Micro 2007 Fall Q I.1

8101

7, 8

- Homotheticity and Utility representation
- $\succsim$ is homothetic if
  $\forall x, x' \in \mathbb{R}^L_+$ $\forall \lambda > 0$, if $x \sim x'$, then $\lambda x \sim \lambda x'$.

$\boxed{\text{Claim}}$ Let consumption set $X = \mathbb{R}^L_+$.
Let $\succsim$ be reflexive, transitive, complete, continuous and strictly increasing (i.e. strongly monotone)
Then, $\succsim$ is homothetic iff $\exists$ utility representation u s.t. u is HD1.

pf

($\Leftarrow$) $\forall x, x' \in \mathbb{R}^L_+$, $\forall \lambda > 0$,

$$\begin{aligned} x \sim x' &\Leftrightarrow u(x) = u(x') \\ &\Leftrightarrow \lambda u(x) = \lambda u(x') \\ &\Leftrightarrow u(\lambda x) = u(\lambda x') \quad (\because \text{HD1}) \\ &\Leftrightarrow \lambda x \sim \lambda x' \end{aligned}$$ □

($\Rightarrow$)

$\boxed{\text{Thm}}$ If $\succsim$ on $X = \mathbb{R}^L_+$ is reflexive, transitive, complete, continuous and strictly increasing, then $\succsim$ has a utility representation.

pf Below. □

Suppose $\succsim$ is homothetic.
In the proof of the thm, we have
$\forall x \in \mathbb{R}^L_+$, $\exists! \alpha_x \geq 0$ s.t. $x \sim \alpha_x e$.
By the homotheticity, $\forall \lambda > 0$,
$$\lambda x \sim \lambda \alpha_x e.$$
Thus, by def. of u,
$$\begin{aligned} u(\lambda x) &= \lambda \alpha_x \\ &= \lambda u(x). \end{aligned}$$
Therefore, u is HD1 □ □

pf of thm

Let $e = \{1, 1, \cdots, 1\} \in \mathbb{R}^L_+$.

Then define $A^+ = \{\alpha \in \mathbb{R}_+ \mid \alpha e \succsim x\}$,

$A^- = \{\alpha \in \mathbb{R}_+ \mid x \succsim \alpha e\}$.

Prove that $\begin{cases} \text{(i)} & A^+ \neq \phi,\ A^- \neq \phi. \\ \text{(ii)} & A^+, A^- \text{ closed} \\ \text{(iii)} & A^+ \cup A^- = \mathbb{R}_+ \end{cases}$

(i) Consider $\bar{\alpha} = \max_{1 \leq \ell \leq L} x_\ell$. Then $\bar{\alpha} e \succsim x$ by strict increasingness.

Thus $\bar{\alpha} \in A^+$.

Also $0 \in A^-$ since $x \succsim 0$ by strict increasingness.

(ii) Let $L = \{\alpha e \mid \alpha \geq 0\}$. L is a closed set in $\mathbb{R}^L_+$.

Since $P(x) \equiv \{y \in \mathbb{R}^L_+ \mid y \succsim x\}$ is closed by the continuity

Thus $P(x) \cap L$ is closed, hence A^+ is also closed.

Same for A^-

(iii) Obvious.

Then, since $\mathbb{R}^L_+$ is connected, $A^+ \cap A^- \neq \phi$.

Take $\alpha_x \in A^+ \cap A^-$.

Prove that α_x is unique. Sps $\exists \bar{\alpha}_x \in A^+ \cap A^-$ and $\bar{\alpha}_x \neq \alpha_x$.

Then, $\alpha_x e \sim x$ and $\bar{\alpha}_x e \sim x$. Thus $\alpha_x e \sim \bar{\alpha}_x e$

Hence $\alpha_x = \bar{\alpha}_x$ by def of e. Contradiction

Define $u(x)$ by $u(x) = \alpha_x$.

Prove that u is indeed a utility representation of $\succsim$.

If $x \succsim x'$, then $\alpha_x e \succsim \alpha_{x'} e$.

If $\alpha_x < \alpha_{x'}$, then $\alpha_{x'} e \succ \alpha_x e$ by the str. increasingness.

Thus $\alpha_x \geq \alpha_{x'}$ and hence $u(x) \geq u(x')$.

If $u(x) \geq u(x')$, then $\alpha_x \geq \alpha_{x'}$.

By the strictly increasingness, $\alpha_x e \succsim \alpha_{x'} e$ and hence $x \succsim x'$.

Therefore u is a utility representation of $\succsim$. Q.E.D.

Micro 2007 Fall Q I. 2

8101

17

- Second-order stochastic dominance
- $Y + Z$ is more risky than Y. $(E[Z \mid Y = y] = 0)$
- $Y + 2Z$ 〃 $Y + Z$.

(a) $Y + Z$ is more risky than Y.

Def.
Y is more risky than Z when Z SOSD Y and $E(Y) = E(Z)$

$E(x) = E[E(x \mid y)]$

pf) Since

$$
\begin{aligned}
E(Y+Z) &= E[E(Y+Z \mid Y)] \\
&= E[Y + E(Z \mid Y)] \\
&= E(Y),
\end{aligned}
$$

★ → it suffices to show that for any nondecreasing concave continuous v,

$$E[v(Y+Z)] \le E[v(Y)].$$

$$
\begin{aligned}
E[v(Y+Z)] &= E\{E[v(Y+Z) \mid Y]\} \\
&= \pi_1 E[v(Y+Z) \mid Y = y_1] + \pi_2 E[v(Y+Z) \mid Y = y_2] \\
&\le \pi_1 v[E(Y+Z \mid Y = y_1)] + \pi_2 v[E(Y+Z \mid Y = y_2)] \\
&\qquad (\because v \text{ is concave}, + \circledast) \\
&= \pi_1 v(y_1 + E(Z \mid Y = y_1)) + \pi_2 v(y_2 + E(Z \mid Y = y_2)) \\
&= \pi_1 v(y_1) + \pi_2 v(y_2) \\
&= E(v(Y))
\end{aligned}
$$

$$
\begin{aligned}
\circledast \quad E[v(Y+Z) \mid Y] &= \int v(Y+Z) f(Z \mid Y)\, dZ \\
&\le v\left[\int (Y+Z) f(Z \mid Y)\, dZ\right] \quad (\text{by concavity}) \\
&= v[E(Y+Z \mid Y)]
\end{aligned}
$$

(b) $Y + 2Z$ is more risky than $Y + Z$.

pf) Since

$$
\begin{aligned}
E(Y+2Z) &= E[E(Y+2Z \mid Y)] \\
&= E[Y + E(Z \mid Y)] \\
&= E(Y),
\end{aligned}
$$

it suffices to show that for any nondecreasing concave continuous v

$$E[v(Y+2Z)] \le E[v(Y+Z)].$$

Since we have

$$E[Y+2Z] = \pi_1 E[v(y_1+2z)] + \pi_2 E[v(y_1+2z)],$$

it suffices to show that

$$\forall_{i=1,2},\quad E[v(y_i+2z)] \leq E[v(y_i+z)].$$

Observe that

★ → $$y_i + z = \frac{1}{2}(y_i+2z) + \frac{1}{2}(y_i+0)$$

$$\Rightarrow v(y_i+z) \geq \frac{1}{2}v(y_i+2z) + \frac{1}{2}v(y_i) \qquad (\because v \text{ is concave})$$

$$\Rightarrow E\,v(y_i+z) \geq E\left[\frac{1}{2}v(y_i+2z) + \frac{1}{2}v(y_i)\right]$$

$$= \frac{1}{2}E\,v(y_i+2z) + \frac{1}{2}v(y_i). \qquad \cdots (*)$$

Since

$$E\,v(y_i+z) \leq v[E(y_i+z)] \qquad (\because v \text{ is concave})$$

$$= v(y_i), \qquad \left(\text{by the argument : } E(Y+Z)=E(Y)\right)$$

$$\text{LHS of } (*) \geq \frac{1}{2}E\,v(y_i+2z) + \frac{1}{2}E\,v(y_i+z)$$

$$\therefore\quad E\,v(y_i+z) \geq E\,v(y_i+2z).$$

Q.E.D.

Micro

2007 Fall Q III . 1

- Weakly dominance
- Finiteness

(a) $\boxed{\text{Claim}}$ $\forall$ finite game G, $\exists s^* \in NE(G)$ s.t. $\nexists i$ s.t. $s_i^*(a) > 0$ and a is WD.

pf

(Sketch)

Step 1. $PE(G)$ exists

Step 2 : $PE(G) \subseteq NE(G)$

Step 3 : If $s^* \in PE(G)$, $\forall i$, $\forall a \in A^i$ s.t. a is WD, $s_i^*(a) = 0$. □

(b) $\boxed{\text{Claim}}$ $\forall$ finite game G, $\exists s^* \in NE(G)$ s.t. $\nexists i$ s.t. s_i^* is WD.

pf

Step 3' : If $s^* \in PE(G)$, $\forall i$, s_i^* is not WD. □

(2008 Mid)

(c)

1 \ 2	1	2	3	⋯
1	1.1	0.0	0.0	
2	1.1	1.1	0.0	⋯
3	1.1	1.1	1.1	
⋮				

For player 2, the only str. which is not WD is 1.

Given this, $\forall n$, $s = (n, 1)$ is a NE.

But $\forall n$, n is WD by $n+1$.

Thus, $\forall s \in NE(G)$, player 1 chooses WD str.

Micro 2007 Spring QI.1

8101

- Equivalence b/w 2 def. of continuity of $\succsim$. (2007, PS 3 Q2.)

$\boxed{7}$

$\succsim$: transitive $\overset{def}{\iff}$ $\forall x, y, z \in X$, if $x \succsim y \wedge y \succsim z$, then $x \succsim z$.

complete $\overset{def}{\iff}$ $\forall x, y \in X$, $x \succsim y$ or $y \succsim x$ holds.

$X = \mathbb{R}^L_+$

C1 : $\forall \{x^n\}, \{y^n\} \subset X$ s.t. $x^n \to x$, $y^n \to y$ and $x^n \succsim y^n, \forall n$, then $x \succsim y$.

C2 : $\forall x \in X$, $U(x) \equiv \{y \in X \mid y \succsim x\}$ and $L(x) \equiv \{y \in X \mid x \succsim y\}$ are closed.

pf $\underline{C1 \Rightarrow C2}$

Fix $x \in X$. Take an arbitrary convergent sequence in $L(x)$, i.e., $\{y_n\}$ and $y_n \to y$.

Let $\{x_n\}$ be $x_n = x, \forall n$. Since $\forall n$, $y_n \in L(x)$,

$$x_n \succsim y_n \quad \forall n.$$

Thus, $x \succsim y$. Therefore, $y \in L(x)$ and hence $L(x)$ is closed.

$U(x)$ is closed by the same argument. Q.E.D.

pf $\underline{C2 \Rightarrow C1}$

Sps not. Then $x^n \to x$, $y^n \to y$, $x^n \succsim y^n$, $\forall n$ but $y \succ x$.

Prove that $\exists z \in X$ s.t. $y \succ z \succ x$. Sps not.

Then $U(y) \cup L(x) = X$. Also since $y \in U(y)$ and $x \in L(x)$, $U(y) \neq \phi \neq L(x)$.

By C2, $U(y)$ and $L(x)$ are closed and $U(y) \cap L(x) = \phi$ since $y \succ x$.

This contradicts that $X = \mathbb{R}^L_+$ is connected. Thus $\exists z \in X$ s.t. $y \succ z \succ x$.

By C2, $U^c(z)$ and $L^c(z)$ are open and $y \in L^c(z)$ and $x \in U^c(z)$.

Since $U^c(z)$ is open, x is an interior point and hence $\exists \varepsilon_x > 0$ s.t. $B_{\varepsilon_x}(x) \subset U^c(z)$.

Since $x^n \to x$, $\exists n_x \in \mathbb{N}$ s.t. $\forall n > n_x$, $x_n \in B_{\varepsilon_x}(x)$. $\therefore$ $z \succ x_n$ $\forall n > n_x$.

