Introduction to Methods of Monotone Comparative Statics

A new approach to comparative statics of constrained optimization problems has been proposed in a paper "Monotone Comparative Statics" by P. Milgrom and Ch. Shannon, *Econometrica* (1994). The approach is based on mathematical theories of supermodularity and vector lattices developed earlier by D.M. Topkis (see the book by Topkis (1998)). A recent contribution to these methods of comparative statics appears in a paper "The Comparative Statics of Constrained Optimization Problems" by J. Quah, *Econometrica* (2007).

I. Normal Demand for Supermodular Concave Utility

First, we present an example of the use of supermodularity in comparative statics of consumer's demand. The presentation follows arguments developed by John Quah.

For two vectors $x, y \in I\!R^L$, we use $x \vee y$ to denote the **supremum** and $x \wedge y$ to denote the **infimum** of x and y. That is,

$$x \vee y = (\max\{x_1, y_1\}, \max\{x_2, y_2\}, \ldots, \max\{x_L, y_L\}),$$

$$x \wedge y = (\min\{x_1, y_1\}, \min\{x_2, y_2\}, \ldots, \min\{x_L, y_L\}),$$

Operations $\vee$ and $\wedge$ are called **lattice operations**.

Note that

$$x + y = x \vee y + x \wedge y, \tag{1}$$

Let $u : \mathbb{R}^L_+ \to \mathbb{R}$ be a utility function. We say that u is **supermodular** if

$$u(x \vee y) - u(x) \geq u(y) - u(x \wedge y), \tag{2}$$

for every $x, y \in \mathbb{R}^L_+$.

Supermodularity is a form of complementarity among all goods. Loosely speaking, it states that the function has increasing differences (see Section II). This can be seen in Figure 1 with two goods. For twice-differentiable function u, supermodularity is equivalent to $\frac{\partial^2 u}{\partial x_i \partial x_j}(x) \geq 0$ for every $i \neq j$, every $x \in \mathbb{R}^L_+$, see Topkis (1998).

Theorem 1: *Suppose that utility function $u : \mathbb{R}^L_+ \to \mathbb{R}$ is supermodular, strictly concave and locally nonsatiated. Then the Walrasian demand function $x^*(\cdot)$ is an increasing function of income w, that is*

$$w' \geq w \quad \text{implies} \quad x^*(p, w') \geq x^*(p, w),$$

for every $w, w' > 0$ and $p >> 0$. In other words, the demand for every good is normal.

Proof: We first prove the following

Lemma 1: *If u is concave and supermodular, then*

$$u(\lambda[x \vee y] + (1-\lambda)y) - u(y) \geq u(x) - u(\lambda[x \wedge y] + (1-\lambda)x), \tag{3}$$

for every $x, y \in \mathbb{R}^L_+$ and $0 \leq \lambda \leq 1$.

Proof of Lemma 1: The following two inequalities follow from concavity of u

$$u(\lambda[x \vee y] + (1-\lambda)y) \geq \lambda u(x \vee y) + (1-\lambda)u(y), \tag{4}$$

$$u(\lambda[x \wedge y] + (1-\lambda)x) \geq \lambda u(x \wedge y) + (1-\lambda)u(x) \tag{5}$$

Inequality (2) holds, too, because u is supermodular. If we multiply (2) by λ and sum the resulting inequality side-by-side with (4) and (5), and do a little algebra, we obtain (3).

We return to the proof of Theorem 1. Of course, we only need to consider $w' > w$. Let $y = x^*(p,w)$ and $x = x^*(p,w')$. Since u is l.n.s., we have $py = w$ and $px = w'$. Clearly, $p[x \wedge y] \leq w$. Since $px > w$, there exists $0 \leq \lambda < 1$ such that $p(\lambda[x \wedge y] + (1-\lambda)x) = w$. Denote $\lambda[x \wedge y] + (1-\lambda)y$ by $\underline{z}_\lambda$ and $\lambda[x \vee y] + (1-\lambda)x$ by $\bar{z}^\lambda$. Since $\underline{z}_\lambda + \bar{z}^\lambda = x + y$ (this follows from (1)), we have $p\bar{z}^\lambda = w'$.

Since y is the unique utility maximizer at w and $p\underline{z}_\lambda = w$, we have $u(y) \geq u(\underline{z}_\lambda)$. Lemma 1 implies that $u(\bar{z}^\lambda) \geq u(x)$. Since x is the unique utility maximizer at w' and $p\bar{z}^\lambda = w'$, it must be $\bar{z}^\lambda = x$. Then also $\underline{z}_\lambda = y$. It can be shown (see Figure 1) that $\bar{z}^\lambda = x$ if and only if $x = x \vee y$. Similarly, $\underline{z}_\lambda = y$ if and only if and $y = x \wedge y$. But if $x = x \vee y$ and $y = x \wedge y$, then $y \leq x$. This concludes the proof.

Remarks:

• Theorem 1 was first proved by Professor John Chipman in (1977). Chipman did not use supermodularity, but instead assumed that $\frac{\partial^2 u}{\partial x_i \partial x_j} \geq 0$ for every $i \neq j$,

• Theorem 1 holds for u concave and supermodular, too. This form requires, however, a way of comparing possibly multivalued demands, see Quah (2007).

II. Monotone Comparative Statics

Let X be a subset of $I\!R^n$. We assume that X is either the entire space $I\!R^n$, or the positive orthant $I\!R^n_+$. Let T be a subset of $I\!R^m$.

