1, c_2, c_3) \geq u(c_1', c_2', c_3') \text{ IFF } \pi_1 v(c_1) + \pi_2 v(c_2) + \pi_3 v(c_3) \geq \pi_1 v(c_1') + \pi_2 v(c_2') + \pi_3 v(c_3')$$

$$u(1,1,0) = u(0,0,1) \Rightarrow u(1,1,0) \geq u(0,0,1) \Rightarrow (\pi_1 + \pi_2) v(1) + \pi_3 v(0) \geq (\pi_1 + \pi_2) v(0) + \pi_3 v(1)$$

$$u(1,1,0) \leq u(0,0,1) \Rightarrow (\pi_1 + \pi_2) v(1) + \pi_3 v(0) \leq (\pi_1 + \pi_2) v(0) + \pi_3 v(1)$$

$$(\pi_1 + \pi_2) v(1) + \pi_3 v(0) = (\pi_1 + \pi_2) v(0) + \pi_3 v(1)$$

$$(\pi_1 + \pi_2)(v(1) - v(0)) = \pi_3 (v(1) - v(0))$$

Subclaim $v(1) > v(0)$.

Suppose $v(0) \geq v(1)$. Then $(\pi_1 + \pi_2 + \pi_3) v(0) \geq (\pi_1 + \pi_2 + \pi_3) v(1) \Rightarrow$
$u(0,0,0) = 0 \geq 4 = u(1,1,1)$. CONTRADICTION, SO $v(1) > v(0)$

Thus $\pi_1 + \pi_2 = \pi_3$.

$$\pi_3 v(4) + \pi_3 v(0) \geq \pi_3 v(4) + \pi_3 v(0)$$

$$(\pi_1 + \pi_2) v(4) + \pi_3 v(0) \geq \pi_3 v(4) + (\pi_1 + \pi_2) v(0) \Rightarrow u(4,4,0) \geq u(0,0,4)$$

$$\Rightarrow 4 \geq 8 \text{ CONTRADICTION}$$

So, $\nexists$ a probability of states for which $u(c_1, c_2, c_3)$ has an expected utility representation. ∎

2 (i) $$v(w-\rho(w,z)) = \frac{1}{2}v(w+z_1) + \frac{1}{2}v(w+z_2)$$

$$\ln(12.5-\rho) = \frac{1}{2}\ln(5) + \frac{1}{2}\ln(20) = \ln(100^{1/2})$$

$$\rho = 12.5 - 10$$

$$\rho = 2.5$$

✓

2. (ii) Agent will solve $\max_a E[v((w-a)\bar{r} + ar)]$

$$\max_a \frac{1}{2}v((w-a)\bar{r}_1 + ar_1) + \frac{1}{2}v((w-a)\bar{r}_2 + ar_2)$$

$$\max_a \frac{1}{2}\ln((w-a)1 + 2a) + \frac{1}{2}\ln\left((w-a)\cdot 1 + \frac{a}{2}\right)$$

$$\max_a \frac{1}{2}\ln(w+a) + \frac{1}{2}\ln\left(w-\frac{a}{2}\right)$$

why is this valid?

FOC $$\frac{1}{2}\frac{1}{w+a} + \frac{1}{2}\frac{-1/2}{w-a/2} = 0$$

$$\frac{1}{2}\frac{1}{w+a} = \frac{1}{4}\frac{1}{w-a/2}$$

$$2\left(w-\frac{a}{2}\right) = w+a$$

$$2w - a = w + a$$

$$w = 2a$$

investment in risky asset $a = \frac{w}{2}$

✓

$a = \frac{w}{2}$ is an increasing function in w

3. (i) From Verner Lecture Notes Theorem 2'

If $\mathbb{R}^2_{++}$ is a lattice, $\pi(z_1,z_2,\alpha)$ is supermodular in (z_1,z_2) and has nonincreasing differences in $(z_1,z_2;\alpha)$, then $(z_1^*(\alpha), z_2^*(\alpha))$ is monotone nonincreasing in α.

Note that $\mathbb{R}^2_{++}$ is a lattice

CLAIM $\pi(z_1,z_2,\alpha)$ has nonincreasing differences in $(z_1,z_2;\alpha)$

Proof Let $(z_1',z_2') \geq (z_1,z_2)$ and $\alpha' \geq \alpha$.