Likewise, $\exists \varepsilon_y > 0$ s.t. $B_{\varepsilon_y}(y) \subset L^c(z)$ and $\exists n_y \in \mathbb{N}$ s.t. $\forall n > n_y$, $y_n \in B_{\varepsilon_y}(y)$.

Thus $y_n \succ z$, $\forall n > n_y$. Let $\bar{n} = \max\{n_x, n_y\}$.

Then $\forall n > \bar{n}$, we have $z \succ x_n$ and $y_n \succ z$. By the transitivity, $y_n \succ x_n$.

This is a contradiction. Thus $x \succsim y$. Q.E.D.

Micro 2007 Spring Q I.2

8101

- Risk compensation and Arrow-Pratt measure of risk aversion. 16
- Application of the Pratt Thm. (Corollary 16.3)

Def Risk compensation for risky claim $z \in \mathbb{R}^S$ w/ $E(z)=0$ at risk-free initial wealth w is $\rho(w,z)$ s.t.

$$E\, v(w+z) = v(w - \rho(w,z))$$

Def Arrow pratt measure of absolute risk aversion at w is

$$A(w) = -\frac{v''(w)}{v'(w)}.$$

(i) $\boxed{\text{Claim}}$ A is weakly decreasing in w iff so is ρ $\forall z$ w/ $E(z)=0$.

pf Fix $\varepsilon > 0$.

Prove that for $\hat{v}(w) \equiv v(w+\varepsilon)$, we get

★ → $$\hat{\rho}(w,z) = \rho(w+\varepsilon, z) \quad \forall z \text{ w/ } E(z)=0,$$
$$\hat{A}(w) = A(w+\varepsilon).$$

By def, $E\,\hat{v}(w+z) = \hat{v}(w - \hat{\rho}(w,z))$

$$\begin{aligned} LHS &= E\, v(w+\varepsilon+z) \\ &= v(w+\varepsilon - \rho(w+\varepsilon, z)) \\ RHS &= v(w+\varepsilon - \hat{\rho}(w,z)) \end{aligned}$$

$$\therefore \hat{\rho}(w,z) = \rho(w+\varepsilon, z)$$

By def, $$\hat{A}(w) = -\frac{\hat{v}''(w)}{\hat{v}'(w)} = -\frac{v''(w+\varepsilon)}{v'(w+\varepsilon)} = A(w+\varepsilon).$$

Then

$$\begin{aligned} & A(w+\varepsilon) \le A(w) \\ \Leftrightarrow\ & \hat{A}(w) \le A(w) \\ \Leftrightarrow\ & \hat{\rho}(w,z) \le \rho(w,z) \quad \forall z \text{ w/ } E(z)=0 \qquad \text{by Pratt Thm} \\ \Leftrightarrow\ & \rho(w+\varepsilon, z) \le \rho(w,z) \quad \forall z \text{ w/ } E(z)=0 \end{aligned}$$

Q.E.D.

(ii) $\boxed{\text{Claim}}$ If $\rho(w,z)$ is independent of w $\forall z$ w/ $E(z)=0$, then v is an affine transformation of $\bar{v}(x) = -e^{-\alpha x}$ for $\alpha > 0$

pf) As in (i), we have

ρ is independent of w $\Leftrightarrow$ so is A.

Thus, let $A(w) = \alpha$, $\forall w$.

Note that $A_{\bar{v}}(w) = \dfrac{-(-\alpha^2 e^{-\alpha x})}{\alpha e^{-\alpha x}} = \alpha$.

Let v be an arbitrary fcn s.t. $A_v(w) = \alpha$, $\forall w$.

Since $A_{\bar{v}}(w) = A_v(w)$, $\forall w$, by Pratt Thm,

$\exists$ increasing concave fcn f & g s.t.

$$v(w) = f(\bar{v}(w)) \qquad \forall w.$$
$$\bar{v}(w) = g(v(w))$$

Note that $\bar{v}(w) = (g \circ f)(\bar{v}(w))$, $\forall w$

It suffices to show that f is linear.

Sps not. Then $\exists$ w_1, w_2 s.t. $x_1 = \bar{v}(w_1)$, $x_2 = \bar{v}(w_2)$ and for some $\lambda \in (0,1)$, $f(\lambda x_1 + (1-\lambda)x_2) > \lambda f(x_1) + (1-\lambda) f(x_2)$.

$$\begin{aligned}
\lambda x_1 + (1-\lambda)x_2 &= (g \circ f)[\lambda x_1 + (1-\lambda)x_2] \\
&> g[\lambda f(x_1) + (1-\lambda) f(x_2)] \quad (\because g \text{ is increasing}) \\
&\geq \lambda (g\circ f)(x_1) + (1-\lambda)(g \circ f)(x_2) \quad (\because g \text{ is concave}) \\
&= \lambda x_1 + (1-\lambda) x_2
\end{aligned}$$

This is a contradiction.

Thus f is linear and hence v is an affine transformation of $\bar{v}$.

Q.E.D.

Micro 2007 Spring Q II. 1

8103

- Linear game.
- Isbell / Kuhn Thm

(a) An EFG is linear if $\forall z \in Z$, $\forall i$, $\forall u^i \in U^i$, $\text{path}(z) \cap u^i$ has at most one element, where $\text{path}(z) = \{x \in K \mid z \succsim x \succsim o\}$.

(b) $\boxed{\text{Claim}}$ If G is linear, Given pure $s^{-i} \in S^{-i}$, $\forall b^i \in B^i$ $\exists \sigma^i \in \Delta(A^i)$ s.t. $P_{(b^i, s^{-i})}(z) = P_{(\sigma^i, s^i)}(z)$, $\forall z \in Z$

pf) Fix s^{-i}, b^i.

Prove that a mixed str.

$$\sigma^i(s^i) = \prod_{u^i \in U^i} b^i(u^i, s^i(u^i))$$

gives the same prob. dist. on Z.

$\forall z \in Z$, path(z) is unique by def. of tree, and $\forall x \in \text{path}(z)$, x is in a different info. set since the game is linear.

$\bar{Z} = \{z \in Z \mid \exists s^i \in S^i \text{ s.t. } P_{(s^i, s^{-i})}(z) \neq 0\}$.

$\bar{U}^i = \{u^i \in U^i \mid u^i \cap \text{path}(z) \neq \phi\}$

E^i : set of actions of i that take $x \in \text{path}(z)$ to $y \in S(x) \cap \text{path}(z)$

$\bar{S}^i = \{s^i \in S^i \mid P_{(s^i, s^{-i})}(z) \neq 0\}$

Then, $\forall z \in Z \setminus \bar{Z}$,

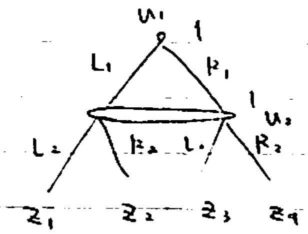
$$P_{(\sigma^i, s^{-i})}(z) = 0 = P_{(b^i, s^{-i})}(z).$$

and $\forall z \in \bar{Z}$,

$$\begin{aligned}
P_{(\sigma^i, s^{-i})}(z) &= \sum_{s^i \in S^i} \sigma^i(s^i) P_{(s^i, s^{-i})}(z) \\
&= \sum_{s^i \in \bar{S}^i} \prod_{u^i \in U^i} b^i(u^i, s^i(u^i)) \\
&= \sum_{s^i \in \bar{S}^i} \prod_{u^i \in \bar{U}^i} b^i(u^i, s^i(u^i)) \prod_{u^i \in U^i \setminus \bar{U}^i} b^i(u^i, s^i(u^i)) \\
&= \prod_{a \in E^i} b^i(a) \sum_{s^i \in \bar{S}^i} \prod_{u^i \in U^i \setminus \bar{U}^i} b^i(u^i, s^i(u^i)) \\
&= \prod_{a \in E^i} b^i(a) \\
&= P_{(b^i, s^{-i})}(z)
\end{aligned}$$

Q.E.D.

(c)



Note that this game is linear (but not perfect recall).

$$\begin{cases} \sigma(L_1, L_2) = \frac{1}{2} \\ \sigma(L_1, R_2) = 0 \\ \sigma(R_1, L_2) = 0 \\ \sigma(R_1, R_2) = \frac{1}{2} \end{cases}$$

Let $b(u_1, L_1) = q_1$

$b(u_1, R_1) = 1 - q_1$

$b(u_2, L_2) = q_2$

$b(u_2, R_2) = 1 - q_2$

Then

$$\begin{cases} P(z_1) = q_1 q_2 \\ P(z_2) = q_1(1-q_2) \\ P(z_3) = (1-q_1) q_2 \\ P(z_4) = (1-q_1)(1-q_2) \end{cases}$$

$$\Rightarrow \left. \begin{aligned} q_1 q_2 &= \frac{1}{2} \\ q_1(1-q_2) &= 0 \\ (1-q_1) q_2 &= 0 \\ (1-q_1)(1-q_2) &= \frac{1}{2} \end{aligned} \right)$$

no $q_1, q_2 \in [0,1]$ satisfy these 4 eqn.

Micro 2006 Fall QI.1

8101

3, 4

- profit fcn / supply correspondence
- Thm 4.1 - 4.3.

Def Profit fcn π^* is a fcn of price vector p s.t. $\pi^*(p) = \max_{y \in Y} py$.

Def Supply correspondence s^* is a correspondence of p s.t. $s^*(p) = \arg\max_{y \in Y} py$.

(a) $\boxed{\text{Claim}}$ π^* is (i) continuous, and (ii) convex.

pf (i) Since $Y \subset \mathbb{R}^n$ and Y is closed and bounded, Y is compact.
The objective fcn is continuous on $\mathbb{R}^n \times \mathbb{R}^n$.
Thus π^* is continuous by the Thm of Maximum

(ii) For $p_1, p_2 \in \mathbb{R}^n$. $\lambda \in (0,1)$, let $y^* \in s^*(\lambda p_1 + (1-\lambda) p_2)$
Then

$$\begin{aligned}
\pi^*(\lambda p_1 + (1-\lambda) p_2) &= [\lambda p_1 + (1-\lambda) p_2]\, y^* \\
&= \lambda p_1 y^* + (1-\lambda) p_2 y^* \\
&\le \lambda \pi^*(p_1) + (1-\lambda) \pi^*(p_2) \qquad (\text{by def. of } \pi^*)
\end{aligned}$$

Thus, π^* is convex.

Q.E.D.

(b) $\boxed{\text{Claim}}$ s^* has closed graph.

pf As in (a)(i), we can apply the Thm of Maximum and hence s^* is u.h.c.
Prove that u.h.c $\Rightarrow$ closed graph.
For any convergent seq. $(p_n, s_n) \in \{(p,s) \in \mathbb{R}^L \times \mathbb{R}^L \mid s \in s^*(p)\} \equiv A(s^*)$
the limit pt $(p, s) \in A(s^*)$. Thus graph is closed.
(u.h.c.)

(c) $\boxed{\text{Claim}}$ π^* : diff $\Rightarrow$ $D\pi^*(p) = s^*(p)$.

(pf) π^* : diff $\Leftrightarrow$ s^* is single-valued at p.

Then we define

$$F(p) \equiv p\, s^*(p) - p\, s^*(\bar{p})$$

with some fixed $\bar{p}$.

By def. of s^*, $F(p) \geq 0$, $\forall p$.

Also we have $F(\bar{p}) = 0$.

Thus, FOC wrt p : $DF(p) = D\pi^*(p) - s^*(\bar{p})$.

Since $\bar{p}$ is the solution,

$$0 = DF(\bar{p}) = D\pi^*(\bar{p}) - s^*(\bar{p})$$

$$\Leftrightarrow \quad D\pi^*(\bar{p}) = s^*(\bar{p}) \qquad \text{Q.E.D.}$$

(d) $\boxed{\text{Claim}}$ π^*, s^* : diff at p $\Rightarrow$ $\frac{\partial s_i^*}{\partial p_i}(p) \geq 0$, $\forall i = 1 \cdots n$

(pf) Since π^* diff, (c) holds.