For a function $f : X \times T \to I\!R$ and a set $S \subset X$, consider the following maximization problem

$$\max_x f(x,t) \tag{6}$$

$$\text{subject to} \qquad x \in S.$$

We denote the **set of solutions** by $\varphi(t)$. That is,

$$\varphi(t) = \text{argmax}_{x \in S} f(x,t). \tag{7}$$

Monotone comparative statics is concerned with conditions on function f and set S so that correspondence $\varphi(t)$ is **monotone nondecreasing** in t. That is

$$\text{if } t \leq t', \text{ then } \varphi(t) \leq \varphi(t'). \tag{8}$$

The meaning of the inequality between sets on the right-hand side of (8) needs an explanation. It is the **strong set order**: for every $x \in \varphi(t)$ and $x' \in \varphi(t')$, it holds $x \wedge x' \in \varphi(t)$ and $x \vee x' \in \varphi(t')$.

If $\varphi(t)$ nad $\varphi(t')$ are singleton sets, then the strong set order coincides with the usual order on vectors, so the inequality in (8) is an inequality between two vectors.

Set $S \subset \mathbb{R}^n$ is said to be a **lattice** if

$$x \wedge x' \in S \quad \text{and} \quad x \vee x' \in S, \tag{9}$$

for every $x, x' \in S$. Note that the set X - being $\mathbb{R}^n$ or $\mathbb{R}^n_+$ - is a lattice.

The main result is

Theorem 2 (Topkis): *If S is a lattice, f is supermodular in x, and has nondecreasing differences in $(x; t)$, then $\varphi(t)$ is monotone nondecreasing in t.*

Function $f : X \times T \to \mathbb{R}$ has **nondecreasing differences** in $(x; t)$ if for every $x' \geq x$,

$$f(x', t) - f(x, t) \tag{10}$$

is monotone nondecreasing in t. That is, if $t' \geq t$, then

$$f(x', t') - f(x, t') \geq f(x', t) - f(x, t). \tag{11}$$

The condition that f is supermodular in x can be characterized in terms of nondecreasing differences among pairs of coordinates of x (see Section I). This is a bit cumbersome; it can be found in Topkis (1998).

A useful result is the following

Proposition 3: *Let $f : I\!R^n \times I\!R^m \to I\!R$ be twice differentiable on an interval (a, b). Then f has nondecreasing differences in $(x; t)$ if and only if*

$$\frac{\partial^2 f}{\partial x_i \partial t_k}(x, t) \geq 0 \tag{12}$$

for every i, k and every (x, t); and f is supermodular in x if and only if

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x, t) \geq 0 \tag{13}$$

for every i, j, $i \neq j$ and every (x, t).

To illustrate applications of Theorem 2, consider the problem of profit maximization for a production function (Section 4, Course Handouts).

$$\max_{x_1 \geq 0, \ldots, x_n \geq 0} [q f(x_1, \ldots, x_n) - \sum_{i=1}^{n} w_i x_i]$$

where q is the price of output and $w = (w_1, \ldots, w_n)$ is a vector of prices of n inputs. Production function f is assumed strictly increasing.

We only consider changes in output price q. It is easy to see that the objective function $F(x, q) = q f(x) - wx$ has nondecreasing differences in $(x; q)$. If production function f is supermodular, then it follows from Theorem 2 that the input demand $x^*(q)$ is monotone nondecreasing in output price q.

Supermodularity of production function f says that inputs are (weak) complements in the production process.

When is $\varphi(t)$ monotone nonincreasing in t?

Monotone nonincreasing φ means that

$$\text{if } t \leq t', \text{ then } \varphi(t) \geq \varphi(t'), \tag{14}$$

where the inequality on the right-hand side is in the strong set order.

A counterpart of Theorem 2 for monotone nonincreasing solutions to maximization problem (6) is

Theorem 2': *If S is a lattice, f is supermodular in x and has nonincreasing differences in $(x;t)$, then $\varphi(t)$ is monotone nonincreasing in t.*

Note that only the monotonicity of differences gets reversed. The assumption of supermodularity remains unchanged.

Function $f : X \times T \to I\!R$ has nonincreasing differences in $(x;t)$ if for every $x' \geq x$ and $t' \geq t$,

$$f(x',t') - f(x,t') \leq f(x',t) - f(x,t). \tag{15}$$

For twice differentiable function f, (15) is equivalent to

$$\frac{\partial^2 f}{\partial x_i \partial t_k}(x,t) \leq 0 \tag{16}$$

for every i, k and every (x,t).

Proof of Theorem 2: Let $x \in \varphi(t)$ and $x' \in \varphi(t')$. First, we prove that $x \vee x' \in \varphi(t')$. Supermodularity of f implies that

$$f(x \vee x', t') \geq f(x', t') + f(x, t') - f(x \wedge x', t') \tag{14}$$

Nondecreasing differences of f imply that

$$f(x, t') - f(x \wedge x', t') \geq f(x, t) - f(x \wedge x', t) \tag{15}$$

The lattice property of S implies that $x \wedge x' \in S$ and hence that

$$f(x, t) \geq f(x \wedge x', t), \tag{16}$$

because $x \in \varphi(t)$. Combining (14), (15) and (16) we obtain

$$f(x \vee x', t') \geq f(x', t') \tag{17}$$

Since $x \vee x' \in S$ and $x' \in \varphi(t')$, (17) implies that $x \vee x' \in \varphi(t')$.

The argument for $x \wedge x' \in \varphi(t)$ is similar:

$$f(x \wedge x', t) \geq f(x, t) + f(x', t) - f(x \vee x', t) \geq$$

$$\geq f(x, t) + f(x', t') - f(x \vee x', t') \geq f(x, t).$$

Since $x \wedge x' \in S$, it follows that $x \wedge x' \in \varphi(t)$.

References:

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