C strictly increasing function $\Rightarrow C(z_1,z_2) < C(z_1',z_2') \Rightarrow C(z_1,z_2)-C(z_1',z_2')<0$

$\alpha' \geq \alpha \Rightarrow \alpha'[C(z_1,z_2)-C(z_1',z_2')] \leq \alpha[C(z_1,z_2)-C(z_1',z_2')]$

$$\Downarrow$$

$$\left[\begin{array}{l} P_1(z_1')z_1'+P_2(z_2')z_2' - \alpha' C(z_1',z_2') - P(z_1)z_1 - P(z_2)z_2 + \alpha' C(z_1,z_2) \leq \\ P_1(z_1')z_1'+P_2(z_2')z_2' - \alpha C(z_1',z_2') - P(z_1)z_1 - P(z_2)z_2 + \alpha C(z_1,z_2) \end{array}\right.$$

$$\Downarrow$$

$$\pi(z_1',z_2',\alpha') - \pi(z_1,z_2,\alpha') \leq \pi(z_1',z_2',\alpha) - \pi(z_1,z_2,\alpha)$$

$\Rightarrow \pi$ has nondecreasing differences in $(z_1,z_2;\alpha)$. ■

CLAIM C submodular in $(z_1,z_2) \Leftrightarrow \pi$ supermodular in (z_1,z_2)

π supermodular in $(z_1,z_2) \Leftrightarrow \pi(x \vee y,\alpha) - \pi(x,\alpha) \geq \pi(y,\alpha) - \pi(x \wedge y,\alpha)$

$$\Updownarrow$$

$$\pi(x \vee y,\alpha) + \pi(x \wedge y,\alpha) \geq \pi(y,\alpha) + \pi(x,\alpha)$$

$$\Updownarrow$$

$$P_1(\max(x_1,y_1))\max(x_1,y_1) + P_1(\min(x_1,y_1))\min(x_1,y_1) + P_2(\max(x_2,y_2))\max(x_2,y_2) + P_2(\min(x_2,y_2))\min(x_2,y_2) - \alpha C(x \vee y) - \alpha C(x \wedge y)$$

$$\geq P_1(y_1)y_1 + P_1(x_1)x_1 + P_2(x_2)x_2 + P_2(y_2)y_2 - \alpha C(x_1,x_2) - \alpha C(y_1,y_2)$$

Note that $P_i(\max(x_i,y_i))\max(x_i,y_i) + P_i(\min(x_i,y_i))\min(x_i,y_i) = P_i(y_i)y_i + P_i(x_i)x_i$

$$-\alpha C(x \vee y) - \alpha C(x \wedge y) \geq -\alpha C(x_1,x_2) - \alpha C(y_1,y_2)$$

Note $\alpha > 0$ 10

$$\Updownarrow$$

$$-C(x \vee y) - C(x \wedge y) \geq -C(x_1,x_2) - C(y_1,y_2)$$

$$\Updownarrow$$

$-C(x \vee y) + C(x) \geq -C(y) + C(x \wedge y) \Rightarrow C$ submodular ■

So C submodular $\Rightarrow (z_1^*(\alpha), z_2^*(\alpha))$ is nonincreasing in α.

Economics 8101 **Problem Set 6**
Fall Semester 2007 **Due: Wednesday, October 24, 1 pm.**
Jan Werner

1. (15 points) There are two states, $s = 1, 2$, with respective probabilities $\pi_1 = 1/3$ and $\pi_2 = 2/3$. Consider two state-contingent consumption plans; $z = (8, 2)$ and $y = (2, 5)$. Note that $E(z) = E(y)$.
   (a) Does y dominate z in the sense of the First-Order Stochastic Dominance? Justify your answer.
   (b) Is z more risky than y (that is, does y dominate z in the sense of the Second-Order Stochastic Dominance)? Justify your answer.

2. (15 points) Consider the optimal portfolio choice problem with one risky asset and a risk-free asset. Suppose that the agent has expected utility function and that her von Neumann-Morgenstern utility function has constant absolute risk aversion equal to $\alpha > 0$. Show that the optimal investment in the risky asset does not depend on the agent's wealth w and is strictly decreasing in risk aversion α.

3. (10 points) Consider two random variables y and z with the same expectations, $E(y) = E(z)$.
   Show that, if z is more risky than y and y is more risky than z, then y and z have the same distribution, i.e., $F_y(t) = F_z(t)$ for every t. You may assume that y and z take only finitely many values.