Also since s^* diff, π^* is twice-differentiable and we have

$$D^2\pi^*(p) = Ds^*(p). \qquad \cdots (*)$$

Since π^* is convex as proved in (a), the LHS of $(*)$ is positive semi-definite and symmetric and hence so is the RHS of $(*)$.

That is, $\forall$ vector $h \in \mathbb{R}^n$,

$$h^T Ds^*(p)\, h \geq 0.$$

Thus for any $i \in \{1, \cdots, n\}$, taking $h_i = (0, \cdots, \underset{i\text{th}}{1}, 0, \cdots, 0)^T$ gives

$$h_i^T Ds^*(p)\, h_i \geq 0$$

$$\Leftrightarrow \quad \frac{\partial s^*}{\partial p_i}(p) \geq 0$$

Q.E.D.

Micro 2006 Fall Q I.2

8|01

15

- Conditions for state-separable representation.
- Concave expected utility representation. and risk averse.

Def U has a state-separable representation if there exist fcn's $v_s : \mathbb{R} \to \mathbb{R}$ s.t. $U(c_1, \cdots, c_S) \geq U(c_1', \cdots, c_S')$ iff $\sum_{s=1}^{S} v_s(c_s) \geq \sum_{s=1}^{S} v_s(c_s')$ for all $c, c' \in \mathbb{R}^S$.

Def U has a concave expected utility representation if $\exists$ concave fcn $v: \mathbb{R} \to \mathbb{R}$ s.t. $U(c_1, \cdots, c_S) \geq U(c_1', \cdots, c_S')$ iff $\sum_{s=1}^{S} \pi_s v(c_s) \geq \sum_{s=1}^{S} \pi_s v(c_s')$ for all $c, c' \in \mathbb{R}^S$.

(a) U has a state-separable representation iff ~~...~~.

Let $c_{-s}y$ be a consumption plan s.t.

$$c_{-s}y = (c_1, \cdots, c_{s-1}, y, c_{s+1}, \cdots, c_S)$$

where $y \in \mathbb{R}$.

Then we say that U satisfies the sure thing principle if

$$U(c_{-s}y) \geq U(c'_{-s}y) \text{ iff } U(c_{-s}z) \geq U(c'_{-s}z)$$

for all $c, c' \in \mathbb{R}^S$, $y, z \in \mathbb{R}$ and $s \in \{1, \cdots, S\}$.

Then, U has a state-separable representation iff U satisfies the sure thing principle.

(b) $\boxed{\text{Claim}}$ U has a concave utility representation w/ π_s iff U has a state-separable representation and is risk averse w.r.t. π.

pf

($\Rightarrow$) Let v be a concave fcn s.t.

$$U(c) \geq U(c') \iff \sum_{s=1}^{S} \pi_s v(c_s) \geq \sum_{s=1}^{S} \pi_s v(c_s') \quad \forall c, c'.$$

Define $v_s : \mathbb{R} \to \mathbb{R}$ as

$$v_s(x) = \pi_s v(x) \quad \forall x \in \mathbb{R}.$$

Then

$$U(c) \geq U(c') \iff \sum_{s=1}^{S} v_s(c_s) \geq \sum_{s=1}^{S} v_s(c_s'). \quad \forall c, c'.$$

Thus U has a state-contingent representation.

Also, since v is concave

$$E(v(c)) \leq v(E(c)) \quad \forall c \in \mathbb{R}^S$$

$$\iff \quad U(c) \leq U(E(c)) \quad \forall c \in \mathbb{R}^S$$

by def of v. Thus U is risk averse. □

($\Leftarrow$) Let $v_s : \mathbb{R} \to \mathbb{R}$ be s.t.

$$U(c) \geq U(c') \iff \sum_{s=1}^{S} v_s(c_s) \geq \sum_{s=1}^{S} v_s(c_s'), \quad \forall c, c'.$$

Assume v_s's are differentiable.

For each $x \in \mathbb{R}$, consider a problem

$$\max_{c \in \mathbb{R}^S} \sum_{s=1}^{S} v_s(c_s)$$

$$\text{s.t.} \quad E(c) = x.$$

Since U is risk averse, $c = (x, \cdots, x)$ solves this problem.

Thus, FOC : $v_s'(x) = \lambda \pi_s \quad \forall s = 1, \cdots, S.$

$$\therefore v_s'(x) = \frac{\pi_s}{\pi_1} v_1'(x)$$

$$\therefore v_s(x) = \frac{\pi_s}{\pi_1} v_1(x) + \Delta_s$$

for some Δ_s. Let $v \equiv v_1$. Then

$$\sum \pi_s v_s(c_s) \geq \sum \pi_s v_s(c_s') \iff \overbrace{\sum \pi_s \left(\frac{1}{\pi_1} v(c_s) + \frac{\Delta_s}{\pi_s}\right) \geq \sum \pi_s \left(\frac{1}{\pi_1} v(c_s') + \frac{\Delta_s}{\pi_s}\right)}^{(*)}$$

$$\iff \sum \pi_s v(c_s) \geq \sum \pi_s v(c_s').$$

($\because$ v is an affine transformation of (*))

Thus U has a expected utility representation.

Since U is risk averse,

$$U(c) \leq U(E(c)) \iff \sum \pi_s v(c_s) \leq \sum \pi_s v(E(c))$$

$$\iff E(v(c)) \leq v(E(c)) \quad \forall c \in \mathbb{R}^S.$$

Thus v is concave.

□ ☒

Micro 2006 Fall Q III . 1

8103

- Perfect eqm
- closedness of PE

(a) $\boxed{\text{Claim}}$ $PE(G) \subseteq NE(G)$.

pf $s^* \in PE(G)$.

$\exists\, \varepsilon_n \to 0$ and $s_n \in NE(G_{\varepsilon_n})$, $\forall n$, s.t. $s^n \to s^*$

$\forall n, s_n \in NE(G_{\varepsilon_n})$

$\Leftrightarrow \forall n, \forall i, s_n^i \in BR^i_{G_{\varepsilon_n}}(s_n)$

$\Leftrightarrow \forall n, \forall i, \forall t^i \in \mathcal{S}^i_{\varepsilon_n}, \; u_i(s_n^i, s_n^{-i}) \geq u_i(t^i, s_n^{-i})$

Since $\mathcal{S}^i$ is compact and $\varepsilon_n \to 0$, $\forall t^i \in \mathcal{S}^i$ $\exists$ convergent seq t_n^i s.t. $t_n^i \to t^i$ and $t_n^i \in \mathcal{S}^i_{\varepsilon_n}$.

Then

$(*) \Leftrightarrow \forall i, \forall n, \forall t_n^i \in \mathcal{S}^i_{\varepsilon_n}, \; u_i(s_n^i, s_n^{-i}) \geq u_i(t_n^i, s_n^{-i})$.

Continuity of u^i implies

$\forall i, \forall t^i \in \mathcal{S}^i. \; u_i(s^{*i}, s^{*-i}) \geq u_i(t^i, s^{*-i})$

Thus, $s^* \in BR(s^*)$ i.e., $s^* \in NE(G)$.

Q.E.D.

(b) $\boxed{\text{Claim}}$ $PE(G)$ is closed

pf Let $s_n \in PE(G)$, $\forall n$ and $s_n \to s$.

WTS : $s \in PE(G)$.

Since $\forall n$, $s_n \in PE(G)$, $\exists\, \varepsilon^n_m \to 0$ as $m \to \infty$ and $S^n_m \in NE(G_{\varepsilon^n_m})$ s.t. $S^n_m \to S_n$ as $m \to \infty$.

Since $\varepsilon^n_m \to 0$ as $m \to \infty$, $\lim_{n \to \infty} \varepsilon^n_n = 0$.

Since $\varepsilon^n_n \in [0,1]^{\#A_1 \times \cdots \times \#A_N}$ and $[0,1]^{\#}$ is compact,

$\exists$ convergent subseq $\varepsilon^{n_k}_{n_k}$ s.t. $\varepsilon^{n_k}_{n_k} \to 0$

Let $\varepsilon_k = \varepsilon^{n_k}_{n_k}$. Then $\varepsilon_k \to 0$

Since $\forall n$, $s^n_n \in NE(G_{\varepsilon^n_n})$, if we take $s_k = s^{n_k}_{n_k}$,

we have $s_k \in NE(G_{\varepsilon_k})$ by construction.

Since $S^n_n \to s$, we have $s_k \to s$. Thus $s \in PE(G)$ and $PE(G)$ is closed.

Micro 2006 Fall Q III . 2

8103

1\2	1	2	3	…	M
1	(-1, 1)	(1, -1)	(1, -1)	…	(1, -1)
2	(1, -1)	(-1, 1)			
3	(1, -1)		⋱		
⋮					
M	(1, -1)				(-1, 1)

Given $s^2 \in \mathcal{S}^2$,

$$BR^1(s^2) = \underset{s^1 \in \mathcal{S}^1}{\arg\max} \sum_{i=1}^{M} s^1(i)\,[-s^2(i) + (1 - s^2(i))]$$

$$= \quad 1 - 2\sum_{i=1}^{M} s^1(i)\,s^2(i)$$

$$= \underset{s^1 \in \mathcal{S}^1}{\arg\min} \sum_{i=1}^{M} s^1(i)\,s^2(i)$$

Given $s^1 \in \mathcal{S}^1$,

$$BR^2(s^1) = \underset{s^2 \in \mathcal{S}^2}{\arg\max} \sum_{i=1}^{M} s^2(i)\,[s^1(i) - (1 - s^1(i))]$$

$$= \underset{s^2 \in \mathcal{S}^2}{\arg\max}\ 2\sum_{i=1}^{M} s^1(i)\,s^2(i) - 1.$$

$$= \underset{s^2 \in \mathcal{S}^2}{\arg\max} \sum_{i=1}^{M} s^1(i)\,s^2(i)$$

Then, since $s^1 \in \mathcal{S}^1$, $s^2 \in \mathcal{S}^2$, we have

$(*)$: $\forall s^1 \in BR^1(s^2)$, if $i \in \arg\max_i s^2(i) \equiv M_1 \wedge M_1 \subsetneq \{1, \cdots, M\}$,
then $s^1(i) = 0$.

$(**)$: $\forall s^2 \in BR^2(s^1)$, if $i \in \arg\min_i s^1(i) \equiv M_2 \wedge M_2 \subsetneq \{1, \cdots, M\}$,
then $s^2(i) = 0$

<u>WTS</u> $M_1^c = \phi$ or $M_2^c = \phi$.

Sps not. Then $M_1^c \neq \phi \wedge M_2^c \neq \phi$.

Then $\forall i \in \{1, \cdots, M\}$, if $i \in M_1$, then $s^1(i) = 0$. (by $M_1^c \neq \phi$ & $(*)$)

$\Rightarrow i \in M_2$

$\Rightarrow M_1 \subseteq M_2$ $\cdots (\#)$

$\forall i \in \{1, \cdots, M\}$, if $i \in M_2$, then $s^2(i) = 0$ (by $M_2^c \neq \phi$ & $(**)$)

$\Rightarrow i \notin M_1$ (since $\exists_{j \neq i} : s^2(j) > 0$)

$\Rightarrow M_2 \subseteq M_1^c$ $\cdots (\#\#)$

Thus $M_1 \subseteq M_1^c$, which is a contradiction.

Thus $M_1^c = \phi$ or $M_2^c = \phi$.

Therefore, $\forall \hat{s} \in NE(G)$,

$$\hat{s}^1(i) = \frac{1}{M} \ \ \forall i \quad \text{or} \quad \hat{s}^2(i) = \frac{1}{M} \ \ \forall i.$$

Sps $\hat{s}^1(i) = \frac{1}{M}, \forall i$.

Then
$$BR^2(\hat{s}^1) = \arg\max_{s^2} \frac{1}{M} \sum_{i=1}^{M} s^2(i)$$
$$= \mathcal{S}^2$$

By def of NE, $\hat{s}^1 \in BR^1(\hat{s}^2)$.

Sps $\exists i, j$ s.t. $\hat{s}^2(i) \neq \hat{s}^2(j)$, WLOG $\hat{s}^2(i) < \hat{s}^2(j)$.

Then
$$\bar{s}^1(k) = \begin{cases} \frac{2}{M} & k = i \\ 0 & k = j \\ \frac{1}{M} & k \neq i, j \end{cases}$$

attains
$$\sum_k \bar{s}^1(k)\, \hat{s}^2(k) < \sum_k \hat{s}^1(k)\, \hat{s}^2(k).$$

This contradicts that $\hat{s}^1 \in BR^1(\hat{s}^2)$.

Thus $\forall i, j$, $\hat{s}^2(i) = \hat{s}^2(j)$.

Sps $\hat{s}^2(i) = \frac{1}{M}, \forall i$.

Then
$$BR^1(\hat{s}^2) = \arg\min_{s^1} \frac{1}{M} \sum s^1(i)$$
$$= \mathcal{S}^1.$$

By def of NE, $\hat{s}^2 \in BR^2(\hat{s}^1)$

Sps $\exists i, j$ s.t. $\hat{s}^1(i) \neq \hat{s}^1(j)$ WLOG $\hat{s}^1(i) < \hat{s}^1(j)$

Then
$$\bar{s}^2(k) = \begin{cases} \frac{2}{M} & k = j \\ 0 & k = i \\ \frac{1}{M} & k \neq i, j \end{cases}$$

attains
$$\sum_k \bar{s}^2(k)\, \hat{s}^1(k) > \sum_k \hat{s}^2(k)\, \hat{s}^1(k)$$

This contradicts that $\hat{s}^2 \in BR^2(\hat{s}^1)$.

Thus, $\forall i, j$, $\hat{s}^1(i) = \hat{s}^1(j)$.

Therefore, the unique NE is

$$\hat{s} = \left(\left(\frac{1}{M}, \cdots, \frac{1}{M}\right), \left(\frac{1}{M}, \cdots, \frac{1}{M}\right) \right).$$

Q.E.D.

Micro

2006 Fall Q III.3

• Equivalence of the games.

(pf)

$\forall\ \hat{s}^{-k} \in S^{-k}$,

$$\hat{s}^{k} \in BR^{k}(\hat{s}^{-k})\,\big|_{G\,new}$$

$$= \underset{s^k \in S^k}{\operatorname{argmax}} \sum_{a^{-k}\in A^{-k}} \hat{s}^{-k}(a^{-k}) \sum_{a^k \in A^k} s^k(a^k)\, v(a^k, a^{-k})$$

$$= \quad \text{〃} \quad \left(c(a^{-k}) + b\, u^k(a^k, a^{-k})\right)$$

$$= \underset{s^k \in S^k}{\operatorname{argmax}}\ b\, u^k(s^k, \hat{s}^{-k}) + \sum_{a^{-k}\in A^{-k}} \hat{s}^{-k}(a^{-k}) \sum_{a^k\in A^k} s^k(a^k)\, c(a^{-k})$$

$$= \underset{s^k \in S^k}{\operatorname{argmax}}\ b\, u^k(s^k, \hat{s}^{-k}) + \sum_{a^{-k}\in A^{-k}} \hat{s}^{-k}(a^{-k})\, c(a^{-k}) \underbrace{\sum_{a^k\in A^k} s^k(a^k)}_{1}$$

$$= \underset{s^k \in S^k}{\operatorname{argmax}}\ b\, u^k(s^k, \hat{s}^{-k}) + 1$$

$$= \underset{s^k \in S^k}{\operatorname{argmax}}\ u^k(s^k, \hat{s}^{-k}) \qquad (\text{since } b>0)$$

$$= BR^k(\hat{s}^{-k})\,\big|_{G}.$$

Q.E.D.

Micro 2006 Spring Q I. 1

8101

[7], [8]

- Special preference : $x \succsim x'$ iff $x_1 \geq x'_1 \wedge \Sigma x_i \geq \Sigma x'_i$
- properties of $\succsim$ / utility representation
- Demand Correspondence

Def $\succsim$ is continuous if the sets $U(x) \equiv \{y \in \mathbb{R}^n_+ \mid y \succsim x\}$ and $L(x) = \{y \in \mathbb{R}^n_+ \mid x \succsim y\}$ are closed for all $x \in \mathbb{R}^n_+$.

Def $\succsim$ is convex if $\forall x \in \mathbb{R}^n_+$, the set $\{y \in \mathbb{R}^n_+ \mid y \succsim x\}$ is convex.

Def $u: \mathbb{R}^n_+ \to \mathbb{R}$ is a utility representation of $\succsim$ if $\forall x, x' \in \mathbb{R}^n_+$, $u(x) \geq u(x')$ iff $x \succsim x'$.

(a) $\boxed{\text{Claim}}$ $\succsim$ is (i) continuous and (ii) convex.

pf (i) Sps not. Then $\exists x \in \mathbb{R}^n_+$ s.t. $U(x)$ or $L(x)$ is not closed.
WLOG, say $U(x)$ is not closed. Then $\exists$ a seq. $\{y^m\}$ s.t.
$y^m \in U(x)$, $\forall m \in \mathbb{N}$ and $y^m \to y \notin U(x)$.
Since $y^m \in U(x)$, $\forall m$, $y^m_1 \geq x_1 \wedge \sum_{i=1}^n y^m_i \geq \sum_{i=1}^n x_i$.
Since $y \notin U(x)$, $x_1 > y_1 \vee \sum_{i=1}^n x_i > \sum_{i=1}^n y_i$.
Sps $x_1 > y_1$. Let $\epsilon = x_1 - y_1$.
Since $y^m \to y$, $\exists \bar{m} \in \mathbb{N}$ s.t. $\forall m > \bar{m}$, $y^m \in B_{\epsilon/2}(y)$.
Thus $x_1 > y^m_1$. Hence $y^m \notin U(x)$. This is a contradiction.
Sps $\sum_{i=1}^n x_i > \sum_{i=1}^n y_i$. Since $x, y \in \mathbb{R}^n_+$, $\exists i \in \{1, \cdots, n\}$ s.t.
$x_i > y_i$. This is a contradiction by the same argument as $i=1$ case.
Therefore, $y \in U(x)$ and hence $U(x)$ is closed.
Similarly, $L(x)$ is also closed. Since x is arbitrary, $\succsim$ is continuous.

(ii) Let $y, z \in U(x)$.
Then $y_1 \geq x_1 \wedge \sum^n y_i \geq \sum x_i \wedge z_1 \geq x_1 \wedge \Sigma z_i \geq \Sigma x_i$.
For any $\lambda \in (0,1)$,

$$\lambda y_1 + (1-\lambda) z_1 \geq \lambda x_1 + (1-\lambda) x_1 = x_1$$

$$\Sigma \lambda y_i + \Sigma (1-\lambda) z_i = \lambda \Sigma y_i + (1-\lambda) \Sigma z_i \geq \Sigma x_i$$

Thus, $\lambda y + (1-\lambda) z \in U(x)$ and hence $\succsim$ is convex.

Q.E.D.

(b) $\boxed{\text{Claim}}$ $\succsim$ does not have a util. rep.

★ If $\succsim$ has u, then $\succsim$ is complete & transitive.

pf) Sps $\succsim$ has a utility rep. $u : \mathbb{R}^n_+ \to \mathbb{R}$ that is,

$$x \succsim x' \text{ iff } u(x) \geq u(x').$$

Let $x, x' \in \mathbb{R}^n_+$ be s.t.

$$x_1 > x_1' \text{ and } \sum_{k=1}^n x_k < \sum_{k=1}^n x_k'.$$

Then we have neither $x \succsim x'$ nor $x' \succsim x$.

But, since $\mathbb{R}$ is complete, we have to have either $u(x) \geq u(x')$ or $u(x) \leq u(x')$.

This is a contradiction. Thus $\succsim$ has no util. rep. Q.E.D.

(c) Derive the demand.

Case 1 : $p_1 > p_2$.

$\boxed{\text{Claim}}$ $x \in d(p, w)$ iff $p_1 x_1 + p_2 x_2 = w$

pf) ($\Rightarrow$) Sps $p_1 x_1 + p_2 x_2 < w$. Then $(x_1 + \varepsilon, x_2) \succ (x_1, x_2) \wedge p_1(x_1+\varepsilon) + p_2 x_2 < w$. with $\varepsilon > 0$ small enough. $\rightarrow\leftarrow$ $x \in d(p, w)$.

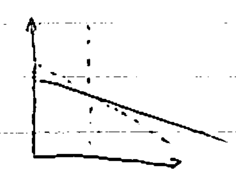
($\Leftarrow$) Sps $\exists x' \in B(p,w)$ s.t. $x' \succ x$.

Since $x' \in B(p,w)$, $p_1 x_1' + p_2 x_2' = w$.

Thus $p_1(x_1' - x_1) = p_2(x_2 - x_2')$ $\Rightarrow$ $x_1' - x_1 < x_2 - x_2'$ $\because p_1 > p_2$

Thus $x_1' + x_2' < x_1 + x_2$, $\rightarrow\leftarrow$ $x' \succ x$. □

Case 2 : $p_1 \leq p_2$



$\boxed{\text{Claim}}$ $x \in d(p, w)$ iff $p_1 x_1 = w$

pf) ($\Rightarrow$) the same as case 1 (sps not, then take ε).

($\Leftarrow$) Sps $\exists x' \in B(p,w)$ s.t. $x' \succ x$.

If $x_1' > x_1$, $p_1 x_1' > p_1 x_1 = w$. $\rightarrow\leftarrow$ $x' \in B(p,w)$

Then $x_1' = x_1$. Since $x' \succ x$, $x_2' > x_2 = 0$

Then $p_1 x_1' + p_2 x_2' > p_1 x_1' = p_1 x_1 = w$. $\rightarrow\leftarrow$ $x' \in B(p,w)$. □■

The demand above is not u.h.c. and hence not continuous.

pf) Let $p^n = (p_1^n, p_2^n)$, $n = 1, \cdots$, $p_1^n = 1$ ∀n, $p_2^n = 1 - \frac{1}{n}$. Then $p^n \to p = (1,1)$.

Let $x^n = (x_1^n, x_2^n)$, $n = 1, \cdots$, $x_1^n = 0$, $x_2^n = \frac{w}{p_2^n}$. Then $x^n \in d(p^n, w)$

and $x^n \to x = (0, w)$. But $x \notin d(p, w) = \{x^*\}$, $x^* = (w, 0)$.

Thus d is not u.h.c. □

Micro 2006 Spring Q I. 2

8101

$\boxed{15}$

- State-separable utility representation
- Why is " $S \geq 3$ " needed?

(a) U has a state-separable utility representation iff ~ ?

Let $c_{-s}y$ be a consumption plan s.t.

$$c_{-s}y = (c_1, \cdots, c_{s-1}, y, c_{s+1}, \cdots, c_S)$$

for $y \in \mathbb{R}_+$

U has a state-separable utility representation iff U satisfies the sure thing principle, i.e.,

$$U(c_{-s}y) \geq U(c'_{-s}y) \text{ iff } U(c_{-s}z) \geq U(c'_{-s}z)$$

for all $c, c' \in \mathbb{R}^S_+$, $y, z \in \mathbb{R}_+$ and $s \in \{1, \cdots, S\}$.