Brian Adams
18 Oct 2007

Micro Problem Set 6

1. (a)

$$y = \begin{cases} 2 & \text{with probability } 1/3 \\ 5 & p = 2/3 \end{cases} \qquad F_y(t) = \begin{cases} 0 & \text{for } 0 \le t \le 2 \\ 1/3 & \text{for } 2 < t \le 5 \\ 1 & \text{for } t > 5 \end{cases}$$

$$z = \begin{cases} 2 & \text{with } p = 2/3 \\ 8 & p = 1/3 \end{cases} \qquad F_z(t) = \begin{cases} 0 & \text{for } 0 \le t \le 2 \\ 2/3 & \text{for } 2 < t \le 8 \\ 1 & \text{for } t > 8 \end{cases}$$

$$F_y(7) = 1 > \tfrac{2}{3} = F_z(7)$$

$$F_y(3) = \tfrac{1}{3} < \tfrac{2}{3} = F_z(3)$$

(b)

$$\left.\begin{aligned} E(y) &= \tfrac{1}{3}2 + \tfrac{2}{3}\cdot 5 = 4 \\ E(z) &= \tfrac{2}{3}2 + \tfrac{1}{3}8 = 4 \end{aligned}\right\} \Rightarrow E(y) = E(z)$$

CLAIM y defined above second-order stochastically dominates z defined above.

Proof (WTS: $E[v(y)] \ge E[v(z)]$ $\forall$ nondecreasing, concave v.)

Let v be a nondecreasing, concave function. v concave $\Rightarrow \tfrac{1}{2}v(2) + \tfrac{1}{2}v(8) \le v(\tfrac{1}{2}2 + \tfrac{1}{2}8) = v(5)$

$\Rightarrow \tfrac{1}{3}v(2) + \tfrac{1}{3}v(8) \le \tfrac{2}{3}v(5) \Rightarrow \tfrac{2}{3}v(2) + \tfrac{1}{3}v(8) \le \tfrac{1}{3}v(2) + \tfrac{2}{3}v(5) \Rightarrow E(v(y)) \ge E(v(z))$

For any nondecreasing, concave v. By Theorem 17.4 in Verner's Lecture Notes, y second-order stochastically dominates z. ■

$E(y) = E(z)$ and y second-order stochastically dominates z $\Rightarrow$ z is more risky than y.

2. Agent solves $\max_a E\left[v(w\bar{r}+(r-\bar{r})a)\right]$

what does this mean? r is a random variable!

-2

Assume $r > \bar{r}$. If $r \leq \bar{r}$, $a^*=0$ for the risk averse agent.

FOC $$E\left[v'(w\bar{r}+(r-\bar{r})a^*)(r-\bar{r})\right]=0 \qquad (1)$$

why is this valid?

$\frac{\partial FOC}{\partial w}$ $$E\left[v''(w\bar{r}+(r-\bar{r})a^*)(r-\bar{r})\left(\bar{r}+(r-\bar{r})\frac{\partial a^*}{\partial w}\right)\right]=0$$

-1

$$-E\left[v''(w\bar{r}+(r-\bar{r})a^*)(r-\bar{r})\right]\bar{r} = E\left[v''(w\bar{r}+(r-\bar{r})a^*)(r-\bar{r})^2\right]\frac{\partial a^*}{\partial w}$$

$$\frac{\partial a^*}{\partial w} = \frac{-E\left[v''(w\bar{r}+(r-\bar{r})a^*)(r-\bar{r})\right]\bar{r}}{E\left[v''(w\bar{r}+(r-\bar{r})a^*)(r-\bar{r})^2\right]} \qquad (2)$$

Since $\alpha = \frac{-v''(x)}{v'(x)} > 0 \ \forall x$ and $v'(x)>0$, $v''(x)<0 \ \forall x$

$(r-\bar{r})^2 > 0$ so the denominator of (2) is negative.

$$\alpha = \frac{-v''(w\bar{r}+(r-\bar{r})a^*)}{v'(w\bar{r}+(r-\bar{r})a^*)}$$

$$-\alpha v'(w\bar{r}+(r-\bar{r})a^*) = v''(w\bar{r}+(r-\bar{r})a^*)$$

$$-\alpha v'(w\bar{r}+(r-\bar{r})a^*)(r-\bar{r}) = v''(w\bar{r}+(r-\bar{r})a^*)(r-\bar{r})$$

$$-\alpha \underbrace{E\left[v'(w\bar{r}+(r-\bar{r})a^*)(r-\bar{r})\right]}_{0 \text{ by Eq. 1}} = E\left[v''(w\bar{r}+(r-\bar{r})a^*)(r-\bar{r})\right]$$

$$E\left[v''(w\bar{r}+(r-\bar{r})a^*)\right] = 0 \Rightarrow 0 = \frac{-E\left[v''(w\bar{r}+(r-\bar{r})a^*\right]}{E\left[v''(w\bar{r}+(r-\bar{r})a^*)(r-\bar{r})^2\right]} = \frac{\partial a^*}{\partial w}$$ ✓