(b) $\boxed{\text{Claim}}$ If U has a state-separable utility representation, then U satisfies the sure thing principle.

pf Let $v_s : \mathbb{R}_+ \to \mathbb{R}$, $s = 1, \cdots, S$ be

$$U(c) \geq U(c') \text{ iff } \sum_s v_s(c) \geq \sum_s v_s(c').$$

Then for any $c, c' \in \mathbb{R}^S_+$, $y, z \in \mathbb{R}_+$ and $\bar{s} \in \{1, \cdots, S\}$

$$U(c_{-\bar{s}}y) \geq U(c'_{-\bar{s}}y) \iff \sum_{s \neq \bar{s}} v_s(c_s) + v_{\bar{s}}(y) \geq \sum_{s \neq \bar{s}} v_s(c'_s) + v_{\bar{s}}(y)$$

$$\iff \sum_{s \neq \bar{s}} v_s(c_s) \geq \sum_{s \neq \bar{s}} v_s(c'_s)$$

$$\iff \sum_{s \neq \bar{s}} v_s(c_s) + v_{\bar{s}}(z) \geq \sum_{s \neq \bar{s}} v_s(c'_s) + v_{\bar{s}}(z)$$

$$\iff U(c_{-\bar{s}}z) \geq U(c_{-\bar{s}}z).$$

Thus, U satisfies the sure thing principle. □

(c) $\boxed{\text{Claim}}$ When $S=2$, the sure thing principle is necessary but not sufficient for U to have a state-separable representation.

(pf) Necessity : Obvious by (b).

Not Sufficient :

Let $U(c) = c_1 c_2$

Prove that U satisfies the sure thing principle
but does not have a state-separable representation.

Note that U is obviously strictly increasing and continuous.

$\forall\, c, c' \in \mathbb{R}^2_+,\ y, z \in \mathbb{R}_+,$

$$U((c_i, y)) \geq U((c_i', y)) \Leftrightarrow y c_i \geq y c_i'$$

$$\Leftrightarrow U((c_i, z)) \geq U((c_i', z))$$

for $i = 1, 2$

Thus, U satisfies the sure thing principle.

Sps U has a state-separable representation : $\exists\, v_1, v_2$ s.t.

$$U(c) \geq U(c') \Leftrightarrow v_1(c_1) + v_2(c_2) \geq v_1(c_1') + v_2(c_2')$$

Let $c = (1, 0)$, $c' = (2, 0)$

$\bar{c} = (1, 1)$, $\bar{c}' = (2, 1)$

We have

$$U(c) = 0 = U(c'),$$

$$U(\bar{c}) = 1 < 2 = U(\bar{c}').$$

But,

$$U(c) = U(c') \Leftrightarrow v_1(1) + v_2(0) = v_1(2) + v_2(0)$$

$$\Leftrightarrow v_1(1) = v_1(2)$$

$$\Leftrightarrow v_1(1) + v_2(1) = v_1(2) + v_2(1)$$

$$\Leftrightarrow U(\bar{c}) = U(\bar{c}').$$

This is a contradiction.

Thus U has no state-separable representation and hence the sure thing principle is not sufficient. Q.E.D.

Micro 2006 Spring Q III.1. 8103

∘ NE & PE.

(a) ex of a game w/ NE that is not PE.

	L	R
U	1,1	0,0
R	0,0	0,0

(b) (i)

$$
\begin{array}{lcccccccccl}
 & (a_{11} & > & a_{21} \wedge a_{12} & > & a_{22} \wedge b_{11} & > & b_{12} \wedge b_{21} & > & b_{22} &) \\
\vee & (& > & & > & & < & & < & &) \\
\vee & (& < & & < & & > & & > & &) \\
\vee & (& < & & < & & < & & < & &)
\end{array}
$$

(ii)

$$
\begin{array}{lcccccccccl}
 & (& > & & < & & < & & > & &) \\
\vee & (& < & & > & & > & & < & &)
\end{array}
$$

(iii) $\neg(1) \wedge \neg(2)$.

(c) Fact : PE exists.

Fact : If NE is fully mixed, that is PE.

$\Rightarrow$ It suffices to prove that not-fully-mixed NE in (iii) are PE.

$\boxed{\text{Claim}}$ $\nexists$ eqm that one plays pure and the other plays mixed.

pf Obvious. □

Micro 2005 Fall Q I.1.

8101

17

- Expectation / Conditional Expectation
- More risky

Def X is more risky than Y if $E(X) = E(Y)$ and Y SOSD X.

(a) $\boxed{\text{Claim}}$ " $\forall X, z$ w/ $E(z) = 0$, $X+z$ is more risky than X " is false.

pf Sps the statement is true.

Then X SOSD $X+z$.

$\Leftrightarrow$ $\forall v$: nondecreasing, continuous, concave $E\,v(X+z) \leq E\,v(X)$.

Let $v(\cdot) = \log(\cdot)$: nondec, cont, concave.

Let $z = \begin{cases} 1 & \text{state } 1 \\ -1 & \text{〃 } 2 \end{cases}$

$X = \begin{cases} 1 & \text{state } 1 \\ 3 & \text{〃 } 2 \end{cases}$

Also Prob (state 1) = Prob (state 2) = $\frac{1}{2}$

Then $E(z) = 0$

Note that $E(z \mid X=1) = 1 \cdot \frac{\frac{1}{2}}{\frac{1}{2}} + -1 \cdot \frac{0}{\frac{1}{2}} = 1 \neq 0$

$\frac{\text{Prob}(z=-1 \wedge X=1)}{\text{Prob}(X=1)}$

But we have

$$E\,v(X+z) = \tfrac{1}{2}v(2) + \tfrac{1}{2}v(2)$$
$$= v(2)$$
$$= \log 2,$$
$$E\,v(X) = \tfrac{1}{2}v(1) + \tfrac{1}{2}v(3)$$
$$= \tfrac{1}{2}\log 3$$
$$< \log 2.$$

This contradicts that $E\,v(X+z) \leq E\,v(X)$. Q.E.D.

(b) Condition : $E(z|X) = 0$.

example of X, z : joint prob

	$z=-1$	$z=1$	Prob(x)
$X=1$	$1/4$	$1/4$	$1/2$
$X=2$	$1/4$	$1/4$	$1/2$
Prob(z)	$1/2$	$1/2$	

Then $E(z|X) = (-1)\,\text{Prob}(z=-1|X) + 1\cdot\text{Prob}(z=1|X)$

$$= (-1)\cdot\frac{\frac{1}{4}}{\frac{1}{2}} + 1\cdot\frac{\frac{1}{4}}{\frac{1}{2}}$$

$$= 0$$

(c) $E(X+z) = E[E(X+z|X)]$

$= E[X + E(z|X)]$

$= E(X)$ $(\because E(z|X)=0.)$

$E\,v(X+z) = E[E[v(X+z)|X]]$

$\leq E[v[E(X+z|X)]]$ $(\because v$ concave.$)$

$= E[v(X + E(z|X))]$

$= E\,v(X)$ $(\because E(z|X)=0)$

Thus, $\forall X, z$ w/ $E(z|X)=0$, $X+z$ is more risky than X.

Micro 2005 Fall Q I. 2

8(01

$\boxed{7}$

- utility representation : $u(x)$: $u(x)e \sim x$.
- Necessity of Continuity & Strict increasingness.

Def $\succsim$ is continuous if $\forall x \in \mathbb{R}^n_+$, $U(x) \equiv \{y \in \mathbb{R}^n_+ \mid y \succsim x\}$ and $L(x) \equiv \{y \in \mathbb{R}^n_+ \mid x \succsim y\}$ are closed.

Def $\succsim$ is strictly increasing if $\forall x, y \in \mathbb{R}^n_+$, $x \geq y \wedge x \neq y$ gives $x \succ y$.

Def u is a utility representation of $\succsim$ if $\forall x, x' \in \mathbb{R}^n_+$
$u(x) \geq u(x')$ iff $x \succsim x'$

(a) Claim If $\succsim$ is cont. & str. inc (SI), then u is well-defined & represents $\succsim$.

(pf) Existence

Define $A^+(x) \equiv \{\alpha \in \mathbb{R}_+ \mid \alpha e \succsim x\}$,

★ → $A^-(x) \equiv \{\alpha \in \mathbb{R}_+ \mid x \succsim \alpha e\}$

Then (i) $A^+(x), A^-(x) \neq \phi$:

Let $\bar{\alpha} = \max_i x_i + 1$. Then $\bar{\alpha} e \geq x \wedge \bar{\alpha} e \neq x \Rightarrow \bar{\alpha} e \succ x$ by (SI). Thus $\bar{\alpha} \in A^+(x)$. Also $0 \in A^-(x)$ since $x \geq 0$.

(ii) $A^+(x), A^-(x)$ closed.

Define $A = \{\alpha e \mid \alpha \geq 0\}$. Note that A is closed.

★ → Thus $A^+(x)$ and $A^-(x)$ are closed.

since $U(x)$ and $L(x)$ are closed by the continuity.

(iii) $A^+(x) \cup A^-(x) = \mathbb{R}_+$.

Thus, since $\mathbb{R}_+$ is connected, $A^+(x) \cap A^-(x) \neq \phi$.

By def, $\forall x \in \mathbb{R}^n_+$ $\exists u(x)$ s.t. $u(x) \in A^+(x) \cap A^-(x)$, i.e., $u(x)e \sim x$

Thus, u is well-defined.

u represents $\succsim$.

$\forall x$, $u(x) = \alpha$ is unique. Sps $\exists \alpha' \in \mathbb{R}_+$ s.t. $\alpha' e \sim x \wedge \alpha' \neq \alpha$.

But since $\alpha e \sim x$, $\alpha e \sim \alpha' e$. Then $\alpha = \alpha'$. Contradiction.

Thus $\forall x \in \mathbb{R}^n_+$, $\exists! u(x)$ s.t. $u(x)e \sim x$.

If $x \succsim x'$, then $u(x)e \succsim u(x')e$. If $u(x) < u(x')$, then $u(x')e \succ u(x)e$ by (SI). Contradiction. Thus $u(x) \geq u(x')$.

If $u(x) \geq u(x')$, then $u(x)e \succsim u(x')e$. Thus $x \succsim x'$. Q.E.D.

(b) Ex of $\succsim$ not (SI) and u does not represent $\succsim$.

Consider $\succsim$ s.t. $\forall x, x' \in \mathbb{R}^n_+$

$$x \succsim x' \quad \text{iff} \quad \|x - e\| \leq \|x' - e\|,$$

where $\|\cdot\|$ is the Euclid norm.

- $\succsim$ is continuous:

  Fix x. Let $\varepsilon_x = \|x - e\|$. Then

  $$U(x) = \{y \in \mathbb{R}^n_+ \mid y \succsim x\} = \{y \in \mathbb{R}^n_+ \mid \|y - e\| \leq \varepsilon_x\},$$

  i.e., $U(x)$ is a closed ball of radius ε_x.

  Also $L(x) = \{y \in \mathbb{R}^n_+ \mid x \succsim y\} = \{y \in \mathbb{R}^n_+ \mid \|y - e\| \geq \varepsilon_x\}$.

  Thus, both $U(x)$ and $L(x)$ are closed. $\Rightarrow$ $\succsim$ continuous.

- $\succsim$ is not SI:

  $$3e > 2e \quad \text{but} \quad 2e \succ 3e.$$

- u does not represent $\succsim$:

  $$u(3) > u(2) \quad \text{but} \quad 2e \succ 3e.$$

(c) Ex of $\succsim$ not continuous and u does not represents $\succsim$.

Consider $\succsim$ s.t. $\forall x, x' \in \mathbb{R}^n_+$

$$x \succsim x' \quad \text{iff} \quad (x_1 > x_1') \text{ or } (x_1 = x_1' \wedge x_2 \geq x_2')$$

- $\succsim$ is not continuous:

  Let $x \in \mathbb{R}^n_+$ s.t. $x_1 = 1 = x_2$. Define

  convergent seq $\{y^m\}_{m=1}^{\infty}$ s.t. $y_1^m = 1 - \frac{1}{m}$, $y_2^m = 2$.

  Since for all $m \in \mathbb{N}$, $x_1 > y_1^m$, we have $y^m \in L(x)$.

  Clearly $y^m \to y$ s.t. $y_1 = 1$, $y_2 = 2$. Thus $y \notin L(x)$.

  Therefore, $L(x)$ is not closed and hence $\succsim$ is not continuous.

- u does not represent $\succsim$:

  Sps not. Consider x s.t. $x_1 = 1$, $x_2 > 1$. Then

  since $x \succ e$, $u(x) > 1$. Also, consider x' s.t.

  $x_1' = 1 + \varepsilon = x_2'$ for some $\varepsilon > 0$. Since $x' \succ x$, $u(x') > u(x)$.

  Also $x' \sim (1+\varepsilon)e$, $u(x') = 1 + \varepsilon$. Thus $1 < u(x) < 1 + \varepsilon$

  But ε can be sufficiently small, this is a contradiction.

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- Optimal insurance problem

<u>Def</u> A agent w/ expected utility fcn u is strictly <u>risk averse</u> if for any risky consumption plan x, $E[u(x)] < u[E(x)]$.

(i) $\boxed{\text{Claim}}$ The opt. insurance $a^* < L$: full insurance.

Pf If we have the corner sol'n, obviously $a^* = 0 < L$. Thus assume interior sol'n.

★ → Pratt's Thm $\Rightarrow$ u is strictly concave. $u' > 0$, $u'' < 0$.

FOC : a^* : optimal insurance

$$\pi u'(\underbrace{w - L + a^*(1-p)}_{=g})(1-p) + (1-\pi) u'(\underbrace{w - pa^*}_{=l})(-p) = 0$$

$$\Leftrightarrow \frac{u'(g)}{u'(l)} = \frac{p(1-\pi)}{\pi(1-p)}$$

$$> 1 \quad (\because p > \pi)$$

Thus $u'(g) > u'(l)$

Since u is strictly concave ($u'' < 0$), we have

$$g < l$$

$$\Leftrightarrow w - L + a^*(1-p) < w - pa^*$$

$$\Leftrightarrow a^* < L.$$

Therefore, the optimal insurance is less than full insurance.

Q.E.D.

(ii) $\boxed{\text{Claim}}$ a^* is an increasing fcn of L.

Pf

Let $F(a^*, L) \equiv \pi u'(w - L + a^*(1-p))(1-p) + (1-\pi) u'(w - pa^*)(-p) = 0$

★ → Then by the implicit fcn thm,

$$\frac{da^*(L)}{dL} = -\frac{\partial F(a^*, L)/\partial L}{\partial F(a^*, L)/\partial a^*} = -\frac{\pi u''(g)(1-p)(-1)}{\pi u''(g)(1-p)^2 + (1-\pi) u''(w - pa^*)(-p)^2}$$

$$> 0,$$

since u is str. concave ($u'' < 0$).

Therefore a^* is an increasing fcn of L.

Q.E.D.

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7 8

- Local non-satiation + Walrasian demand.

Def $\succsim$ on $\mathbb{R}^L_+$ is strictly convex if for all $x \in \mathbb{R}^L_+$ we have if $y \succsim x$ & $z \succsim x$ & $y \neq z$, then $\alpha y + (1-\alpha) z \succ x$ $\forall \alpha \in (0,1)$.

Def $\succsim$ on $\mathbb{R}^L_+$ is locally non-satiated if $\forall x \in \mathbb{R}^L_+$, $\forall \varepsilon > 0$, $\exists y \in \mathbb{R}^L_+$ s.t. $\|x - y\| < \varepsilon \wedge y \succ x$, where $\|\cdot\|$ is the Euclid norm

Def Walrasian demand $x^*(p, M)$ is a set s.t.

$$x^*(p,M) = \{ x \in \mathbb{R}^L_+ \mid px \leq M \wedge \nexists x' \in \mathbb{R}^L_+ \text{ s.t. } (x' \succ x \wedge px' \leq M)\}$$

(a) $\boxed{\text{Claim}}$ $\succsim$: str. convex + increasing $\Rightarrow$ l.n.s.

pf) Fix an arbitrary $x \in \mathbb{R}^L_+$ and $\varepsilon > 0$.

Since $\mathbb{R}^L_+$ is complete, $\exists y, z \in \mathbb{R}^L_+$ s.t. $y \geq x$, $z \geq x$, $\|y - x\| < \varepsilon$, $\|z - x\| < \varepsilon$ and $y \neq z$.

Since $\succsim$ is complete and increasing, we have $y \succsim x$, $z \succsim x$.

Then by the str. convexity, for some $\alpha \in (0,1)$,

$$\alpha y + (1-\alpha) z \succ x.$$

Since $\mathbb{R}^L_+$ is convex, $\alpha y + (1-\alpha) z \in \mathbb{R}^L_+$.

And

$$\begin{aligned}
\|\alpha y + (1-\alpha) z - x\| &= \|\alpha y - \alpha x + (1-\alpha) z - (1-\alpha) x\| \\
&\leq \|\alpha y - \alpha x\| + \|(1-\alpha) z - (1-\alpha) x\| \\
&\qquad (\because \text{Triangle inequality}) \\
&= \alpha \|y - x\| + (1-\alpha) \|z - x\| \\
&< \varepsilon.
\end{aligned}$$

Since x and ε are arbitrary, this completes the proof. Q.E.D.

(b) $\boxed{\text{Claim}}$ If $\succsim$ is l.n.s, then $\forall x \in x^*(p,M)$, $px = M$.

(pf) Sps not. Then $\exists x^* \in x^*(p,M)$ s.t. $px^* < M$.
Since $\succsim$ is l.n.s, $\forall \varepsilon > 0$, $\exists x \in \mathbb{R}^L_+ \cap B_\varepsilon(x^*)$ s.t. $x \succ x^*$.
Since ε can be sufficiently small,
$\exists x \in \mathbb{R}^L_+ \cap B_\varepsilon(x)$ s.t. $x \succ x^* \wedge px < M$.
This contradicts $x^* \in x^*(p,M)$. Thus $px^* = M$. Q.E.D.

(c) $\boxed{\text{Claim}}$ The following statement is not true :
" $\forall p \in \mathbb{R}^L_{++}$, $\forall M > 0$, if $x \in x^*(p,M)$ satisfies $px = M$, then $\succsim$ is l.n.s."

Counter example :
$L = 1$, $p = 1$, $M = 1$.
$\succsim$: $\forall x, x' \in \mathbb{R}_+$,
$x \succsim x'$ iff $|x-2| \leq |x'-2|$.

Then,
$x^*(1,1) \equiv \{x \in \mathbb{R}_+ \mid x \leq 1 \wedge (\nexists x' \in \mathbb{R}_+ \text{ s.t. } x' \leq 1 \wedge |x'-2| \leq |x-2|)\}$
$= \{1\}$.
Thus, $\forall x \in x^*(1,1)$, $x = 1$, i.e., $px = M$ is satisfied.
But $\succsim$ is not l.n.s :
Take $x = 2$ and $\varepsilon > 0$. Then $\forall y \in \mathbb{R}_+ \cap B_\varepsilon(x)$, $|x-2| \leq |y-2|$
$\Leftrightarrow x \succsim y$.

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◦ Indirect utility fcn

Def Indirect utility fcn is $u^*(p, M) = u(x^*)$ where $x^* \in x^*(p, M)$ and $x^*(p, M)$ is the Walrasian demand for $p \in \mathbb{R}^L_{++}$, $M > 0$.

Def u^* is quasi-convex in p if $\forall \bar{u} \in \mathbb{R}$, $M \in \mathbb{R}_+$,
$A(M, \bar{u}) \equiv \{ p \in \mathbb{R}^L_{++} \mid u^*(p, M) \le \bar{u} \}$ is convex, i.e.,
$\forall p, p' \in A(M, \bar{u})$, $\forall \lambda \in (0,1)$, $\lambda p + (1-\lambda) p' \in A(M, \bar{u})$.

(a) $\boxed{\text{Claim}}$ u^* is quasi-convex.

* pf Let $p, p' \in A(M, \bar{u})$.
Consider $\lambda p + (1-\lambda) p' = p_\lambda$.
$\forall x$ s.t. $p_\lambda x \le M$,

$$
\begin{aligned}
& p_\lambda x \le M \\
\Leftrightarrow\ & \lambda p x + (1-\lambda) p' x \le \lambda M + (1-\lambda) M \\
\Rightarrow\ & \text{at least, } p x \le M \text{ or } p' x \le M \text{ holds.} \\
\Rightarrow\ & \text{at least, } u(x) \le u^*(p, M) \text{ or } u(x) \le u^*(p', M) \text{ holds}
\end{aligned}
$$

(since x is affordable)

$p, p' \in A(M, \bar{u})$ implies
$u(x) \le \bar{u}$.

Therefore, $\forall x$ s.t. $p_\lambda x \le M$, we have $u(x) \le \bar{u}$.
This means that $p_\lambda \in A(M, \bar{u})$. Q.E.D.

(b) $\boxed{\text{Claim}}$ If u is quasi-linear, then u^* is linear in M on the domain of (p, M) where the Walrasian demand is interior.

pf See HW3 Q1 for v diff. case.
Consider a maximization problem:

$$
(*) \begin{cases} \max\limits_{x_1, \cdots, x_L} & u(x_1, \cdots, x_L) = x_1 + v(x_2, \cdots, x_L) \\ \text{s.t.} & p x \le M \\ & x \ge 0 \end{cases}
$$

Since u is strictly increasing ($\because$ v is strictly increasing),
the BC holds w/ equality.
Thus (*) becomes

$$\max_{\{x_2,\cdots,x_L\}} \quad \frac{M}{p_1} - \frac{p_2}{p_1}x_2 - \cdots - \frac{p_L}{p_1}x_L + v(x_2,\cdots,x_L)$$

$$\text{s.t.} \quad \frac{M}{p_1} - \frac{p_2}{p_1}x_2 - \cdots - \frac{p_L}{p_1}x_L \geq 0$$

$$x_2,\cdots,x_L \geq 0.$$

Here we are using $p_1 > 0$ since if $p_1 = 0$ then the Walrasian demand is not well-defined.

Focusing on (p, M) s.t. the Walrasian demand is interior, we know that the all constraints are not binding.

Thus,

$$u^*(p,M) = \max_{\{x_2,\cdots,x_L\}} \frac{M}{p_1} - \frac{p_2}{p_1}x_2 - \cdots - \frac{p_L}{p_1}x_L + v(x_2,\cdots,x_L)$$

Therefore, u^* is a linear fcn of M.

Q.E.D.

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- Independence axiom
- Part of the proof of vNM Thm

Def $\succsim$ on $\mathcal{L}$ is continuous if $\forall L, L', L'' \in \mathcal{L}$,
$A^+(L) \equiv \{\alpha \in [0,1] \mid \alpha L' + (1-\alpha)L'' \succsim L\}$ and $A^-(L) \equiv \{\alpha \in [0,1] \mid L \succsim \alpha L' + (1-\alpha)L''\}$ are closed.

Def Let $X = \{x_1, \cdots, x_K\}$, $L_i = (\pi^i_1, \cdots, \pi^i_K)$, $i = 1, 2$

Compound lottery $L_1 \alpha L_2$ is a lottery that puts a probability $\alpha \pi^1_i + (1-\alpha)\pi^2_i$ on the outcome x_i for every $i = 1, \cdots, K$.

(a) $\succsim$ satisfies the independence axiom if $\forall L, L', L'' \in \mathcal{L}$, $\forall \alpha \in [0,1]$, we have

$$L \succsim L' \quad \text{iff} \quad \alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''.$$

(b) Example of $\succsim$ which does not satisfy (IA).

Let $K = 2$. Consider $\succsim$ on $\mathcal{L}$:

$$L \succsim L' \iff \min\{\pi_1, \pi_2\} \geq \min\{\pi_1', \pi_2'\}$$

Consider 2 lotteries

$$L = \left(\frac{1}{3}, \frac{2}{3}\right), \quad L' = \left(\frac{3}{4}, \frac{1}{4}\right).$$

Then $L \succ L'$ since $\min\{\frac{1}{3}, \frac{2}{3}\} = \frac{1}{3} > \frac{1}{4} = \min\{\frac{3}{4}, \frac{1}{4}\}$.