(cont'd)

2. (cont'd)

$$\alpha = -\frac{V''(x)}{V'(x)}$$

$$V''(x) + \alpha V'(x) = 0$$

$$V(x) = -Ce^{-\alpha x} + D$$

Agent solves $\max_a E\left[Ce^{-\alpha(w\bar{r}+(r-\bar{r})a)} + D\right]$

equivalent to $\max_a E\left[-e^{-\alpha(w\bar{r}+(r-\bar{r})a)}\right]$

FOC $$E\left[+\alpha(r-\bar{r})\, e^{-\alpha(w\bar{r}+(r-\bar{r})a)}\right] = 0 \qquad (1)$$

$\frac{\partial FOC}{\partial \alpha}$ $$E\left[+(r-\bar{r})e^{-\alpha(w\bar{r}+(r-\bar{r})a^*)} - \alpha(r-\bar{r})e^{-\alpha(w\bar{r}+(r-\bar{r})a^*)}\left(w\bar{r}+(r-\bar{r})a^* + \alpha(r-\bar{r})\frac{\partial a^*}{\partial \alpha}\right)\right] = 0$$

$$\underbrace{E\left[+(r-\bar{r})e^{-\alpha(w\bar{r}+(r-\bar{r})a^*)}\right]}_{0 \text{ by eq } (1)} = E\left[\alpha(r-\bar{r})e^{-\alpha(w\bar{r}+(r-\bar{r})a^*)}\left(w\bar{r}+(r-\bar{r})a^* + \alpha(r-\bar{r})\frac{\partial a^*}{\partial \alpha}\right)\right]$$

$$E\left[\underbrace{\alpha}_{\text{positive}}\underbrace{e^{-\alpha(w\bar{r}+(r-\bar{r})a^*)}}_{\text{positive}}\underbrace{(r-\bar{r})\left(w\bar{r}+(r-\bar{r})a^*\right)}_{\text{positive}}\right] = E\left[\underbrace{-\alpha^2}_{\text{negative}}\underbrace{(r-\bar{r})^2}_{\text{positive}}\underbrace{e^{-\alpha(w\bar{r}+(r-\bar{r})a^*)}}_{\text{positive}}\right]\frac{\partial a^*}{\partial \alpha}$$

$$\frac{\partial a^*}{\partial \alpha} < 0$$

3. y is more risky than z $\xrightarrow[\text{def 17.3}]{\text{by VLN}}$ $\int_a^w F_z(t)\,dt \le \int_a^w F_y(t)\,dt \quad \forall w \in [a,b]$

z is more risky than y $\Rightarrow$ $\int_a^w F_y(t)\,dt \le \int_a^w F_z(t)\,dt \quad \forall w \in [a,b]$

Thus, $\int_a^w F_z(t)\,dt = \int_a^w F_y(t)\,dt \quad \forall w \in [a,b]$

$$\int_a^q F_z(t)\,dt = \int_a^q F_y(t)\,dt \quad \forall q \in [a,b]$$

Subtraction gives $\int_q^w F_z(t)\,dt = \int_q^w F_y(t)\,dt \quad \forall q,w \in [a,b]$

Since $\underline{F_z, F_y \text{ continuous}}$ on $[a,b]$, $F_z(t) = F_y(t) \quad \forall t \in [a,b]$ by lemma below. ■

you are adding this assumption!

Lemma Let $f(x), g(x)$ be continuous real-valued functions on $[a,b]$.
Then $\int_q^w f(t)\,dt = \int_q^w g(t)\,dt \quad \forall q,w \in [a,b] \Rightarrow f(x) = g(x) \quad \forall x \in [a,b]$.

Proof by contradiction.

Let $\int_q^w f(x)\,dx = \int_q^w g(x) \quad \forall q,w$ with f, g continuous on $[a,b]$.

Suppose $f(z) \neq g(z)$ for some $z \in [a,b]$. Then for some $2\varepsilon > 0$ $|f(z) - g(z)| > 2\varepsilon$. WLOG assume $f(z) > g(z)$. Since f, g continuous, $f(y) > f(z) - \varepsilon$ and $g(y) < g(z) + \varepsilon$ where $y \in (z-\delta, z+\delta)$.
Then $f(y) > g(y) \quad \forall y \in (z-\delta, z+\delta) \Rightarrow \int_{z-\delta/2}^{z} f(t)\,dt > \int_{z-\delta/2}^{z} g(t)\,dt$. (or $\int_z^{z+\frac{\delta}{2}}$

CONTRADICTION, so $f(x) = g(x) \quad \forall x \in [a,b]$. ■