Taking $\alpha = \frac{1}{2}$ and $L'' = (0, 1)$, we have

$$\alpha L' + (1-\alpha) L'' \succ \alpha L + (1-\alpha) L''$$

since

$$\begin{aligned} &\min\left\{\frac{1}{2}\frac{1}{3} + \frac{1}{2}\cdot 0, \frac{1}{2}\cdot\frac{2}{3} + \frac{1}{2}\cdot 1\right\} \\ =\ &\min\left\{\frac{1}{6}, \frac{5}{6}\right\} \\ =\ &\frac{1}{6} \\ <\ &\frac{3}{8} \\ =\ &\min\left\{\frac{3}{8}, \frac{5}{8}\right\}. \end{aligned}$$

Thus, $\succsim$ does not satisfy the (IA).

(c) $\boxed{\text{Claim}}$ If $\succsim$ is continuous and satisfies (IA), then $\forall L_1, L_2, L$ s.t. $L_1 \succ L \succ L_2$ $\exists! \alpha \in [0,1]$ s.t. $L \sim L_1 \alpha L_2$.

(pf) Since $\succsim$ is continuous,

$$A^+(L) = \{\alpha \in [0,1] \mid \alpha L_1 + (1-\alpha) L_2 \succsim L\}$$

and

$$A^-(L) = \{\alpha \in [0,1] \mid L \succsim \alpha L_1 + (1-\alpha) L_2\}$$

are closed.

Since $1 \in A^+(L)$ and $0 \in A^-(L)$, both are nonempty.

Clearly $A^+(L) \cup A^-(L) = [0,1]$.

Thus, since $[0,1]$ is connected, $\exists \alpha_L \in [0,1]$ s.t. $\alpha_L \in A^+(L) \cap A^-(L)$,

i.e., $L \sim L_1 \alpha_L L_2$.

Next, prove the uniqueness of α_L

Sps $\exists \alpha, \alpha' \in A^+(L) \cap A^-(L)$ s.t. $\alpha \neq \alpha'$.

WLOG, let $\alpha > \alpha'$.

$$L_1 \alpha L_2 = \beta L_1 + (1-\beta)(L_1 \alpha' L_2)$$

where $\beta = \frac{\alpha - \alpha'}{1-\alpha'}$

Note that $\beta \in [0,1]$ since $\alpha > \alpha'$.

Then,

$$\begin{aligned} L_1 \alpha L_2 &= \beta L_1 + (1-\beta)(L_1 \alpha' L_2) \\ &\succ \beta (L_1 \alpha' L_2) + (1-\beta)(L_1 \alpha' L_2) \\ &= L_1 \alpha' L_2, \end{aligned}$$

where the 2nd line holds by the (IA) and the fact that

$$L_1 \succ L_1 \alpha' L_2$$

given by

$$\begin{aligned} L_1 &= L_1 \alpha' L_1 \\ &\succ L_1 \alpha' L_2 \qquad (\text{by (IA)}, \ L_1 \succ L_2). \end{aligned}$$

But this contradicts that $\alpha, \alpha' \in A^+(L) \cap A^-(L)$, i.e.,

$$L_1 \alpha L_2 \sim L \sim L_1 \alpha' L_2.$$

Therefore, $\alpha_L \in A^+(L) \cap A^-(L)$ is unique.

Q.E.D.

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- Dual Theory
- Examples of util. fcn which does not satisfy each property.

<u>Def</u> Walrasian demand $x^*(\cdot,\cdot)$ is a set s.t. for $p \gg 0$, $w > 0$,
$$x^*(p,w) = \{x \in \mathbb{R}^n_+ \mid px \le w \wedge \nexists x' \in \mathbb{R}^n_+ \text{ s.t. } u(x') > u(x) \wedge px' \le w\}$$
<u>Def</u> Hicksian demand $h(\cdot,\cdot)$ is a set s.t. for $p \gg 0$, $\bar{u} \in \mathbb{R}$,
$$h(p,\bar{u}) = \{x \in \mathbb{R}^n_+ \mid u(x) \ge \bar{u} \wedge \nexists x' \in \mathbb{R}^n_+ \text{ s.t. } px' < px \wedge u(x) \ge \bar{u}\}.$$
<u>Def</u> $u : \mathbb{R}^n_+ \to \mathbb{R}$ is a quasi-concave utility fcn if $\forall x, x' \in \mathbb{R}^n_+$, $\forall \alpha \in (0,1)$
$$u(\alpha x + (1-\alpha)x') \ge \min\{u(x), u(x')\}$$

(i) $\boxed{\text{Claim}}$ If u is continuous and locally non-satiated, then
$$h(p,\bar{u}) = x^*(p, e(p,\bar{u})) \text{ and}$$
$$x^*(p,w) = h(p, u^*(p,w)).$$

pf

<u>Lemma 1</u> : u is l.n.s $\Rightarrow$ $px = w$, $\forall x \in x^*(p,w)$

(pf) Sps not. Then $\exists x \in x^*(p,w)$ s.t. $px < w$.

Let $\varepsilon > 0$ s.t. $\varepsilon \le w - px$. Then $px + \varepsilon \le w$ or $\sum_{i=1}^n p_i(x_i + \frac{\varepsilon}{np_i}) \le w$.

Let $\bar{\varepsilon} \equiv \min_i \frac{\varepsilon}{np_i}$.

Since u is l.n.s, $\exists x' \in \mathbb{R}^n_+$ s.t. $\|x' - x\| < \bar{\varepsilon} \wedge u(x') > u(x)$

But we have $px' \le w$ since
$$px' \le p(x + \bar{\varepsilon}e) \quad \text{where } e : \text{unit vector}$$
$$\le px + \varepsilon \le w$$
This contradicts that $x \in x^*(p,w)$ □

<u>Lemma 2</u> : u is continuous $\Rightarrow$ $u(x) = \bar{u}$, $\forall x \in h(p,\bar{u})$.

(pf) Sps not. Then $\exists x \in h(p,\bar{u})$ s.t. $u(x) > \bar{u}$.

Let $\varepsilon = u(x) - \bar{u}$.

Since u is continuous, $\exists \delta > 0$ s.t. $\forall x'$ s.t. $\|x' - x\| < \delta$, $|u(x') - u(x)| < \varepsilon$

Let $x' \equiv \alpha x$ w/ $\alpha \in (0,1)$. Clearly $p \cdot x' < px$

But we have $u(x') > \bar{u}$ for α sufficiently close to 1 since
$$\|x' - x\| < \delta \Rightarrow |u(x') - u(x)| < \varepsilon$$
$$\Rightarrow 0 < u(x') - u(x) < \varepsilon \text{ or } 0 < u(x) - u(x') < \varepsilon = u(x) - \bar{u}$$
$$\Rightarrow u(x') > u(x) \ge \bar{u} \text{ or } u(x') > \bar{u}. \rightarrow\leftarrow x \in h(p,\bar{u}). \ \square$$

Prove that $h(p,\bar{u}) \subset x^*(p, e(p,\bar{u}))$. $\cdots$ (*)

Pick $x \in h(p,\bar{u})$.

Then we know $e(p,\bar{u}) = px$ (by def) & $u(x) = \bar{u}$ (by Lemma 2).

Sps not. Then $x \notin x^*(p, e(p,\bar{u}))$.

$\exists\, x' \in \mathbb{R}^n_+$ s.t. $u(x') > u(x) \wedge px' \leq e(p,\bar{u})$.

$\Rightarrow u(x') > \bar{u} \wedge px' \leq px$.

Since u is continuous, $\exists\, x'' \in \mathbb{R}^n_+$ s.t. $u(x'') \geq \bar{u} \wedge px'' < px' \leq px$.

This contradicts that $x \in h(p,\bar{u})$. Thus $x \in x^*(p, e(p,\bar{u}))$.

* → Also $u^*(p, e(p,\bar{u})) = u(x) = \bar{u}$. (#)

Prove that $x^*(p,w) \subset h(p, u^*(p,w))$. $\cdots$ (**)

Pick $x \in x^*(p,w)$. Then $u(x) = u^*(p,w)$ (by def) and $px = w$ (by Lemma 1)

Sps not. Then $x \notin h(p, u^*(p,w))$.

$\exists\, x' \in \mathbb{R}^n_+$ s.t. $px' < px \wedge u(x') \geq u^*(p,w)$.

$\Rightarrow px' < w \wedge u(x') \geq u(x)$

Since u is l.n.s, $\exists\, x'' \in \mathbb{R}^n_+$ s.t. $px'' \leq w \wedge u(x'') > u(x') \geq u(x)$

This contradicts that $x \in x^*(p,w)$. Thus $x \in h(p, u^*(p,w))$

* → Also $e(p, u^*(p,w)) = px = w$. (##)

Prove that $x^*(p, e(p,\bar{u})) \subset h(p,\bar{u})$.

Pick $x \in x^*(p, e(p,\bar{u}))$.

Then $x \in x^*(p, e(p,\bar{u})) \subset h(p, u^*(p, e(p,\bar{u})))$ by (**)

$= h(p,\bar{u})$ by (#).

Prove that $h(p, u^*(p,w)) \subset x^*(p,w)$

Pick $x \in h(p, u^*(p,w))$.

Then $x \in h(p, u^*(p,w)) \subset x^*(p, e(p, u^*(p,w)))$ by (*)

$= x^*(p,w)$ by (##)

Thus we get

$$h(p,\bar{u}) = x^*(p, e(p,\bar{u})).$$

and $$x^*(p,w) = h(p, u^*(p,w)).$$

Q.E.D.

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(ii) Ex of : ① u is not continuous $\Rightarrow x \neq h$

② 〃 l.n.s. $\Rightarrow$ 〃

① $n=2$.

$$u(x) = \begin{cases} x_1+x_2 & \text{if } x_1+x_2<1 \\ 1 & \text{〃 } =1 \wedge x_1 \le \frac{1}{2} \\ 2 & \text{〃 } =1 \wedge x_1 > \frac{1}{2} \\ x_1+x_2+1 & \text{〃 } >1 \end{cases}$$

Then u is not continuous.

Set $p=(1,1)$, $\bar{u}=1$. Then

$h(p,\bar{u}) = \{x \in \mathbb{R}^2_+ \mid x_1+x_2=1\}$, $e(p,\bar{u})=1$.

$x(p, e(p,\bar{u})) = \{x \in \mathbb{R}^2_+ \mid x_1+x_2=1, x_1>\frac{1}{2}\}$.

② $n=1$

$$u(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 1 & x \ge 1. \end{cases}$$

Then u is not l.n.s.

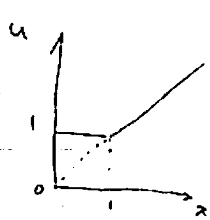
Set $p=1$, $w=2$ Then

$x(p,w) = [1,2]$, $u^*(p,w) = 1$

$h(p, u^*(p,w)) = \{1\}$.

(iii). Ex of : u : quasi-concave but not concave.

$$u(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ x & x \ge 1 \end{cases}$$



$\forall x, x' \in \mathbb{R}_+$, $\forall \lambda \in (0,1)$,

$u(\lambda x + (1-\lambda)x') \ge \min\{u(x), u(x')\} \Rightarrow$ quasi-concave

But for $x=\frac{1}{2}$, $x'=\frac{3}{2}$, $\lambda=\frac{1}{2}$,

$u(\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{3}{2}) = u(1) = 1$

$\frac{1}{2}u(\frac{1}{2}) + \frac{1}{2}u(\frac{3}{2}) = \frac{1}{2} + \frac{3}{4} = \frac{5}{4}$ $\Rightarrow$ not concave.

$\boxed{\text{Claim}}$ u : quasi-concave $\Rightarrow$ x^* : convex valued.

pf) Take $x, x' \in x^*(p,w)$

Prove that $\alpha x + (1-\alpha)x' \equiv x_\alpha \in x^*(p,w)$, $\forall \alpha \in (0,1)$.

Since $x, x' \in x^*(p,w)$, $u^*(p,w) = u(x) = u(x')$.

Since u is quasi-concave,

$$u(x_\alpha) \geq \min\{u(x), u(x')\}$$
$$= u^*(p,w).$$

Also, x_α is affordable :

$$p x_\alpha = \alpha p x + (1-\alpha) p x'$$
$$\leq w.$$

Therefore, $x_\alpha \in x^*(p,w)$.

Since x, x' are arbitrary, x^* is convex-valued. Q.E.D.

Micro 2003 Spring Q II . 2

∘ Lexicographic Preference and the utility representation

<u>Def</u> Lexicographic preference on $\mathbb{Q}_+ \times \mathbb{Q}_+$ is a preference $\succsim$ s.t. $\forall x, y \in \mathbb{Q}_+^2$,

$$x \succsim y \iff x_1 > y_1 \text{ or } (x_1 = y_1 \wedge x_2 \geq y_2)$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$.

<u>Def</u> $\succsim$ has a utility (function) representation $u : \mathbb{Q}_+^2 \to \mathbb{R}$ if $\forall x, y \in \mathbb{Q}_+^2$,

$$x \succsim y \iff u(x) \geq u(y).$$

$\boxed{\text{Claim}}$ $\succsim$ does not have a utility representation.

pf Sps $\succsim$ has a utility rep. u, i.e.,

$$x \succsim y \iff u(x) \geq u(y).$$

Consider x, x' s.t. $x = (r, 1)$, $x' = (r, 0)$ where r is an arbitrary rational number.

By def of lexicographic pref, we have

$$x \succ x'.$$

Thus $$u(x) > u(x').$$

Since $u(x), u(x') \in \mathbb{R}$ and $\mathbb{R}$ is a complete space,

$\exists\, p \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $u(x) > p > u(x')$.

Similarly, consider y, y' s.t. $y = (r', 1)$, $y' = (r', 0)$ where $r' \in \mathbb{Q}_+$ s.t. $r' < r$.

We also have $\exists\, p' \in \mathbb{R} \setminus \mathbb{Q}$ s.t.

$$u(y) > p' > u(y').$$

But since we have $p > u(x') > u(y) > p'$,

we have $\forall r, r' \in \mathbb{Q}$, $\exists\, p, p' \in \mathbb{R} \setminus \mathbb{Q}$ s.t.

$$r > r' \Rightarrow p > p'.$$

Micro 2003 Spring Q II.3

8101

$\boxed{16}$

∘ Corollary of Pratt's Thm

(a) <u>Def</u> Sps v is strictly increasing and twice differentiable.
Then Arrow-Pratt measure of absolute risk aversion at wealth w is defined by
$$A(w) = \frac{-v''(w)}{v'(w)}.$$

<u>Def</u> Risk compensation for v with a deterministic wealth w and a risk z s.t. $E(z) = 0$ is defined by $\rho(w, z)$ s.t.
$$E[v(w+z)] = v(w - \rho(w,z)).$$

(b) $\boxed{\text{Claim}}$ $A(w)$ is increasing in w iff $\rho(w,z)$ is increasing in w for z w/ $E(z)=0$.

pf (Sketch : see 2007 spring Q I.2 for the detail)

Take $\tilde{v}(w) = v(w+\varepsilon)$, $\forall w$.
Then prove that $A_{\tilde{v}}(w) = A_v(w+\varepsilon)$ and $\rho_{\tilde{v}}(w,z) = \rho_v(w+\varepsilon, z)$.
Then apply the Pratt's Thm.

Q.E.D

$\boxed{\text{Pratt's Thm}}$ The followings are equivalent : (v_1, v_2 : strictly inc & twice diff')

(1) $A_1(w) \geq A_2(w)$, $\forall w \in \mathbb{R}$

(2) $\rho_1(w, z) \geq \rho_2(w, z)$, $\forall w \in \mathbb{R}$, $\forall$ risky plan z w/ $E(z) = 0$

(3) $v_1(w) = f(v_2(w))$ $\forall w \in \mathbb{R}$ w/ f : concave & strictly increasing

pf) (1) $\Rightarrow$ (3) : LN p.36

(3) $\Rightarrow$ (2) : LN p.36

(2) $\Rightarrow$ (1) :

Sps not. Then $A_1(w) < A_2(w)$, $\exists w \in \mathbb{R}$

Thus by (1) $\Rightarrow$ (3), $\exists f$ strictly concave s.t. $v_2(w) = f\, v_1(w)$

For some z w/ $E(z) = 0$,

$$E\, v_2(w, z) = v_2(w - \rho_2(w, z))$$

$$\shortparallel$$

$$E\, f(v_1(w, z)) < f[E(v_1(w, z))] \quad \text{(by Jenson's inequality)}$$

$$= f[v_1(w - \rho_1(w, z))]$$

$$= v_2(w - \rho_1(w, z)) \quad \text{(by def of } f\text{)}$$

Since v_2 is strictly increasing,

$$\rho_1(w, z) < \rho_2(w, z)$$

This is a contradiction. Thus $A_1(w) \geq A_2(w)$.

Q.E.D.

Micro 2002 Fall Q II.1

8/01

- Hicksian demand fcn.
- Slutsky matrix

Def $h: \mathbb{R}^n_{++} \times \mathbb{R} \rightrightarrows \mathbb{R}^n_+$ is a Hicksian demand if
$$h(p,\bar{u}) = \{ h \in \mathbb{R}^n_+ \mid u(h) \geq \bar{u} \text{ and } \nexists h' \in \mathbb{R}^n_+ \text{ s.t. } u(h') \geq \bar{u} \wedge ph' < ph \}$$

(a) $\boxed{\text{Claim}}$ h is monotonically decreasing in p:
$$[p - p'][h(p,\bar{u}) - h(p',\bar{u})] \leq 0. \quad \forall p, p', \bar{u}.$$

pf By def of h (h is a fcn), $\forall p, p', \bar{u}$,
$$p\, h(p,\bar{u}) \leq p\, h(p',\bar{u}),$$
$$p'\, h(p',\bar{u}) \leq p'\, h(p,\bar{u}).$$
Adding the both sides gives us
$$[p - p'][h(p,\bar{u}) - h(p',\bar{u})] \leq 0 \qquad \square$$

(b) $\boxed{\text{Claim}}$ $D_p h(p,\bar{u})$ is negative semi-definite.

pf Define the expenditure fcn:
$$e(p,\bar{u}) \equiv p\, h(p,\bar{u}).$$
Prove that e is concave in p:
$$\begin{aligned} e(\lambda p + (1-\lambda)p', \bar{u}) &= [\lambda p + (1-\lambda)p']\, h(\lambda p + (1-\lambda)p', \bar{u}) \\ &\geq \lambda p\, h(p,\bar{u}) + (1-\lambda)p'\, h(p',\bar{u}) \\ &= \lambda e(p,\bar{u}) + (1-\lambda) e(p', u). \end{aligned}$$

★ → Prove that $D_p e(p,\bar{u}) = h(p,\bar{u})$:

pf 1) It is easy to prove that h is HD 0.
So $\forall \alpha > 0$, $h_i(\alpha p, \bar{u}) = h_i(p,\bar{u})$, $\forall i \in \{1, \dots, n\}$
Differentiate by α: $\quad p_i h_i'(\alpha p, \bar{u}) = 0$
Set $\alpha = 1$: $\quad p_i h_i'(p,\bar{u}) = 0$
$$\therefore \quad D_p h(p,\bar{u})\, p = 0$$
Thus
$$\begin{aligned} D_p e(p,\bar{u}) &= h(p,\bar{u}) + D_p h(p,\bar{u})\, p \\ &= h(p,\bar{u}). \end{aligned} \qquad \square$$

$$\text{pf:} \quad \text{Define} \quad F(p) = \overset{e(p,\bar{u})}{\overbrace{p\,h(p,\bar{u})}} - p\,h(\bar{p},\bar{u}) \quad \text{for some } \bar{p} \in \mathbb{R}^n_{++}$$

By def of h, $\quad F(p) \le 0$

and $\quad F(\bar{p}) = 0.$

Then $\quad DF(p) = D_p e(p,\bar{u}) - h(\bar{p},\bar{u})$

Since $\bar{p}$ is a sol'n to $F(p) = 0$,

$$DF(\bar{p}) = 0 = D_p e(\bar{p},\bar{u}) - h(\bar{p},\bar{u})$$

$$\therefore D_p e(p,\bar{u}) = h(p,\bar{u}) \qquad (\because \bar{p} \text{ is arbitrary})$$

▷

Since h is diff',

$$D_p^2 e(p,\bar{u}) = D_p h(p,\bar{u})$$

And since e is concave, we have $D_p h(p,\bar{u})$ is negative semi-definite.

(c) Dual Theory:

$$\begin{cases} h(p,\bar{u}) = x^*(p, e(p,\bar{u})) \\ x^*(p,w) = h(p, u^*(p,w)) \end{cases}$$

Define $w = e(p,\bar{u})$

Then we have

$$\star \rightarrow \quad D_p h(p,\bar{u}) = D_p x^*(p,w) + D_w x^*(p,w) \cdot x^*(p,w)$$

Thus the Slutsky matrix

$$S(p,w) \equiv D_p x^*(p,w) + D_w x^*(p,w)\, x^*(p,w)$$

is negative semi-definite.

Moreover, we have

$$\frac{\partial x^*_i(p,w)}{\partial p_i} + \frac{\partial x^*_i(p,w)}{\partial w} x^*_i(p,w) \le 0 \qquad \forall i \in \{1,\cdots,n\}.$$

Micro

F 2007	Allen	~~Rustichini~~	~~Werner*~~		
S 2007	Allen	~~Rustichini*~~	~~Werner~~		
F 2006	Allen	~~Rustichini*~~	~~Werner~~		
S 2006	Allen	McLennan	Rustichini	~~Werner*~~	
F 2005	Allen	Hurwicz	McLennan	Rustichini	~~Werner*~~
S 2005	Allen*	Hurwicz	McLennan	Rustichini	~~Werner~~
F 2004	Allen*	Hurwicz	McLennan	Rustichini	~~Werner~~
S 2004	Allen	Hurwicz	Richter	Rustichini*	
F 2003	Allen	Hurwicz	Richter*	Rustichini	
S 2003	Allen	Hurwicz	Richter	Rustichini	~~Werner*~~
F 2002	Allen	Hurwicz*	Rustichini	~~Werner~~	
S 2002	Allen	Rustichini	Richter	Werner*	
~~F 2001~~	Allen	McLennan	Richter*	Werner	
~~S 2001~~	Allen	McLennan	Richter	Werner*	
~~F 2000~~	Allen	McLennan*	Richter	Werner	
~~S 2000~~	Allen	McLennan	Richter	Werner*	

Macro

F 2007	Chari	Jones*	P.Kehoe	Perri	
S 2007	Chari*	Jones	T. Kehoe	Kocherlakota	
S 2006	Chari	Jones	T. Kehoe		
F 2005	Chari	Jones*	T. Kehoe		
S 2005	Chari*	Jones	T.Kehoe		
F 2004	Chari*	Jones	T. Kehoe		
S 2004	Chari*	Jones	T.Kehoe		
F 2003	Chari*	Jones	T.Kehoe		
S 2003	Chari	Jones	Prescott*		
F 2002	Chari	Jones	Prescott*	Kocherlakota	
S 2002	Chari*	Jones	Prescott	Kocherlakot	
F 2001	Chari*	Jones	Prescott	Kocherlakota	
S 2001	Jones*	T. Kehoe	Kocherlakota	Prescott	
F 2000	T. Kehoe*	Jones	Kocherlakota	Prescott*	
S 2000	T. Kehoe*	Jones	Kocherlakota	Santos	